EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 2, 2014, 201-209
ISSN 1307-5543 – www.ejpam.com

Primary Decomposition in Lattice Modules

C S Manjarekar [1], U N Kandale[2,*]

[1] *Department of Mathematics, Shivaji University, Kolhapur, Maharashtra, India*
[2] *General Engineering Department, Sharad Institute of Technology College of Engineering, Shivaji University, Kolhapur, India*

Abstract. In this paper, we study primary decomposition of elements in lattice modules. A necessary and sufficient condition for a prime element p of a multiplicative lattice L to be equal to some associated prime of an element in a lattice module having primary decomposition is obtained.

2010 Mathematics Subject Classifications: 13A99

Key Words and Phrases: Prime element,primary element,lattice modules,primary decomposition

---

1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $a \in L$ is called proper if $a < 1$. A proper element p of L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. If $a \in L, b \in L$, $(a : b)$ is the join of all elements c in L such that $cb \leq a$. A proper element p of L is said to be primary if $ab \leq p$ implies $a \leq p$ or $b^n \leq p$ for some positive integer n. If $a \in L$, the radical of a denoted by $\sqrt{a} = \vee\{x \in L \mid x^n \leqslant a, n \in Z_+\}$. An element $a \in L$ is called compact if $a \leqslant \underset{\alpha}{\vee} b_\alpha$ implies $a \leqslant b_{\alpha 1} \vee b_{\alpha 2} \vee \ldots \vee b_{\alpha n}$ for some finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Throughout this paper, L denotes a multiplicative lattice which satisfies the ACC so that each element of L is compact. If q is a primary element of L then

$$p_q = \vee\{x \in L \mid x^n \leqslant q, \text{ for some integer } n\}$$

is a prime element containing q. It is easily verified that, p_q is a minimal prime containing q [2]. The prime p_q which is same as $\sqrt{q}$ is called the prime associated with q and has the properties, $p_q^k \leqslant q \leqslant p_q$ for some integer k and $ab \leqslant q$ implies $a \leqslant q$ or $b \leqslant p_q$.

An element a is said to have a primary decomposition if there exist primary elements $q_1, q_2, \ldots, q_n$ such that $a = q_1 \wedge q_2 \wedge \ldots \wedge q_n$. If some q_i contains the meet of remaining ones

*Corresponding author.

Email addresses: csmanjrekar@yahoo.co.in (C. Manjarekar), ujwalabiraje@gmail.com (U. Kandale)

then this q_i can be dropped from the primary decomposition. After deleting such primary components and combining the primary components with same associated prime we get a reduced primary decomposition in which distinct primaries are associated with distinct primes.Such a primary decomposition is also called an irredundant primary decomposition, reduced primary decomposition or normal primary decomposition. Let $a = q_1 \wedge q_2 \ldots \wedge q_n$ be a reduced primary decomposition of a and let $p_1, p_2, \ldots, p_n$ denotes the associated primes of $q_1, q_2, \ldots, q_n$ respectively,which are also called associated primes of a. A subset C of $\{p_1, p_2, \ldots, p_n\}$ is called isolated if $p_i \in C$ *implies* $p_j \in C$ whenever $p_j \leqslant p_i$.

Let M be a complete lattice and L be a multiplicative lattice. Then M is called L-module or module over L if there is a multiplication between elements of L and M written as aB where $a \in L$ and $B \in M$ which satisfies the following properties,

i) $(\underset{\alpha}{\vee} a_\alpha)A = \underset{\alpha}{\vee} a_\alpha A \ \forall a_\alpha \in L, A \in M$

ii) $a(\underset{\alpha}{\vee} A_\alpha) = \underset{\alpha}{\vee} aA_\alpha \ \forall a \in L, A_\alpha \in M$

iii) $(ab)A = a(bA) \ \forall a, b \in L, A \in M$

iv) $IB = B$

v) $0B = 0_M$, for all $a, a_\alpha, b \in L$ and $A, A_\alpha \in M$,

where I is the supremum of L and 0 is the infimum of L. We denote by 0_M and I_M the least element and the greatest element of M. The elements of L will generally be denoted by $a, b, c, \ldots$ and elements of M will generally be denoted by $A, B, C \ldots$.

Let M be a L-module. If $N \in M$ and $a \in L$ then $(N : a) = \vee\{X \in M \mid aX \leqslant N\}$. If $A, B \in M$, then $(A : B) = \vee\{x \in L \mid xB \leqslant A\}$. An L-module M is called a multiplication L-module if for every element $N \in M$ there exists an element $a \in L$ such that $N = aI_M$ [4].

