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Some Relations Between Crossed Modules and Simplicial Objects in Categories of Interest

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Abstract. We introduce a simplicial object in a category of interest and determine relations between crossed modules and simplicial objects in a category of interest.

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1. Introduction

Categories of interest were introduced in order to study properties of different algebraic categories and different algebras simultaneously. Roughly speaking, category of interest can be seen as a gadget which unifies many algebraic constructions. The idea comes from P.G. Higgins [10] and the definition is due to M. Barr and G. Orzech [11]. The categories of groups, modules over a ring, vector spaces, associative algebras, associative commutative algebras, Lie algebras and Leibniz algebras are categories of interest [11]. The categories of crossed modules and precrossed modules in the category of groups, respectively, are equivalent to the categories of interests (see e.g. [3, 4]).

The functorial relation between crossed modules and simplicial objects with Moore complex of length 1 in groups, commutative algebras, Lie algebras, Leibniz n-algebras were given in [1, 2, 5, 8, 9]. In this paper, we will define simplicial objects in categories of interest and unify the stated results under the name of categories of interest.

2. Category of Interest

We will have the main definitions and the statements given for category of interest in [4, 7, 11].

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Let $\mathbb{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group laws and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) the group operations (written additively : $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \setminus \{+\}$, $\Omega_1' = \Omega_1 \setminus \{-\}$. Assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$ and assume $\Omega_0 = \{0\}$;

(c) for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;

(d) for each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x * y) = \omega(x) * y$.

Let C be an object of $\mathbb{C}$ and $x_1, x_2, x_3 \in C$:

Axiom 1: $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$.

Axiom 2: For each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that

$$\begin{aligned}(x_1 * x_2)\overline{*}x_3 = &W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1,\\ &(x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),\end{aligned}$$

where each juxtaposition represents an operation in Ω_2'.

Definition 1. *A category of groups with operations satisfying Axiom 1 and Axiom 2 is called a category of interest by Orzech [11].*

Example 1. *Some examples of categories of interest that are given in [4]: In the example of groups $\Omega_2' = \varnothing$. In the case of associative algebras with multiplication represented by $*$, we have $\Omega_2' = \{*, *^\circ\}$. For Lie algebras $\Omega_2' = ([,],[,]^\circ)$ (where $[a,b]^\circ = [b,a] = -[a,b]$). For Leibniz algebras $\Omega_2' = ([,],[,]^\circ)$ (here $[a,b]^\circ = [b,a]$).*

Definition 2. *Let $C \in \mathbb{C}$. A subobject of C is called an ideal if it is the kernel of some morphism.*

Theorem 1. *Let A be a subobject of B in $\mathbb{C}$. Then A is an ideal of B if and only if the following conditions hold:*

i) A is a normal subgroup of B;

ii) $a * b \in A$, for all $a \in A$, $b \in B$ and $* \in \Omega_2'$.

Proof. Follows from Theorem 1.7 given in [11]. □

Definition 3. *Let A, $B \in \mathbb{C}$. An extension of B by A is a sequence*

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B \longrightarrow 0 \qquad (1)$$

in which p is surjective and i is the kernel of p. We say that an extension is split if there is a morphism $s : B \longrightarrow E$ such that $ps = 1_B$.

Definition 4. *For $A, B \in \mathbb{C}$ we will say that we have a set of actions of B on A, whenever there is a map $f_* : A \times B \longrightarrow A$, for each $* \in \Omega_2$.*

Definition 5. *A split extension of B by A induces an action of B on A corresponding to the operations in $\mathbb{C}$. For a given split extension (1), we have*

$$b \cdot a = s(b) + a - s(b), \tag{2}$$
$$b * a = s(b) * a, \tag{3}$$

for all $b \in B$, $a \in A$ and $* \in \Omega_2{}'$.

Actions defined by (2) and (3) are called derived actions of B on A. Given an action of B on A, the semidirect product $A \rtimes B$ is a universal algebra whose underlying set is $A \times B$ and the operations are defined by

$$\begin{aligned}\omega(a, b) &= (\omega(a), \omega(b)),\\ (a', b') + (a, b) &= (a' + b' \cdot a, b' + b),\\ (a', b') * (a, b) &= (a' * a + a' * b + b' * a, b' * b),\end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$.

