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On Generalized Ideals of Left Almost Semigroups

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Abstract. In this paper, we study (m,n)-ideals of an $\mathcal{LA}$-semigroup in detail. We characterize $(0,2)$-ideals of an $\mathcal{LA}$-semigroup S and prove that A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S. We also show that an $\mathcal{LA}$-semigroup S is $0-(0,2)$-bisimple if and only if S is right 0-simple. Furthermore we study 0-minimal (m,n)-ideals in an $\mathcal{LA}$-semigroup S and prove that if R, (L) is a 0-minimal right $(left)$ ideal of S, then either $R^mL^n = \{0\}$ or R^mL^n is a 0-minimal (m,n)-ideal of S for $m,n \geq 3$. Finally we discuss (m,n)-ideals in an (m,n)-regular $\mathcal{LA}$-semigroup S and show that S is $(0,1)$-regular if and only if $L = SL$ where L is a $(0,1)$-ideal of S.

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1. Introduction

A left almost semigroup ($\mathcal{LA}$-semigroup) is a groupoid Ssatisfying the left invertive law $(ab)c = (cb)a$ for all $a,b,c \in S$. This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc = cba$. The concept of an $\mathcal{LA}$-semigroup was first given by Kazim and Naseeruddin in 1972 [3]. An $\mathcal{LA}$-semigroup satisfies the medial law $(ab)(cd) = (ac)(bd)$ for all $a,b,c,d \in S$. Since $\mathcal{LA}$-semigroups satisfy medial law, they belong to the class of entropic groupoids which are also called abelian quasigroups [12]. If an $\mathcal{LA}$-semigroup S contains a left identity (unitary $\mathcal{LA}$-semigroup), then it satisfies the paramedial law $(ab)(cd) = (dc)(ba)$ and the identity $a(bc) = b(ac)$ for all $a,b,c,d \in S$ [7].

An $\mathcal{LA}$-semigroup is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An $\mathcal{LA}$-semigroup is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures. It has been investigated in [7] that if an $\mathcal{LA}$-semigroup contains a right identity, then it becomes a commutative semigroup. The connection of a commutative inverse semigroup

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with an $\mathscr{LA}$-semigroup has been given by Yousafzai *et al.* in [16] as, a commutative inverse semigroup $(S,\cdot)$ becomes an $\mathscr{LA}$-semigroup $(S,*)$ under $a*b = ba^{-1}r^{-1}$, $\forall a,b,r \in S$. An $\mathscr{LA}$-semigroup S with left identity becomes a semigroup under the binary operation "$\circ_e$", defined as $x \circ_e y = (xe)y$ for all $x, y \in S$ [17]. An $\mathscr{LA}$-semigroup is the generalization of a semigroup theory [7] and has vast applications in collaboration with semigroups like other branches of mathematics. Khan *et al.* studied an intra-regular class of an $\mathscr{LA}$-semigroup in [4] and proved some interesting problems by using different ideals. They proved that the set of all two-sided ideals of intra-regular $\mathscr{LA}$-semigroup forms a semilattice structure. They characterized an intra-regular $\mathscr{LA}$-semigroup by using left, right, two-sided and bi-ideals. An $\mathscr{LA}$-semigroup is the generalization of a semigroup theory [7]. Many interesting results on $\mathscr{LA}$-semigroups have been investigated in [5, 9–11, 15].

Yaqoob, Corsini and Yousafzai [13] extended the concept of *LA*-semigroups and introduced a new structure called left almost semihypergroup. Further Yaqoob and Gulistan [14] defined partial ordering on left almost semihypergroups. Gulistan *et al.* [2] defined H_v-$\mathscr{LA}$-semigroups which is a new generalization of $\mathscr{LA}$-semigroups and $\mathscr{LA}$-semihypergroups.

2. Preliminaries and Examples

If S is an $\mathscr{LA}$-semigroup with product $\cdot : S \times S \longrightarrow S$, then $ab \cdot c$ and $(ab)c$ both denote the product $(a \cdot b) \cdot c$.

If there is an element 0 of an $\mathscr{LA}$-semigroup $(S,\cdot)$ such that $x \cdot 0 = 0 \cdot x = x$ $\forall x \in S$, we call 0 a zero element of S.