A proper element N of M is said to be prime if $aX \leqslant N$ implies $X \leqslant N$ or $aI_M \leqslant N$ that is $a \leqslant (N : I_M)$ for every $a \in L$, $X \in M$. An element $N < I_M$ in M is said to be primary if $aX \leqslant N$ implies $X \leqslant N$ or $a^n I_M \leqslant N$ that is $a^n \leqslant (N : I_M)$ for some integer n. An element N of M is called a radical element if $(N : I_M) = \sqrt{(N : I_M)}$. Noether lattice is a modular multiplicative lattice satisfying ascending chain condition in which every element is the join of principal elements. Let N be an element of a lattice module M. Then N is said to have a primary decomposition if there exist primary element $Q_1, Q_2, \ldots, Q_n$ such that $N = Q_1 \wedge Q_2 \wedge \ldots Q_n$. If some Q_i contains the meet of remaining ones then this Q_i can be dropped from the primary decomposition. Similarly, any other primary components which contains the meet of remaining ones can be dropped from the primary decomposition. If no Q_i can be dropped further we get a reduced primary decomposition of N. Such a primary decomposition is also called an irredundant primary decomposition. If Q is primary then $\sqrt{Q} = \sqrt{(Q : I_M)}$ is prime. We note that, $\sqrt{(N : I_M)}$ may also be denoted by $\sqrt{N}$.

2. Primary Decomposition of Elements

The following result gives the relation between a primary element Q and $\sqrt{(Q : I_M)}$.

Theorem 1. *If Q is a primary element of a lattice module M then $\sqrt{(Q:I_M)}$ is a prime element of L. If a is an element of L and if $a \leqslant p$ where p is a prime element of L then $\sqrt{a} \leqslant p$.*

Proof. Let $ab \leqslant \sqrt{(Q:I_M)}$ and suppose $b \nleqslant \sqrt{(Q:I_M)}$. Then $(ab)^n = a^n b^n \leqslant (Q:I_M)$ for some positive integer n. Now $b \nleqslant \sqrt{(Q:I_M)}$ implies $b^m \nleqslant (Q:I_M)$ for any positive integer m. In particular $b^n \nleqslant (Q:I_M)$. As Q is primary, $(a^n)^k \leqslant (Q:I_M)$ for some positive integer k. That is $a^t \leqslant (Q:I_M)$ for some positive integer t and $a \leqslant \sqrt{(Q:I_M)}$. Therefore $\sqrt{(Q:I_M)}$ is prime. Let $a \leqslant p$. Take any $x \leqslant \sqrt{a}$. Then $x^n \leqslant a \leqslant p$ for some positive integer n. As p is prime, $x \leqslant p$ and hence $\sqrt{a} \leqslant p$. □

The following theorem gives the relation between meet of primary elements and their equal associated primes.

Theorem 2. *If $Q_1, Q_2, \ldots, Q_k$ are p-primary elements of a lattice module M then $Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$ is p-primary.*

Proof. By hypothesis $\sqrt{(Q_i:I_M)} = p$, $i = 1,2,\ldots,k$. Let $Q = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$. We have, $\sqrt{\wedge Q_i} = \sqrt{((\wedge Q_i):I_M)} = \sqrt{(Q_1:I_M)} \wedge \sqrt{(Q_2:I_M)} \wedge \ldots \wedge \sqrt{(Q_k:I_M)} = p$. We show that $\wedge Q_i$ is primary, where $i = 1,2,\ldots,k$. Let $aX \leqslant Q = \wedge Q_i$ where $a \in L$, $X \in M$. Suppose, $X \nleqslant Q$. Then $X \nleqslant Q_i$ for some i $(1 \leqslant i \leqslant k)$. As Q_i is primary, $aX \leqslant Q_i$ and $X \nleqslant Q_i$ implies $a \leqslant \sqrt{(Q_i:I_M)} = p$. That is $a \leqslant \sqrt{(Q:I_M)}$. Therefore, Q is primary. □

It is shown by Thakare and Manjarekar [6] that the radical of any element a of a multiplicative lattice satisfing the ACC can be written as the meet of minimal prime divisors of a. Hence, we have the following result.

