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On a Proper Subclass of Primeful Modules Which Contains the Class of Finitely Generated Modules Properly

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Abstract. Let R be a commutative ring with identity and M a unital R-module. Moreover, let $PSpec(M)$ denote the primary-like spectrum of M and $Spec(R/Ann(M))$ the prime spectrum of $R/Ann(M)$. We define an R-module M to be a ϕ-module, if $\phi : PSpec(M) \to Spec(R/Ann(M))$ given by $\phi(Q) = \sqrt{(Q : M)}/Ann(M)$ is a surjective map. The class of ϕ-modules is a proper subclass of primeful modules, called ψ-modules here, and contains the class of finitely generated modules properly. Indeed, ϕ and ψ are two sides of a commutative triangle of maps between spectrums. We show that if R is an Artinian ring, then all R-modules are ϕ-modules and the converse is true when R is a Noetherian ring.

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1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N : M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and annihilator of M, denoted by $Ann(M)$, is the ideal $(0 : M)$. A submodule P of an R-module M is said to be prime (or p-prime) if $P \neq M$ and for $p = (P : M)$, whenever $rm \in P$ (where $r \in R$ and $m \in M$) then $m \in P$ or $r \in p$ [5, 11, 12]. The collection of all prime (resp. p-prime) submodules of M, denoted by $Spec(M)$ (resp. $Spec_p(M)$), is called the prime (resp. p-prime) spectrum of M. Also the intersection of all prime submodules of M containing a submodule N is called the radical of N and is denoted by $\operatorname{rad} N$. In the ideal case, we denote the radical of an ideal I of R by $\sqrt{I}$. An R-module M is said to be a primeful module or a ψ-module if either $M = (0)$ or $M \neq (0)$ and the map $\psi : Spec(M) \to Spec(R/Ann(M))$, defined by $\psi(P) = (P : M)/Ann(M)$, is surjective [9]. If M/N is a ψ-module over R, then $\sqrt{(N : M)} = (\operatorname{rad} N : M)$ [9, Proposition 5.3]. A submodule Q of M is said to be primary-like if $Q \neq M$ and whenever $rm \in Q$ (where $r \in R$ and $m \in M$) implies $r \in (Q : M)$ or

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$m \in \text{rad}\, Q$ [7]. The primary-like spectrum $PSpec(M)$ is defined to be the set of all primary-like submodules N of M such that M/N is a ψ-module over R. In [7, Lemma 2.1] it is shown that, if $Q \in PSpec(M)$, then $(Q : M)$ is a primary ideal of R and so $p = \sqrt{(Q : M)}$ is a prime ideal of R. In this case, the primary-like submodule Q is also called a p-primary-like submodule of M.

Definition 1. *We say that an R-module M is a ϕ-module if either $M = (0)$ or $M \neq (0)$ and the map $\phi : PSpec(M) \rightarrow Spec(R/Ann(M))$ defined by $\phi(Q) = \sqrt{(Q : M)}/Ann(M)$ is surjective.*

The saturation of a submodule N of an R-module M with respect to a prime ideal p of R is the contraction of N_p in M and designated by $S_p(N)$. It is known that [4, 10]

$$S_p(N) = \{m \in M \mid rm \in N \text{ for some } r \in R \backslash p\}.$$

If $p \in Spec(R)$ and N is a submodule of an R-module M such that $(N : M) \subseteq p$ and M/N is a ψ-module over R, then $S_p(N + pM)$ is a p-prime submodule of M [9, Proposition 4.4]. Therefore $\rho : PSpec(M) \rightarrow Spec(M)$ defined by $\rho(Q) = S_p(Q + pM)$ is a well-defined map, where $p = \sqrt{(Q : M)}$. It is easy to see that $\phi = \psi \circ \rho$, ψ composed with ρ. Thus, if ϕ is a surjective map, so is ψ. This means that every ϕ-module is a ψ-module. We give an example of a ψ-module module which is not a ϕ-module (Example 1). An R-module M is said to be multiplication module if every submodule N of M is of the form IM for some ideal I of R [6]. We show that the multiplication ψ-modules, finitely generated modules, free modules (of finite or infinite rank), faithful projective modules over domains and modules over Artinian rings are ϕ-modules (Theorem 1, Corollary 1, Theorem 2, Theorem 3 and Theorem 4).