Example 1. *Let $S = \{a,b,c,d,e\}$ with a left identity d. Then the following multiplication table shows that $(S,\cdot)$ is a unitary $\mathscr{LA}$-semigroup with a zero element a.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Example 2. *Let $S = \{a,b,c,d\}$. Then the following multiplication table shows that $(S,\cdot)$ is an $\mathscr{LA}$-semigroup with a zero element a.*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	d	d	c
c	a	c	c	c
d	a	c	c	c

The above $\mathscr{LA}$-semigroup S has commutative powers, that is $aa \cdot a = a \cdot aa$ for all $a \in S$ which is called a locally associative $\mathscr{LA}$-semigroup [8]. Note that S has no associative powers for all $a \in S$ because $(bb \cdot b)b \neq b(bb \cdot b)$ for $b \in S$.

Assume that S is an $\mathscr{LA}$-semigroup. Let us define $a^1 = a$, $a^{m+1} = a^m a$ and $a^m = ((((aa)a)a)\ldots a)a = a^{m-1}a$ for all $a \in S$ where $m \geq 1$. It is easy to see that $a^m = a^{m-1}a = aa^{m-1}$ for all $a \in S$ and $m \geq 3$ if S has a left identity. Also, we can show by induction, $(ab)^m = a^m b^m$ and $a^m a^n = a^{m+n}$ hold for all $a, b \in S$ and $m, n \geq 3$.

A subset A of an $\mathscr{LA}$-semigroup S is called a *right* (*left*) ideal of S if $AS \subseteq A$ ($SA \subseteq A$), and is called an *ideal* of S if it is both left and right ideal of S.

A subset A of an $\mathscr{LA}$-semigroup S is called an $\mathscr{LA}$*-subsemigroup* of S if $A^2 \subseteq A$.

The concept of (m,n)-ideals of a semigroup and an $\mathscr{LA}$-semigroup was given in [6] and [1] respectively.

An $\mathscr{LA}$-subsemigroup A of an $\mathscr{LA}$-semigroup S is said to be an (m,n)*-ideal* of S if $A^m S \cdot A^n \subseteq A$ where m, n are non-negative integers such that $m = n \neq 0$. Here A^m or A^n are suppressed if $m = 0$ or $n = 0$, that is $A^0 S = S$ or $SA^0 = S$. Note that if $m = n = 1$, then an (m,n)-ideal A of an $\mathscr{LA}$-semigroup S is called a *bi-ideal* of S. If we take $m = 0$ or $n = 0$, then an (m,n)-ideal A of an $\mathscr{LA}$-semigroup S becomes a left or a right ideal of S.

An (m,n)-ideal A of an $\mathscr{LA}$-semigroup S with zero is said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only (m,n)-ideal of S properly contained in A.

An $\mathscr{LA}$-semigroup S with zero is said to be 0-$(0,2)$-*bisimple* if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-ideal of S.

An $\mathscr{LA}$-semigroup S with zero is said to be *nilpotent* if $S^l = \{0\}$ for some positive integer l.

Let m, n be non-negative integers and S be an $\mathscr{LA}$-semigroup. We say that S is (m,n)-*regular* if for every element $a \in S$ there exists some $x \in S$ such that $a = (a^m x)a^n$. Note that a^0 is defined as an operator element such that $a^0 y = y$ and $za^0 = z$ for any $y, z \in S$.

3. 0-Minimal $(0,2)$-Bi-Ideals in Unitary $\mathscr{LA}$-Semigroups

If S is a unitary $\mathscr{LA}$-semigroup, then it is easy to see that $S^2 = S$, $SA^2 = A^2 S$ and $A \subseteq SA$ $\forall A \subseteq S$. Note that every right ideal of a unitary $\mathscr{LA}$-semigroup S is a left ideal of S but the converse is not true in general. Example 1 shows that there exists a subset $\{a, b, e\}$ of S which is a left ideal of S but not a right ideal of S. It is easy to see that SA and SA^2 are the left and right ideals of a unitary $\mathscr{LA}$-semigroup S. Thus SA^2 is an ideal of a unitary $\mathscr{LA}$-semigroup S.