Theorem 3. *Let L be a multiplicative lattice satisfing the ACC and N be an element of M then $\sqrt{(N:I_M)} = \wedge\{p \mid p$ is minimal prime containing $(N:I_M)\}$.*

An element N of a lattice module M is said to be meet irreducible if for any two elements A_1 and A_2 of M, $N = A_1 \wedge A_2$ implies either $A_1 = N$ or $A_2 = N$.

Theorem 4. *If a lattice module M satisfies ACC the chain $A_1 \leqslant A_2 \leqslant \ldots$ implies there exist positive integer m such that $A_n = A_m$ for all $n \geq m$. Then every element of M can be written as the meet of a finite number of meet irreducible elements of M.*

Proof. Let τ be the set of all elements of M which can not be written as a meet of a finite number of meet irreducible elements of M. If τ is empty we have nothing to prove. Suppose τ is not empty. As M satifies ACC, τ has a maximal element say N. As $N \in \tau$, N is not irreducible. So there exists elements A_1 and A_2 of M such that $N = A_1 \wedge A_2$ where $N \neq A_1$, $N \neq A_2$. So, $N < A_1, N < A_2$. This shows that both A_1 and A_2 can be written as the meet of a finite number of meet irreducible elements of M. So there are irreducible elements $K_1, K_2, \ldots, K_m$ and $K_1^{'}, K_2^{'}, \ldots, K_n^{'}$ of M such that $A_1 = K_1 \wedge K_2 \wedge \ldots \wedge K_m$ and $A_2 = K_1^{'} \wedge K_2^{'} \wedge \ldots \wedge K_n^{'}$. But then $N = K_1 \wedge K_2 \ldots \wedge K_m \wedge K_1^{'} \wedge K_2^{'} \wedge \ldots K_n^{'}$. That is N is the meet of a finite number of meet irreducible elements. This contradicts the fact that $N \in \tau$. Hence, τ is empty. □

The study of primary elements and their associated primes for modules is carried out by P J Mc Carthy and Larsen [5]. We give eqivalent formulation in the next theorems for lattice modules.

Theorem 5. *Let Q be a p-primary element of lattice module M and N be an element of M. If $N \nleqslant Q$ then $(Q:N)$ is a p-primary element.*

Proof. First we show that $(Q:N)$ is a p-primary element. Let $a,b \in L$, $ab \leqslant (Q:N)$ and suppose, $a \nleqslant (Q:N)$. As $a \nleqslant (Q:N)$, $aN \nleqslant Q$. Also as $ab \leqslant (Q:N)$, $abN \leqslant Q$. But $aN \nleqslant Q$ and Q is a primary element implies that $b^n \leqslant (Q:I_M)$ for some integer n. But $b^n I_M \leqslant Q$ implies $b^n N \leqslant Q$. Hence, $b \leqslant \sqrt{(Q:N)}$. Therefore, $(Q:N)$ is a primary element of L. Now since $N \nleqslant Q$, there exists $A \in M$ and $A \leqslant N$ such that $A \nleqslant Q$. Let $a \leqslant \sqrt{(Q:N)}$. Then $a^n N \leqslant Q$. Hence, $a^n A \leqslant Q$. But $A \nleqslant Q$ and Q is primary implies that $(a^n)^k = a^m \leqslant (Q:I_M)$ for some integer m. That is $a \leqslant \sqrt{(Q:I_M)} = p$ and $\sqrt{(Q:N)} \leqslant p$. Conversely, let $a \leqslant \sqrt{(Q:I_M)} = p$. Hence, $a^n I_M \leqslant Q$ for some integer n. So $a^n N \leqslant Q$ for some integer n. Thus $a^n \leqslant (Q:N)$ and $a \leqslant \sqrt{(Q:N)}$. This shows that $p \leqslant \sqrt{(Q:N)}$ and we have $\sqrt{(Q:N)} = p$. Therefore,$(Q:N)$ is a p-primary element. □

Theorem 6. *Let M be a lattice module and a be an element of L, p be a prime element of L and Q be p-primary element of M. If $a \nleqslant p$ then $(Q:a) = Q$.*

Proof. Suppose $a \nleqslant p$ where $p = \sqrt{(Q:I_M)}$. Since, $a \nleqslant p$ there is some $b \leqslant a$ such that $b \nleqslant p$. Let $X \leqslant (Q:a)$. Then $aX \leqslant Q$ and hence $bX \leqslant Q$ where $b \nleqslant \sqrt{(Q:I_M)} = p$. As Q is a p-primary, $X \leqslant Q$. Hence, $(Q:a) \leqslant Q$. Conversely let $X \leqslant Q$. Since $a \leqslant 1$, $aX \leqslant Q$. So $X \leqslant (Q:a)$ and hence $Q \leqslant (Q:a)$. Therefore,$Q = (Q:a)$. □

The following theorem gives the characterisation of a prime element p of L to be equal to some associated prime of an element which has a primary decomposition.