2. ϕ-Modules

We will use $\mathscr{X}$, X_p and $\mathscr{X}_p$ to represent $PSpec(M)$, $Spec_p(M)$ and $\{Q \in PSpec(M) \mid \sqrt{(Q : M)} = p\}$ respectively. Also $V(Ann(M))$ will be the set of all prime ideals containing $Ann(M)$. We begin with a lemma which will be referred to in the rest of this section.

Lemma 1 (cf. [9, Theorem 2.1])**.** *Let M be a non-zero R-module. Then the following statements are equivalent.*

(1) M is a ψ-module;

(2) $X_p \neq \emptyset$ for every $p \in V(Ann(M))$;

(3) $pM_p \neq M_p$ for every $p \in V(Ann(M))$;

(4) $S_p(pM)$ is a p-prime submodule for every $p \in V(Ann(M))$.

Theorem 1. *Every ϕ-module M over a ring R is a ψ-module, and the converse is true in each of the following cases.*

(1) M is a multiplication R-module.

(2) M is a non-zero faithfully flat (or in particular a projective) R-module.

(3) $M/S_p(pM)$ is a ψ-module over R for every $p \in V(Ann(M))$.

Proof. Since $\phi = \psi \circ \rho$, every ϕ-module is a ψ-module. (1) Let $p \in V(Ann(M))$. Then there exists a prime submodule P such that $(P : M) = p$. Since M is a multiplication module $P = pM$. Suppose $q \in Spec(R)$ and $p \subseteq q$. By Lemma 1, there exists a prime submodule P' such that $(P' : M) = q$. It follows that $P = pM \subseteq qM = P'$. Hence M/P is a ψ-module and so $P \in PSpec(M)$. Now from $\phi(P) = p/Ann(M)$, we conclude that ϕ is surjective, i.e., M is a ϕ-module. (2) Let $p \in V(Ann(M))$ and $(P : M) = p$. If M is a projective module, then pM is a prime submodule of M by [1, Corollary 2.3]. Also if M is a faithfully flat module, then pM is a prime submodule by [3, Corollary 2.6]. On the other hand M/pM is a ψ-module and $(pM : M) = p$ by [9, Corollary 4.3 and Proposition 4.5]. Consequently $pM \in \mathscr{X}_p$. Thus M is a ϕ-module. (3) Since M is a ψ-module, $S_p(pM)$ is a p-prime submodule of M by Lemma 1. Hence $S_p(pM) \in \mathscr{X}_p$. Thus M is a ϕ-module. □

The following example shows that a ψ-module is not necessarily a ϕ-module.

Example 1 (cf. [9, Example 1]). *Let Ω be the set of all prime integers, $M = \prod_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $M' = \bigoplus_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p runs through Ω. Hence M is a faithful ψ-module over $\mathbb{Z}$ and $Spec(M) = \{M' = S_0(0)\} \cup \{pM : p \in \Omega\}$. Now if ϕ is surjective, then there exists $N \in \mathscr{X}$ such that $\phi(N) = \sqrt{(N : M)} = 0$. It follows that $(N : M) = 0$. Since M/N is a ψ-module, we have $N \subseteq \cap_{p\in\Omega} pM = 0$. But 0 is not prime and so is not primary-like because* rad 0 = 0. *Hence $N \notin \mathscr{X}$, a contradiction. Thus M is not a ϕ-module.*

Corollary 1. *Every finitely generated R-module M is a ϕ-module, hence so is the factor module M/N of M by any submodule N of M.*

Proof. Follows from Lemma 1 and Theorem 1. □

Corollary 2. *Let R be a ring of (Krull) dimension* 0 *and M be a non-zero R-module. Then the following statements are equivalent.*

(1) $mM \neq M$ for every $m \in V(Ann(M)) \cap Max(R)$;

(2) M is a ψ-module;

(3) M is a ϕ-module.