Lemma 1. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a $(0,2)$-ideal of S if and only if A is an ideal of some left ideal of S.*

Proof. Let A be a $(0,2)$-ideal of S, then $SA \cdot A = AA \cdot S = SA^2 \subseteq A$ and $A \cdot SA = S \cdot AA = SS \cdot AA = SA^2 \subseteq A$. Hence A is an ideal of a left ideal SA of S.

Conversely, assume that A is a left ideal of a left ideal L of S, then

$$SA^2 = AA \cdot S = SA \cdot A \subseteq SL \cdot A \subseteq LA \subseteq A,$$

and clearly A is an $\mathscr{LA}$-subsemigroup of S, therefore A is a $(0,2)$-ideal of S. □

Corollary 1. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S.*

Lemma 2. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a $(0,2)$-bi-ideal of S if and only if A is an ideal of some right ideal of S.*

Proof. Let A be a $(0,2)$-bi-ideal of S, then $SA^2 \cdot A = A^2S \cdot A = AS \cdot A^2 \subseteq SA^2 \subseteq A$ and $A \cdot SA^2 = SS \cdot AA^2 = A^2A \cdot SS = SA \cdot A^2 \subseteq SA^2 \subseteq A$. Hence A is an ideal of some right ideal SA^2 of S.

Conversely, assume that A is an ideal of a right ideal R of S, then

$$SA^2 = A \cdot SA = A \cdot (SS)A = A \cdot (AS)S \subseteq A \cdot (RS)R \subseteq AR \subseteq A,$$

and $(AS)A \subseteq (RS)A \subseteq RA \subseteq A$, which shows that A is a $(0,2)$-ideal of S. □

Theorem 1. *Let S be a unitary $\mathscr{LA}$-semigroup. Then the following statements are equivalent.*

(i) *A is a $(1,2)$-ideal of S;*

(ii) *A is a left ideal of some bi-ideal of S;*

(iii) *A is a bi-ideal of some ideal of S;*

(iv) *A is a $(0,2)$-ideal of some right ideal of S;*

(v) *A is a left ideal of some $(0,2)$-ideal of S.*

Proof. $(i) \Longrightarrow (ii)$. It is easy to see that $SA^2 \cdot S$ is a bi-ideal of S. Let A be a $(1,2)$-ideal of S, then

$$\begin{aligned}(SA^2 \cdot S)A =& (SA^2 \cdot SS)A = (SS \cdot A^2S)A = (S \cdot A^2S)A = A^2S \cdot A \\ =& AS \cdot A^2 \subseteq A,\end{aligned}$$

which shows that A is a left ideal of a bi-ideal $SA^2 \cdot S$ of S.

$(ii) \Longrightarrow (iii)$. Let A be a left ideal of a bi-ideal B of S, then

$$\begin{aligned}(A \cdot SA^2)A =& (S \cdot AA^2)A \subseteq [S(SA \cdot AA)]A = [S(AA \cdot AS)]A \\ =& [AA \cdot S(AS)]A = [\{S(AS) \cdot A\}A]A = [(AS \cdot A)A]A \\ \subseteq& [(BS \cdot B)A]A \subseteq BA \cdot A \subseteq A,\end{aligned}$$

which shows that A is a bi-ideal of an ideal SA^2 of S.

$(iii) \Longrightarrow (iv)$. Let A be a bi-ideal of an ideal I of S, then

$$\begin{aligned}SA^2 \cdot A^2 =& (A^2 \cdot AA)S = (A \cdot A^2A)S \subseteq [A \cdot (AI)A]S = AA \cdot S \\ =& SA \cdot A \subseteq SI \cdot S \subseteq I,\end{aligned}$$

which shows that A is a $(0,2)$-ideal of a right ideal SA^2 of S.

$(iv) \Longrightarrow (v)$. It is easy to see that SA^3 is a $(0,2)$-ideal of S. Let A be a $(0,2)$-ideal of a right ideal R of S, then

$$\begin{aligned} A \cdot SA^3 =& A(SS \cdot A^2A) = A(AA^2 \cdot S) \subseteq A[(SA \cdot AA)S] = A[(AA \cdot AS)S] \\ =&(AA)[(A \cdot AS)S] = [S \cdot A(AS)]A^2 = [A \cdot S(AS)]A^2 \\ \subseteq& RS \cdot A^2 \subseteq RA^2 \subseteq A, \end{aligned}$$

which shows that A is a left ideal of a $(0,2)$-ideal SA^3 of S.