Theorem 7. *Let $N \neq I_M$ be an element of a lattice module M and assume that N has a primary decomposition. Let $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$ be a reduced primary decomposition of N and p be prime element of L. Then $p = \sqrt{Q_i}$ for some i if and only if $(N:X)$ is a p-primary element of L for some $X \nleqslant N$.*

Proof. Let $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$ be a reduced primary decomposition of N. First suppose that, $p = \sqrt{Q_i}$ for some i. Without loss of generality we can assume that $p = \sqrt{(Q_1:I_M)}$ where $p_i = \sqrt{(Q_i:I_M)}$ $i = 1,2,\ldots,k$. We prove that, $(N:X)$ is a p-primary element of L for some $X \nleqslant N$. Since the decomposition is reduced $Q_i \ngeqslant Q_1 \wedge Q_2 \wedge \ldots \wedge Q_{i-1} \wedge Q_{i+1} \wedge \ldots \wedge Q_k$ for $i = 1,2,\ldots,k$. In particular,$Q_1 \ngeqslant Q_2 \wedge Q_3 \wedge \ldots \wedge Q_k$. So there exists $X \leqslant Q_2 \wedge Q_3 \wedge \ldots \wedge Q_k$ such that $X \nleqslant Q_1$ and hence $X \nleqslant N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$. Also

$$(N:X) = (Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k):X = (Q_1:X) \wedge (Q_2:X) \wedge \ldots \wedge (Q_k:X).$$

For $i = 2,3,\ldots,k$ we show that $(Q_i:X) = 1$. Since $X \leqslant Q_2 \wedge Q_3 \ldots \wedge Q_k$, we have $X \leqslant Q_i$ for all $i = 2,\ldots,k$. Then $aX \leqslant Q_i$ for all $a \in L$ and for all $i = 2,3,\ldots,k$. That is $a \leqslant (Q_i:X)$ for all

$i = 2, 3, \ldots, k$. So $1 \leqslant (Q_i : X)$. But $(Q_i : X) \leqslant 1$ implies $(Q_i : X) = 1$ for $i = 2, 3, \ldots, k$. Hence, $(N : X) = (Q_1 : X) \wedge 1 \wedge \ldots \wedge 1 = (Q_1 : X)$. So by above result, $(Q_1 : X)$ is p-primary element implies $(N : X)$ is a p-primary element of L where $X \nleqslant N$. Conversely assume that $(N : X)$ is a p-primary element of L for some $X \nleqslant N, X \in M$. We prove that $\sqrt{Q_i} = p$ for some i. We have, $p = \sqrt{(N : X)} = \sqrt{[(Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k) : X]} = \sqrt{(Q_1 : X)} \wedge \sqrt{(Q_2 : X)} \wedge \ldots \wedge \sqrt{(Q_k : X)}$. We claim that for each i, $\sqrt{(Q_i : X)} = p_i$ or 1 and equal to p_i for at least one i. We have $X \nleqslant N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_k$ implies $X \nleqslant Q_i$ for at least one i $(1 \leqslant i \leqslant k)$. Suppose $X \nleqslant Q_r$ $(1 \leqslant r \leqslant k)$ and $X \leqslant Q_1 \wedge Q_2 \wedge \ldots \wedge Q_{r-1} \wedge Q_{r+1} \wedge \ldots Q_k$ that is $X \leqslant \wedge Q_i$, where $(i \neq r)$. We have, $aX \leqslant Q_i$ for all $i \neq r$ and $a \in L$. Hence, $a \leqslant \sqrt{(Q_i : X)}$ for all $a \in L$. In particular, $1 \leqslant \sqrt{(Q_i : X)}$ for all $i \neq r$. But, $\sqrt{(Q_i : X)} \leqslant 1$ for all $i \neq r$. Therefore, $\sqrt{(Q_i : X)} = 1$ for all $i \neq r$. For $i = r, X \nleqslant Q_r$. Let $a \leqslant \sqrt{(Q_r : X)}$. Hence,$a^n X \leqslant Q_r$, for some positive integer n, where $X \nleqslant Q_r$. As Q_r is primary, $a^n \leqslant \sqrt{(Q_r : I_M)} = p_r$. Thus, $a \leqslant p_r$, since p_r is prime and we have, $\sqrt{(Q_r : X)} \leqslant p_r$. On the other hand, let $a \leqslant p_r = \sqrt{Q_r} = \sqrt{(Q_r : I_M)}$. Hence, $a^n \leqslant (Q_r : I_M)$ for some positive integer n. That is $a^n I_M \leqslant Q_r$ and therefore, $a^n X \leqslant Q_r$, for some positive integer n. Consequently, $a^n \leqslant (Q_r : X)$ and hence $a \leqslant \sqrt{(Q_r : X)}$. This gives $p_r \leqslant \sqrt{(Q_r : X)}$. Hence, $\sqrt{(Q_r : X)} = p_r$ where $X \nleqslant Q_r$. We have shown that for each i, $\sqrt{(Q_i : X)} = p_i$ or 1 and is equal to p_i for at least one i, since $X \nleqslant N$. Then,