Proof. (1) ⟺ (2) follows from [9, Result 3].
(2) ⇒ (3) Suppose M is a ψ-module. We show that $M/S_p(pM)$ is a ψ-module for every $p \in V(Ann(M))$. Assume $(S_p(pM) : M) \subseteq q$ for a prime ideal q of R. Hence $p \subseteq q$. Since $dim(R) = 0$, then $p = q$. Hence $S_q(qM)$ is a q-prime submodule containing $S_p(pM)$. Thus M is a ϕ-module by Theorem 1.
(3) ⇒ (2) follows from Theorem 1. □

Corollary 3. *Let R be a domain which is not a field. If a non-zero R-module M is either a divisible module or a faithful torsion module, then M is not a ϕ-module.*

Proof. Use Theorem 1 and [9, Proposition 2.6]. □

Theorem 2. *Every free module is a ϕ-module.*

Proof. Suppose F is a free R-module and $\overline{p} \in Spec(R/Ann(F))$. It is easy to see that pF is a prime, and hence a primary-like submodule, of F. Now we show that F/pF is a ψ-module. Assume q is a prime ideal of R containing $(pF : F)$. It follows from [13, Proposition 2.2] that $(qF : F) = q$ and hence $qF \neq F$. Thus qF is a q-prime submodule of F containing pF [11, Theorem 3]. It implies that F/pF is a ψ-module. □

Theorem 3. *Let R be a domain and M be a faithful projective R-module. Then M is a ϕ-module.*

Proof. Assume $M \neq (0)$ and $p \in Spec(R)$. We show that $pM \in \mathscr{X}$. By [9, Corollary 3.4], M is a ψ-module and hence $pM \neq M$ by [9, Result 2]. It follows from [11, Theorem 3] that pM is a p-prime, and hence a p-primary-like, submodule of M. It remains to show that M/pM is a ψ-module. Suppose q is a prime ideal of R containing $p = (pM : M)$. Therefore $pM \subseteq qM$ and $qM \in X_q$. Thus M/pM is a ψ-module and so M is a ϕ-module. □

Proposition 1. *Let M be a non-zero ϕ-module over a ring R. Then the following statements hold.*

(1) Let I be a radical ideal of R. Then $(IM : M) = I$ if and only if $I \supseteq Ann(M)$.

(2) $mM \in \mathscr{X}$ for every $m \in V(Ann(M)) \cap Max(R)$.

(3) If M is faithful, then M is flat if and only if M is faithfully flat.

Proof. (1) follows from [9, Proposition 3.1] and Theorem 1.
(2) By Theorem 1, M is a ψ-module. Hence by [9, Result 2], $mM \neq M$. Thus mM is a m-prime, and hence m-primary-like, submodule of M. It remains to show that M/mM is a ψ-module. Assume p is a prime ideal of R containing $(mM : M)$. Since $m \in Max(R)$, then $m = p$ and so M/mM is a ψ-module. Thus $mM \in \mathscr{X}$.
(3) The sufficiency is clear. Suppose that M is flat. Hence by part (2), we have $mM \neq M$ for every $m \in Max(R)$. This implies that M is faithfully flat. □

We give an elementary example of a module which is not a ϕ-module.

Example 2. *The $\mathbb{Z}$-module $\mathbb{Q}$ is flat and faithful, but not faithfully flat. So, $\mathbb{Q}$ is not a ϕ-module, by Proposition* 1.

Proposition 2. *Let M be a non-zero ϕ-module over a ring R. Then M_p is a non-zero ϕ-module over R_p for every $p \in V(Ann(M))$.*

Proof. Suppose M is a non-zero ϕ-module over R. Hence $M_p \neq (0)$ for every $p \in V(Ann(M))$. Assume $q' \in Spec(R_p/Ann(M_p))$. We set $q = (q')^c$, the contraction of q' in R. It is easy to check that q is a prime ideal of R. We show that there exists a q'-primary-like submodule Q_p of M_p such that M_p/Q_p is a ψ-module. Since R_p is a local ring, $p_p \supseteq q' \supseteq Ann(M_p) \supseteq (Ann(M))_p$. Taking the contraction of each term of this sequence of ideals in R, we have that

$$p \supseteq q \supseteq Ann(M_p) \cap R \supseteq S_p(Ann(M)) \supseteq Ann(M).$$

Hence $q \in Spec(R/Ann(M))$. Since M is a ϕ-module over R, there exists $Q \in \mathscr{X}$ such that $\sqrt{(Q:M)} = q$. Thus by [7, Theorem 3.8] Q_p is a q'-primary-like submodule in M_p such that M_p/Q_p is a ψ-module and hence M_p is a ϕ-module over R_p for every $p \in V(Ann(M))$. □

Theorem 4. *Let R be a ring. Consider the following statements.*

(1) R is an Artinian ring.

(2) Every R-module is a ϕ-module.

(3) Every R-module is a ψ-module.