$(v) \Longrightarrow (i)$. Let A be a left ideal of a $(0,2)$-ideal O of S, then

$$AS \cdot A^2 = (AA \cdot SS)A = SA^2 \cdot A \subseteq SO^2 \cdot A \subseteq OA \subseteq A,$$

which shows that A is a $(1,2)$-ideal of S. □

Lemma 3. *Let S be a unitary $\mathscr{LA}$-semigroup and A be an idempotent subset of S. Then A is a $(1,2)$-ideal of S if and only if there exist a left ideal L and a right ideal R of S such that $RL \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-ideal of S such that A is idempotent. Setting $L = SA$ and $R = SA^2$, then

$$\begin{aligned} RL =& SA^2 \cdot SA = A^2S \cdot SA = (SA \cdot SS)A^2 = (SS \cdot AS)A^2 \\ =&[S(AA \cdot SS)]A^2 = [S(SS \cdot AA)]A^2 = [S\{A(SS \cdot A)\}]A^2 \\ =&[A(S \cdot SA)]A^2 \subseteq AS \cdot A^2 \subseteq A. \end{aligned}$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right ideal and L be a left ideal of S such that $RL \subseteq A \subseteq R \cap L$, then $AS \cdot A^2 = AS \cdot AA \subseteq RS \cdot SL \subseteq RL \subseteq A$. □

Assume that S is a unitary $\mathscr{LA}$-semigroup with zero. Then it is easy to see that every left (right) ideal of S is a $(0,2)$-ideal of S. Hence if O is a 0-minimal $(0,2)$-ideal of S and A is a left (right) ideal of S contained in O, then either $A = \{0\}$ or $A = O$.

Lemma 4. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Assume that A is a 0-minimal ideal of S and O is an $\mathscr{LA}$-subsemigroup of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Proof. Let O be a $(0,2)$-ideal of S contained in a 0-minimal ideal A of S. Then $SO^2 \subseteq O \subseteq A$. Since SO^2 is an ideal of S, therefore by minimality of A, $SO^2 = \{0\}$ or $SO^2 = A$. If $SO^2 = A$, then $A = SO^2 \subseteq O$ and therefore $O = A$. Let $SO^2 = \{0\}$, then $O^2S = SO^2 = \{0\} \subseteq O^2$, which shows that O^2 is a right ideal of S, and hence an ideal of S contained in A, therefore by minimality of A, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$, then $O = A$.

Conversely, let $O^2 = \{0\}$, then $SO^2 = O^2S = \{0\}S = \{0\} = O^2$. Now if $O = A$, then $SO^2 = SS \cdot OO = SA \cdot SA \subseteq A = O$, which shows that O is a $(0,2)$-ideal of S contained in A. □

Corollary 2. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Assume that A is a 0-minimal left ideal of S and O is an $\mathscr{LA}$-subsemigroup of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Lemma 5. *Let S be a unitary $\mathscr{LA}$-semigroup with zero and O be a 0-minimal $(0,2)$-ideal of S. Then $O^2 = \{0\}$ or O is a 0-minimal right $(left)$ ideal of S.*

Proof. Let O be a 0-minimal $(0,2)$-ideal of S, then

$$S(O^2)^2 = SS \cdot O^2O^2 = O^2O^2 \cdot S = SO^2 \cdot O^2 \subseteq OO^2 \subseteq O^2,$$

which shows that O^2 is a $(0,2)$-ideal of S contained in O, therefore by minimality of O, $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then $OS = OO \cdot SS = SO^2 \subseteq O$, which shows that O is a right ideal of S. Let R be a right ideal of S contained in O, then $R^2S = RR \cdot S \subseteq RS \cdot S \subseteq R$. Thus R is a $(0,2)$-ideal of S contained in O, and again by minimality of O, $R = \{0\}$ or $R = O$. □

The following Corollary follows from Lemma 4 and Corollary 2.