$$p = \sqrt{(N : X)} = \sqrt{(Q_1 : X)} \wedge \ldots \wedge \sqrt{(Q_k : X)}$$

is the meet of some of the prime elements $p_1, p_2, \ldots, p_l$ $(1 \leqslant l \leqslant k)$. That is

$$p = \sqrt{(N : X)} = p_1 \wedge p_2 \wedge \ldots \wedge p_l.$$

We show that $p = p_i$ for some i. We have, $p \leqslant p_i$ $i = 1, 2, \ldots, l$. If for each i, $p \neq p_i$ then $p_i \nleqslant p$ for all $i = 1, 2, \ldots, l$. This implies that there exist $x_i \leqslant p_i$ such that $x_i \nleqslant p$ for all $i = 1, 2, \ldots, l$ Then,$x_1 x_2 \ldots x_l \leqslant p_1 \wedge p_2 \wedge \ldots \wedge p_l = p$. This shows that $x_i \leqslant p$ for at least one i $(1 \leqslant i \leqslant k)$ a contradiction. Hence, $p = p_i$ for at least one i. □

This leads us to the following result.

Theorem 8. *Let $N \neq I_M$ be an element of a lattice module M and assume that N has a primary decomposition. If $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m = S_1 \wedge S_2 \wedge \ldots \wedge S_n$ are two reduced primary decompositions of N then $n = m$ and the Q_i and S_i can be so numbered that $\sqrt{(Q_i : I_M)} = \sqrt{(S_i : I_M)}$ for $i = 1, 2, \ldots, n$.*

The above theorem proves the uniqueness of associated primes in reduced primary decomposition. The next result gives the relation between zero divisors of L and associated primes of zero.

Theorem 9. *Let L be a Noetherian lattice and $p_1, p_2, \ldots, p_k$ be the prime divisors of the element 0 that is associated prime elements of element 0. Then every zero divisors of L is contained in $p_1 \vee p_2 \vee \ldots \vee p_k$.*

Proof. Let $0 = q_1 \wedge q_2 \wedge \ldots \wedge q_k$ be a reduced primary decomposition of 0 and $p_i = \sqrt{q_i}$, $i = 1, 2, \ldots, k$. Suppose a is a zero divisor. Then if $a = 0$ obviously $a \leqslant p_1 \vee p_2 \vee \ldots \vee p_k$. Suppose, a is a proper zero divisor that is $a \neq 0$ and let $ab = 0$ where $b \neq 0$. Now, $ab = 0 \leqslant q_1 \wedge q_2 \wedge \ldots \wedge q_k = \{0\}$. Hence, $ab \leqslant q_i$ for all i and $b \nleqslant q_i$ for at least one i. Because, $b \leqslant q_i$ for all i implies $b \leqslant q_1 \wedge\ q_2 \wedge \ldots \wedge q_k = \{0\}$ and hence $b = 0$, a contradiction. Let $b \nleqslant q_j$. Then, $ab \leqslant q_j, b \nleqslant q_j$ and q_j is a primary element. Therefore, $a \leqslant \sqrt{q_j} = p_j$, which shows that $a \leqslant p_1 \vee p_2 \vee \ldots \vee p_k$. □

Theorem 10. *Let M be a lattice module and $N \neq I_M$ be an element of M which has a reduced primary decomposition $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m$. If every Q_i $(1 \leqslant i \leqslant m)$ is a prime element then $(N : I_M) = \sqrt{(N : I_M)}$ and the converse holds if $(Q_i : I_M)$ are prime elements.*