Definition 6. *A precrossed module in $\mathbb{C}$ is a triple (C_1, C_0, ∂), where $C_0, C_1 \in \mathbb{C}$, the object C_0 has a derived action on C_1 or shortly C_0 acts on C_1 and $\partial : C_1 \longrightarrow C_0$ is a morphism in $\mathbb{C}$ with the conditions:*

CM 1) $\partial(c_0 \cdot c_1) = c_0 + \partial(c_1) - c_0$, $\partial(c_0 * c_1) = c_0 * \partial(c_1)$, *for all $c_0 \in C_0$, $c_1 \in C_1$, and $* \in \Omega_2{}'$.*

In addition, if $\partial : C_1 \longrightarrow C_0$ satisfies the conditions

CM 2) $\partial(c_1) \cdot c_1' = c_1 + c_1' - c_1$, $\partial(c_1) * c_1' = c_1 * c_1'$,

for all $c_1, c_1' \in C_1$, and $* \in \Omega_2'$, then the triple (C_1, C_0, ∂) is called a crossed module in $\mathbb{C}$.

Definition 7. *A morphism between two crossed modules $(C_1, C_0, \partial) \longrightarrow (C_1', C_0', \partial')$ is a pair of morphisms (μ_1, μ_0) in $\mathbb{C}$, $\mu_0 : C_0 \longrightarrow C_0'$, $\mu_1 : C_1 \longrightarrow C_1'$, such that*

i) $\mu_0 \partial(c) = \partial' \mu_1(c)$,

ii) $\mu_1(r \cdot c) = \mu_0(r) \cdot \mu_1(c)$,

iii) $\mu_1(r * c) = \mu_0(r) * \mu_1(c)$,

for all $r \in C_0$, $c \in C_1$ and $* \in \Omega_2{}'$.

With this definition, we have a category whose objects are crossed modules and morphisms are morphisms of crossed modules defined above.

The category of crossed modules will be denoted by $\mathfrak{Xmod}(\mathbb{C})$.

3. Simplicial Objects in a Category of Interest

Let $\triangle$ be the category of finite ordinals. A simplicial object in a category of interest $\mathbb{C}$ is a functor from the opposite category $\triangle^{op}$ to $\mathbb{C}$. In other words, a simplicial object **C** in $\mathbb{C}$ is a sequence

$$\mathbf{C} = \{C_0, C_1, \ldots, C_n, \ldots\}$$

together with face and degeneracy maps

$$\begin{array}{llll} d_i^n: & C_n & \longrightarrow & C_{n-1}, 0 \leq i \leq n \ (n \neq 0) \\ s_i^n: & C_n & \longrightarrow & C_{n+1}, 0 \leq i \leq n \end{array}$$

which are homomorphisms of objects in $\mathbb{C}$ satisfying the following simplicial identities;

$$\begin{array}{lll} d_i d_j & = & d_{j-1} d_i \quad \text{for } i < j \\ d_i s_j & = & \begin{cases} s_{j-1} d_i & \text{for } i < j \\ id & \text{for } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases} \\ s_i s_j & = & s_{j+1} s_i \quad \text{for } i \leq j \end{array}$$

for $0 \leq i \leq n$ (Here the superscripts of maps are dropped for shortness).

3.1. The Moore Complex

The Moore complex **NC** of a simplicial object **C** in a category of interest $\mathbb{C}$ is the complex

$$\mathbf{NC}: \cdots \longrightarrow NC_n \xrightarrow{\partial_n} NC_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} NC_1 \xrightarrow{\partial_1} NC_0$$

where $NC_0 = C_0$, $NC_n = \bigcap_{i=0}^{n-1} Ker d_i$ and ∂_n is the restriction of d_n to NC_n.

We say that the Moore complex **NC** of a simplicial object **C** is of length k if $NC_n = 0$, for all $n \geq k+1$. Now define a category whose objects are simplicial objects with Moore complex of length k and the morphisms are families of homomorphisms compatible with face and degeneracy maps. We denote this category by $\mathfrak{Simp}_{\leq k}(\mathbb{C})$.

3.2. Truncated Simplicial Objects

The following terminology is adapted from [6]. Details of the group case can be found in [6]. For each $k \geq 0$ we have a subcategory of $\triangle$, denoted by $\triangle_{\leq k}$ obtained by the objects $[j]$ of $\triangle$ with $j \leq k$. A k-truncated simplicial object is a functor from $\triangle^{op}_{\leq k}$ to $\mathbb{C}$. Consequently, a k-truncated simplicial object is a family of objects $\{C_0, C_1, \ldots, C_k\}$ and homomorphism $d_i: C_n \longrightarrow C_{n-1}$, $s_i: C_n \longrightarrow C_{n+1}$, for each $0 \leq i \leq n$ which satisfy the simplicial identities. We denote the category of k-truncated simplicial objects by $\mathfrak{Tr}_k\mathfrak{Simp}(\mathbb{C})$. There is a truncation functor tr_k from the category $\mathfrak{Simp}(\mathbb{C})$ to the category $\mathfrak{Tr}_k\mathfrak{Simp}(\mathbb{C})$ given by restrictions. This truncation functor has a left adjoint st_k and a right adjoint $cost_k$ called as k-skeleton and k-coskeleton respectively. These adjoints can be pictured as follows;