Corollary 3. *Let S be a unitary $\mathscr{LA}$-semigroup. Then O is a minimal $(0,2)$-ideal of S if and only if O is a minimal left ideal of S.*

Theorem 2. *Let S be a unitary $\mathscr{LA}$-semigroup. Then A is a minimal $(2,1)$-ideal of S if and only if A is a minimal bi-ideal of S.*

Proof. Let A be a minimal $(2,1)$-ideal of S. Then

$$\begin{aligned}
[(A^2S \cdot A)^2S](A^2S \cdot A) =& [\{(A^2S \cdot A)(A^2S \cdot A)\}S](A^2S \cdot A) \\
\subseteq& [\{(AS \cdot A)(AS \cdot A)\}S](AS \cdot A) \\
=& [\{(AS \cdot AS)(AA)\}S](AS \cdot A) \\
=& [(A^2S \cdot AA)S](AS \cdot A) \\
\subseteq& [(AS \cdot AS)S](AS \cdot A) \\
=& (A^2S \cdot S)(AS \cdot A) \\
\subseteq& (AS \cdot S)(AS \cdot A) = (AS \cdot AS)(SA) \\
=& A^2S \cdot SA = AS \cdot SA^2 = (SA^2 \cdot S)A \\
=& (A^2S \cdot S)A = (SS \cdot AA)A = A^2S \cdot A,
\end{aligned}$$

and similarly we can show that $(A^2S \cdot A)^2 \subseteq A^2S \cdot A$. Thus $A^2S \cdot A$ is a $(2,1)$-ideal of S contained in A, therefore by minimality of A, $A^2S \cdot A = A$. Now

$$\begin{aligned}
AS \cdot A =& (AS)(A^2S \cdot A) = [(A^2S \cdot A)S]A = (SA \cdot A^2S)A \\
=& [A^2(SA \cdot S)]A \subseteq A^2S \cdot A = A,
\end{aligned}$$

It follows that A is a bi-ideal of S. Suppose that there exists a bi-ideal B of S contained in A, then $B^2S \cdot B \subseteq BS \cdot B \subseteq B$, so B is a $(2,1)$-ideal of S contained in A, therefore $B = A$.

Conversely, assume that A is a minimal bi-ideal of S, then it is easy to see that A is a $(2,1)$-ideal of S. Let C be a $(2,1)$-ideal of S contained in A, then

$$\begin{aligned}[(C^2S\cdot C)S](C^2S\cdot C) =&(SC\cdot C^2S)(C^2S\cdot C)=(SC^2\cdot CS)(C^2S\cdot C)\\ =&[C(SC^2\cdot S)](C^2S\cdot C)=[(C^2S\cdot C)(SC^2\cdot SS)]C\\ =&[(C^2S\cdot C)(S\cdot C^2S)]C=[(C^2S\cdot C)(C^2S)]C\\ =&[C^2\{(C^2S\cdot C)S\}]C\subseteq C^2S\cdot C.\end{aligned}$$

This shows that $C^2S\cdot C$ is a bi-ideal of S, and by minimality of A, $C^2S\cdot C=A$. Thus $A=C^2S\cdot C\subseteq C$, and therefore A is a minimal $(2,1)$-ideal of S. □

Theorem 3. *Let A be a 0-minimal $(0,2)$-bi-ideal of a unitary $\mathscr{LA}$-semigroup S with zero. Then exactly one of the following cases occurs:*

(i) $A=\{0,a\}$, $a^2=0$;

(ii) $\forall a\in A\backslash\{0\}$, $Sa^2=A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-ideal of S. Let $a\in A\backslash\{0\}$, then $Sa^2\subseteq A$. Also Sa^2 is a $(0,2)$-bi-ideal of S, therefore $Sa^2=\{0\}$ or $Sa^2=A$.

Let $Sa^2=\{0\}$. Since $a^2\in A$, we have either $a^2=a$ or $a^2=0$ or $a^2\in A\backslash\{0,a\}$. If $a^2=a$, then $a^3=a^2a=a$, which is impossible because $a^3\in a^2S=Sa^2=\{0\}$. Let $a^2\in A\backslash\{0,a\}$, we have

$$S\cdot\{0,a^2\}\{0,a^2\}=SS\cdot a^2a^2=Sa^2\cdot Sa^2=\{0\}\subseteq\{0,a^2\},$$

and

$$[\{0,a^2\}S]\{0,a^2\}=\{0,a^2S\}\{0,a^2\}=a^2S\cdot a^2\subseteq Sa^2=\{0\}\subseteq\{0,a^2\}.$$

Therefore $\{0,a^2\}$ is a $(0,2)$-bi-ideal of S contained in A. We observe that $\{0,a^2\}\neq\{0\}$ and $\{0,a^2\}\neq A$. This is a contradiction to the fact that A is a 0-minimal $(0,2)$-bi-ideal of S. Therefore $a^2=0$ and $A=\{0,a\}$.