Proof. Suppose each Q_i is a prime element. Let $a \leqslant \sqrt{(N : I_M)}$. Then

$$a^n I_M \leqslant N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m$$

for some positive integer n. This implies that, $a^n I_M \leqslant Q_i$ for each i. As Q_i is a prime element, $aI_M \leqslant Q_i$ or $a^{n-1} I_M \leqslant Q_i$. If $aI_M \leqslant Q_i$ then $a \leqslant (Q_i : I_M)$. Otherwise $a^{n-1} I_M \leqslant Q_i$ implies $aI_M \leqslant Q_i$ or $a^{n-2} I_M \leqslant Q_i$. Continuing in this way we obtain, $a \leqslant (Q_i : I_M)$ for each i. Therefore $a \leqslant (Q_1 : I_M) \wedge (Q_2 : I_M) \wedge \ldots \wedge (Q_m : I_M)$. That is $a \leqslant (N : I_M)$ and hence $(N : I_M) = \sqrt{(N : I_M)}$. Conversely assume that, $(N : I_M) = \sqrt{(N : I_M)}$. We show that $(Q_i : I_M) = p_i$. Let $y \leqslant p_i = \sqrt{(Q_i : I_M)}$. As $\overset{m}{\underset{i=1}{\wedge}} p_i$ is irredundant(reduced) there exists $z \leqslant \wedge p_j$ such that $z \nleqslant p_i$ in L. Now $yz \leqslant \overset{m}{\underset{i=1}{\wedge}} p_i = \overset{m}{\underset{i=1}{\wedge}} \sqrt{(Q_i : I_M)} = \overset{m}{\underset{i=1}{\wedge}} (Q_i : I_M)$ implies $yzI_M \leqslant Q_i$ for each i. Since Q_i is primary, $z \nleqslant p_i$ gives $y \leqslant (Q_i : I_M)$. Hence, $p_i \leqslant (Q_i : I_M)$. Consequently, $p_i = (Q_i : I_M)$. □

Now we obtain a characterization of a prime element p of L containing some associated prime p_i of $N \neq I_M$ in a lattice module M.

Theorem 11. *Let $N \neq I_M$ have a reduced primary decomposition $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m$ and $p_i = \sqrt{(Q_i : I_m)}$ be the associated primes of N. For a prime element p of L to contain $(N : I_M)$ it is necessary and sufficient that p contains p_i for some i.*

Proof. Suppose $p_i \leqslant p$ for some i. Then $(N : I_M) = \overset{m}{\underset{i=1}{\wedge}} (Q_i : I_M)$ implies $(N : I_M) \leqslant p$. Conversely assume that $(N : I_M) \leqslant p$. Then $(Q_1 : I_M) \wedge (Q_2 : I_M) \wedge \ldots \wedge (Q_m : I_M) \leqslant p$ implies $(Q_i : I_M) \leqslant p$ for some i. But $\sqrt{(Q_i : I_M)} = p_i$ is the smallest prime containing $(Q_i : I_M)$. Hence, $p_i \leqslant p$ for some i. □

In our next result we show that those $Q_i^{'s}$ can be uniquely determined which are isolated primary components of $N \neq I_M$.

Theorem 12. *Let $N \neq I_M$ have a reduced primary decomposition $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m$ and $p_1, p_2, \ldots, p_m$ be the associated primes of $Q_1, Q_2, \ldots, Q_m$ respectively. The element*

$$Q_i^{'} = \vee \{X \in M \mid (N : X) \nleqslant p_i\}$$

is an element of M which is contained in Q_i. If Q_i is an isolated primary component of N then $Q_i = Q_i'$.