(4) $mM \neq M$ for every R-module M and $m \in V(Ann(M)) \cap Max(R)$.

(5) $dim(R) = 0$

Then (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5). Furthermore, if R is a Noetherian ring, then the above statements are equivalent.

Proof. (1) ⇒ (2) Let M be a non-zero R-module. Then $Ann(M) \neq R$. Since R is Artinian, we have $R = R_1 \times \cdots \times R_n$, where $n \in \mathbb{N}$ and each R_i is an Artinian local ring. First we assume that $n = 1$, i.e., R is an Artinian local ring with maximal ideal m. Since $J(R)$, the Jacobson radical of R, equals to m and $J(R)$ is T-nilpotent, $mM \neq M$ by [8, Theorem 23.16]. Thus every R-module is a ϕ-module, by Corollary 2. Now assume $n \geq 2$ and let m_i be the maximal ideal of the local ring R_i for every $1 \leq i \leq n$. Let m be a maximal ideal of R containing $Ann(M)$. Clearly m is the form $R_1 \times \cdots R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_n$ for some i. Without loss of generality we may assume that $i = 1$, i.e., $m = m_1 \times R_2 \times \cdots \times R_n$. Again, by Corollary 2, it suffices to show that $mM = (m_1 \times R_2 \times \cdots \times R_n)M \neq M$. On the contrary, suppose that $(m_1 \times R_2 \times \cdots \times R_n)M = M$. Take $M_1 = (R_1 \times (0) \times \cdots \times (0))M$. It is easy to verify that $R_1 \cong R/(0) \times R_2 \times \cdots \times R_n$ and hence M_1 can be expressed as an R_1-module by defining $r_1 x_1 = r_1(1, 0, \cdots, 0)x_1$ for $r_1 \in R_1$ and $x_1 \in M_1$. We may assume that $M_1 \neq 0$, for otherwise we have

$$R_1 \times (0) \times \cdots \times (0) \subseteq Ann(M) \subseteq m_1 \times R_2 \times \cdots \times R_n,$$

a contradiction. Thus $m_1 M_1 \neq M_1$ by using case $n = 1$. On the other hand, for each $x \in M$, $(1, 0, \cdots, 0)x \in M = (m_1 \times R_2 \times \cdots \times R_n)M$. Thus for each $x \in M$, $(1, 0, \cdots, 0)x = \sum_{j=1}^{s}(p_{1j}, r_{2j}, \cdots, r_{nj})x_j$ for some $s \in \mathbb{N}$, $x_j \in M$, $p_{1j} \in M_1$ and $r_{ij} \in R$, where $2 \leq i \leq n$ and $1 \leq j \leq s$. Multiplying the former equation by $(1,0,\cdots,0)$, we get $(1,0,\cdots,0)x \in (m_1 \times (0) \times \cdots \times (0))M$ for each $x \in M$. It follows that

$$(R_1 \times (0) \times \cdots \times (0))M \subseteq (m_1 \times (0) \times \cdots \times (0))M$$

and so $m_1M_1 = M_1$, a contradiction.
(2) ⇒ (3) follows from Theorem 1.
(3) ⇒ (4) follows from [9, Result 2].
(4) ⇒ (5) Suppose p be a prime ideal of R and K the quotient field of R/p. We know that K is a non-zero divisible R/p-module. Let $0 \neq r + p \in R/p$. Then $(r + p)K = K$ implies that $Ann(K) + R/p(r + p) = R/p$. Otherwise, if $Ann(K) + R/p(r + p) \neq R/p$, then there is a maximal ideal m/p of R/p containing $Ann(K) + R/p(r + p)$. Thus $K = (r + p)K \subseteq (m/p)K$ follows that $(m/p)K = K$, contradicting the assumption in (4). Now, let $Ann(K) \neq (0)$. Take $r+p \in Ann(K)$ and hence by the above argument $Ann(K) = R/p$, i.e., $K = (0)$, a contradiction. Thus $Ann(K) = (0)$. Hence $R/p(r + p) = R/p$ for any $0 \neq r + p \in R/p$. Thus $dim(R) = 0$.
(4) ⇒ (5) follows from [2, Theorem 8.5]. □

The following is now immediate.

Corollary 4. *Let R be a domain. Then the following statements are equivalent.*

(1) Every R-module is a ϕ-module;

(2) Every R-module is a ψ-module;

(3) R is a field.

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