If $Sa^2\neq\{0\}$, then $Sa^2=A$. □

Corollary 4. *Let A be a 0-minimal $(0,2)$-bi-ideal of a unitary $\mathscr{LA}$-semigroup S with zero such that $A^2\neq 0$. Then $A=Sa^2$ for every $a\in A\backslash\{0\}$.*

Lemma 6. *Let S be a unitary $\mathscr{LA}$-semigroup. Then every right ideal of S is a $(0,2)$-bi-ideal of S.*

Proof. Assume that A is a right ideal of S, then

$$SA^2=AA\cdot SS=AS\cdot AS\subseteq AA\subseteq AS\subseteq A,\ AS\cdot A\subseteq A,$$

and clearly $A^2\subseteq A$, therefore A is a $(0,2)$-bi-ideal of S. □

The converse of Lemma 6 is not true in general. Example 1 showed that there exists a $(0,2)$-bi-ideal $A=\{a,c,e\}$ of S which is not a right ideal of S.

Theorem 4. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Then $Sa^2 = S$ $\forall a \in S\backslash\{0\}$ if and only if S is 0-$(0,2)$-bisimple if and only if S is right 0-simple.*

Proof. Assume that $Sa^2 = S$ for every $a \in S\backslash\{0\}$. Let A be a $(0,2)$-bi-ideal of S such that $A \neq \{0\}$. Let $a \in A\backslash\{0\}$, then $S = Sa^2 \subseteq SA^2 \subseteq A$. Therefore $S = A$. Since $S = Sa^2 \subseteq SS = S^2$, we have $S^2 = S \neq \{0\}$. Thus S is $0-(0,2)$-bisimple. The converse statement follows from Corollary 4.

Let R be a right ideal of 0-$(0,2)$-bisimple S. Then by Lemma 6, R is a $(0,2)$-bi-ideal of S and so $R = \{0\}$ or $R = S$.

Conversely, assume that S is right 0-simple. Let $a \in S\backslash\{0\}$, then $Sa^2 = S$. Hence S is 0-$(0,2)$-bisimple. □

Theorem 5. *Let A be a 0-minimal $(0,2)$-bi-ideal of a unitary $\mathscr{LA}$-semigroup S with zero. Then either $A^2 = \{0\}$ or A is right 0-simple.*

Proof. Assume that A is 0-minimal $(0,2)$-bi-ideal of S such that $A^2 \neq \{0\}$. Then by using Corollary 4, $Sa^2 = A$ for every $a \in A\backslash\{0\}$. Since $a^2 \in A\backslash\{0\}$ for every $a \in A\backslash\{0\}$, we have $a^4 = (a^2)^2 \in A\backslash\{0\}$ for every $a \in A\backslash\{0\}$. Let $a \in A\backslash\{0\}$, then

$$\begin{aligned}(Aa^2)S \cdot Aa^2 =& a^2A \cdot S(Aa^2) = [(S \cdot Aa^2)A]a^2 \subseteq [(S \cdot A)A]a^2\\ =& (AA \cdot SS)a^2 = SA^2 \cdot a^2 \subseteq Aa^2,\end{aligned}$$

and

$$\begin{aligned}S(Aa^2)^2 =& S(Aa^2 \cdot Aa^2) = S(a^2A \cdot a^2A) = S[a^2(a^2A \cdot A)]\\ =& (aa)[S(a^2A \cdot A)] = [(a^2A \cdot A)S]a^2\\ \subseteq& (AA \cdot SS)a^2 = SA^2 \cdot a^2 \subseteq Aa^2,\end{aligned}$$

which shows that Aa^2 is a $(0,2)$-bi-ideal of S contained in A. Hence $Aa^2 = \{0\}$ or $Aa^2 = A$. Since $a^4 \in Aa^2$ and $a^4 \in A\backslash\{0\}$, we get $Aa^2 = A$. Thus by using Theorem 4, A is right 0-simple. □