Proof. Take any element $A \in \{X \in M \mid (N : X) \nleq p_i\}$. Then $(N : A) \nleq p_i$. So there exists $a \in L$ such that $aA \leqslant N$ and $a \nleq p_i = \sqrt{(Q_i : I_M)}$. Hence, $a^n I_M \nleq Q_i$ for any integer n. Now $aA \leqslant Q_i$, $a^n I_M \nleq Q_i$ and Q_i is primary gives $A \leqslant Q_i$. Hence $Q_i' \leqslant Q_i$ and the first part is proved. If p_i is a minimal associated primes of N it follows that $p_j \nleq p_i$ for $i \neq j$. Then there exists $b_j \leqslant p_j$ in L such that $b_j \nleq p_i$. We have $b_j \leqslant p_j = \sqrt{(Q_j : I_M)} = \vee\{a_j \in L \mid a_j^{s_j} I_M \leqslant Q_j$ for some integer $s_j\}$. Since each element of L is compact, we have $b_j \leqslant p_j = \overset{n}{\underset{j=1}{\vee}}\{a_j \mid a_j^{s_j} I_M \leqslant Q_j$ for some integer $s_j\}$. Put $s_1 + s_2 + \ldots + s_n = k(j)$. Then $b_j^{k(j)} I_M \leqslant (a_1 \vee a_2 \vee \ldots \vee a_n)^{k(j)} I_M \leqslant Q_j$. Clearly $b = \prod_{j \neq i} b_j^{k(j)} \nleq p_i$ as p_i is prime. However, $bI_M \leqslant \underset{j\neq i}{\wedge} Q_j$. Next take any $X \leqslant Q_i$. Then $X bI_M \leqslant \overset{m}{\underset{i=1}{\wedge}} Q_i = N$. So $b \leqslant (N : X) \nleq p_i$. This implies that $X \in \{X \in M \mid (N : X) \neq p_i\}$. Hence $X \leqslant \vee\{X \in M \mid (N : X) \nleq p_i\} = Q_i'$ and we have $Q_i \leqslant Q_i'$. Consequently, $Q_i = Q_i'$. ☐

We now relate the radical of N with the isolated primes of $N \in M$. In that direction we have:

Theorem 13. *Let M be a lattice module and $N \neq I_M$ have an irredudent(reduced) primary decomposition $N = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_n$ then $\sqrt{(N : I_M)}$ is the meet of isolated prime elements of N.*

Proof. We have

$$\begin{aligned}\sqrt{(N : I_M)} =& \sqrt{(Q_1 \wedge Q_2 \wedge \ldots Q_n) : I_M} = \sqrt{(Q_1 : I_M)} \wedge \sqrt{(Q_2 : I_M)} \ldots \wedge \sqrt{(Q_n : I_M)}\\ =& p_1 \wedge p_2 \wedge \ldots \wedge p_n,\end{aligned}$$

where $p_i = \sqrt{(Q_i : I_M)}$ are associated primes of N. If some p_k is not isolated then $p_k \geq p_i$ for some p_i and hence we can delete such elements from the above primary decomposition and we are through. ☐

We note that an element $A = aI_M$ of M where $a \in L$ is said to be nilpotent if $a^n I_M = 0_M$ for some positive integer n. If a lattice module M satisfies the ACC and if every element of L is the join of meet prncipal elements then every element of M can be written as a meet of finite number of primary elements [1].

Theorem 14. *Let M be a lattice module satisfying the ACC over a multiplicative lattice L in which every element is the join of meet principal elements. Then the join of the set of all elements $a \in L$ such that aI_M is nilpotent is the meet of the isolated primes of 0_M.*

Proof. Let $0_M = \overset{n}{\underset{i=1}{\wedge}} Q_i$ be a reduced primary decomposition of 0_M and $p_i = \sqrt{(Q_i : I_M)}$ be an associated prime of $Q_i, i = 1, 2, \ldots n$. We have

$$\sqrt{(0_M : I_M)} = \vee\{a \in L \mid a^n I_M = 0_M \text{ for some positive integer } n\}$$

and $\sqrt{(0_M : I_M)} = p_1 \wedge p_2 \wedge \ldots \wedge p_k$ where $p_1, p_2, \ldots, p_k$ are the isolated primes of 0_M. Hence, $\sqrt{(0_M : I_M)}$ is the meet of isolated primes of 0_M. □

The primeness of the radical of $A \in M$ is characterized in the following result.

Theorem 15. *For $A \in M$, $\sqrt{(A : I_M)}$ is prime if and only if A has a single isolated prime element.*

Proof. Let $A = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_n$ be primary decomposition of A. If A has a single isolated prime element p then $\sqrt{(A : I_M)} = p$. Conversely, assume that $\sqrt{(A : I_M)}$ is prime and $\sqrt{(A : I_M)} = p_1 \wedge p_2$ where p_1, p_2 are isolated primes of A. Then there are $x, y \in L$ such that $x \leqslant p_1$, $x \nleq p_2$ and $y \leqslant p_2$, $y \nleq p_1$. Hence $xy \leqslant \sqrt{(A : I_M)}$ which is prime. But then $x \leqslant p_2$ or $y \leqslant p_1$ which is a contradiction. Thus A has a single isolated prime element. □

In the remaining part we assume that a lattice module M satisfies the ACC over a multiplicative lattice L in which every element is the join of meet principal elements. This condition assures that any element M has a reduced primary decomposition.