$$\mathfrak{Tr}_k\mathfrak{Simp}(\mathbb{C}) \quad \underset{cost_k}{\overset{tr_k}{\rightleftarrows}} \quad \mathfrak{Simp}(\mathbb{C}) \quad \underset{st_k}{\overset{tr_k}{\rightleftarrows}} \quad \mathfrak{Tr}_k\mathfrak{Simp}(\mathbb{C}).$$

See [6] for details about the functors $cost_k$ and st_k.

Theorem 2. *The category* $\mathfrak{Xmod}(\mathbb{C})$ *of crossed modules is naturally equivalent to the category* $\mathfrak{Simp}_{\leq 1}(\mathbb{C})$ *of simplicial objects with Moore complex of length* 1.

Proof. Let **C** be a simplicial object with Moore complex of length 1. Take $G = \ker d_0$ and ∂ is the restriction of d_1 to G. Define the actions of C_0 on G by

$$\begin{aligned} c_0 \cdot g =& s_0(c_0) + g - s_0(c_0), \\ c_0 * g =& s_0(c_0) * g, \end{aligned}$$

for all $c_0 \in C_0$ and $g \in G$. By using this action $\partial : G \longrightarrow C_0$ is a crossed module. Indeed,

CM 1: Since $d_1 s_0 = id$, we have

$$\begin{aligned} \partial(c_0 \cdot g) =& \partial(s_0(c_0) + g - s_0(c_0)) \\ =& c_0 + \partial(g) - c_0, \\ \partial(c_0 * g) =& \partial(s_0(c_0) * g) \\ =& c_0 * \partial(g), \end{aligned}$$

for all $c_0 \in C_0$ and $g \in G$.

CM 2: Since $s_0 d_1 = d_2 s_0, d_2 s_1 = id$, we have

$$\begin{aligned} \partial(g') * g =& s_0 d_1(g') * g \\ =& (s_0 d_1(g') - g' + g') * g \\ =& \left(s_0 d_1(g') - g'\right) * g + g' * g \\ =& \left(d_2 s_0 g' - d_2 s_1 g'\right) * \left(d_2 s_1 g\right) + g' * g \\ =& d_2\left(\left(s_0 g' - s_1 g'\right) * \left(s_1 g\right)\right) + g' * g \\ =& g' * g, \end{aligned}$$

for all $g, g' \in G$.

By a similar way, we have

$$\partial(g') \cdot g = g' + g - g'$$

for all $g, g' \in G$.

So we obtain the functor

$$N_1 : \mathfrak{Simp}_{\leq 1}(\mathbb{C}) \longrightarrow \mathfrak{Xmod}(\mathbb{C}).$$

Conversely, let $\partial : G \longrightarrow H$ be a crossed module. By using the action of H on G, we can form the semi-direct product $C_1 := G \rtimes H = \{(g,h) : h \in H,\ g \in G\}$. We have the homomorphisms

$$\begin{aligned} d_0 :& G \rtimes H \longrightarrow H \\ & (g,h) \longmapsto h \\ d_1 :& G \rtimes H \longrightarrow H \\ & (g,h) \longmapsto \partial(g) + h \\ s_0 :& H \longrightarrow G \rtimes H \\ & h \longmapsto (0,h) \end{aligned}$$

which satisfy the simplicial identities. Finally

$$C_1 \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} C_0$$

is a 1-truncated simplicial object. Thus we have the functor

$$s_1 : \mathfrak{Xmod}(\mathbb{C}) \longrightarrow \mathfrak{Tr}_1\mathfrak{Simp}(\mathbb{C}).$$

By using the functor st_k from the category of k-truncated simplicial objects to that of simplicial objects with Moore complex of length 1, we have

$$M : \mathfrak{Xmod}(\mathbb{C}) \longrightarrow \mathfrak{Simp}_{\leq 1}(\mathbb{C})$$

defined as the composition of s_1 and st_1. Finally we have the natural equivalence between the category of simplicial objects with Moore complex of length 1 and that of crossed modules in a category of interest $\mathbb{C}$.

The main result of the paper can be diagramized as

$$\mathfrak{Simp}_{\leq 1}(\mathbb{C}) \underset{M}{\overset{N}{\rightleftarrows}} \mathfrak{Xmod}(\mathbb{C}).$$

□

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