Theorem 16. *Let A be any element of M and $b \in L$ be such that $A \neq I_M$. Then $A = (A : b)$ if and only if b is contained in no associated prime element of A.*

Proof. Let $A = Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m$ be an irredundant primary decomposition of A and let $p_i = \sqrt{(Q_i : I_M)}$. Suppose $b \nleq p_i$ for any $i = 1, 2, \ldots, m$. This leads us to the fact $b^n \nleq p_i$ for any positive integer n. We know that $(A : b)b \leqslant A$ [3] and thus $(A : b)b \leqslant Q_i$ for all i. Since Q_i is primary and $b \nleq p_i$ we have $(A : b) \leqslant Q_i$. That is $(A : b) \leqslant \overset{m}{\underset{i=1}{\wedge}} Q_i = A$. But $A \leqslant (A : b)$ gives $(A : b) = A$. Conversely, suppose $(A : b) = A$ and if possible without loss of generality assume that $b \leqslant p_1$. Then $(Q : b^s) = I_M$ for some integer s. We have $(A : b) : b = A : b^2$ [3]. Continuing in this way $A : b = A : b^s$. But $A : b = A$ implies $A : b^s = A$. Finally,

$$
\begin{aligned}
A = &(A : b^s) = ((Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m) : b^s) = ((Q_1 : b^s) \wedge (Q_2 : b^s) \wedge \ldots \wedge (Q_m : b^s) \\
&= \underset{j \neq 1}{\wedge} (Q_j : b^s) \geq \underset{j \neq 1}{\wedge} Q_j \geq A.
\end{aligned}
$$

That is $A = \underset{j \neq 1}{\wedge} Q_j$. This contradicts the fact that $A = \overset{m}{\underset{i=1}{\wedge}} Q_i$ is a reduced primary decomposition of A. □

The above theorem can be restated in the following form.

Theorem 17. *Let $N \neq I_M$ have a reduced primary decomposition $Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m$ and $p_1, p_2, \ldots, p_m$ be the associated primes of $Q_i^{'s}$. For an element b of L to be contained in some associated prime element of N it is necessary and sufficient that $(N : b) \neq N$.*

Direct application of the above theorem gives the following result.

Theorem 18. *For an element b of L to be contained in some associated prime element of N, it is necessary and sufficient that there is an element $Y \nleq N$ such that $bY \leqslant N$.*

An element $X \in M$ is called a zero divisor if $(0_M : X) \neq 0$ so there exists $a \neq 0$ in L such that $aX = 0_M$.

Theorem 19. *Let M be a lattice module where M satisfies the ACC and every element of M is the join of meet principal elements. If $X \in M$ then the join of all $a \in L$ such that $a \neq 0$ and $aX = 0_M$ is contained in the join of all associated prime elements of 0_M.*

Proof. Let X be a zero divisor of M. Then $0_M : X \neq 0$ that is there exists $a \neq 0$ in L such that $aX = 0_M$. We know that for an element b of L $(b \neq 0, bX = 0_M)$ to be contained in some associated prime of 0_M it is necessary and sufficient that

$$(0_M : b) = \vee\{X \in M \mid bX = 0_M\} \neq 0_M.$$

Hence the join of all elements a of L such that $a \neq 0$ and $aX = 0_M$ is contained in the join of all associated prime elements of 0_M, by Theorem 16.

ACKNOWLEDGEMENTS The authors thank the readers of European Journal of Pure and Applied Mathematics, for making our journal successful.

References

[1] D.D. Anderson. Multiplicative Latices. Ph.D. thesis, Chicago University, 1974.

[2] R.P. Dilworth. Abstract Commutative Ideal Theory, Pacific Journal of Mathematics, 12, 481-498. 1962.

[3] J.A. Johnson. a-adic.Completions of Noetherian Lattice Modules, Fundamenta Mathematicae, 66, 341-371. 1970.

[4] F. Callialp and U. Tekir. Multiplication Lattice Modules, Iranian Journal of Science and Technology, 309-313. 2011.

[5] M.D. Larsen and P.J. McCarthy. Multiplicative theory of ideals, Academic press, New York, USA. 1970.

[6] N.K. Thakare and C.S. Manjarekar. Radicals and uniqueness theorem in multiplicative lattices with chain conditions, Studia Scientifica Mathematicarum Hungarica 18, 13-19. 1983