4. (m,n)-Ideals in Unitary $\mathscr{LA}$-Semigroups

In this section, we characterize a unitary $\mathscr{LA}$-semigroup in terms of (m,n)-ideals with the assumption that $m,n \geq 3$. If we take $m,n \geq 2$, then all the results of this section can be trivially followed for a locally associative unitary $\mathscr{LA}$-semigroup. If S is a unitary $\mathscr{LA}$-semigroup, then it is easy to see that $SA^m = A^mS$ and $A^mA^n = A^nA^m$ for $m,n \geq 3$ such that $A^0 = e$ if occurs, where e is a left identity of S.

Lemma 7. *Let S be a unitary $\mathscr{LA}$-semigroup. If R and L are the right and left ideals of S respectively, then RL is an (m,n)-ideal of S.*

Proof. Let R and L be the right and left ideals of S respectively, then

$$\begin{aligned}(RL)^m S\cdot(RL)^n =&(R^mL^m\cdot S)(R^nL^n)=(R^mL^m\cdot R^n)(SL^n)\\ =&(L^mR^m\cdot R^n)(SL^n)=(R^nR^m\cdot L^m)(SL^n)\\ =&(R^mR^n\cdot L^m)(SL^n)=(R^{m+n}L^m)(SL^n)\\ =&S(R^{m+n}L^m\cdot L^n)=S(L^nL^m\cdot R^{m+n})\\ =&SS\cdot L^{m+n}R^{m+n}=SL^{m+n}\cdot SR^{m+n}\\ =&R^{m+n}S\cdot L^{m+n}S=SR^{m+n}\cdot SL^{m+n},\end{aligned}$$

and

$$\begin{aligned}SR^{m+n}\cdot SL^{m+n}=&(S\cdot R^{m+n-1}R)(S\cdot L^{m+n-1}L)\\ =&[S(R^{m+n-2}R\cdot R)][S(L^{m+n-2}L\cdot L)]\\ =&[S(RR\cdot R^{m+n-2})][S(LL\cdot L^{m+n-2})]\\ \subseteq&(SS\cdot RR^{m+n-2})(SS\cdot LL^{m+n-2})\\ \subseteq&(SR\cdot SR^{m+n-2})(SL\cdot SL^{m+n-2})\\ \subseteq&(R^{m+n-2}S\cdot RS)(L\cdot SL^{m+n-2})\\ \subseteq&(R^{m+n-2}S\cdot R)(S\cdot LL^{m+n-2})\\ =&(RS\cdot R^{m+n-2})(SL^{m+n-1})\\ \subseteq&RR^{m+n-2}\cdot SL^{m+n-1}\\ \subseteq&SR^{m+n-1}\cdot SL^{m+n-1},\end{aligned}$$

therefore

$$\begin{aligned}(RL)^m S\cdot(RL)^n\subseteq&SR^{m+n}\cdot SL^{m+n}\subseteq SR^{m+n-1}\cdot SL^{m+n-1}\subseteq\ldots\subseteq SR\cdot SL\\ \subseteq&(SS\cdot R)L=(RS\cdot S)L\subseteq RL,\end{aligned}$$

and also

$$RL\cdot RL=LR\cdot LR=(LR\cdot R)L=(RR\cdot L)L\subseteq(RS\cdot S)L\subseteq RL.$$

This shows that RL is an (m,n)-ideal of S. □

Theorem 6. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. If S has the property that it contains no non-zero nilpotent (m,n)-ideals and R (L) is a 0-minimal right $(left)$ ideal of S, then either $RL=\{0\}$ or RL is a 0-minimal (m,n)-ideal of S.*

Proof. Assume that $R(L)$ is a 0-minimal right $(left)$ ideal of S such that $RL\neq\{0\}$, then by Lemma 7, RL is an (m,n)-ideal of S. Now we show that RL is a 0-minimal (m,n)-ideal of S. Let $\{0\}\neq M\subseteq RL$ be an (m,n)-ideal of S. Note that since $RL\subseteq R\cap L$, we have $M\subseteq R\cap L$. Hence $M\subseteq R$ and $M\subseteq L$. By hypothesis, $M^m\neq\{0\}$ and $M^n\neq\{0\}$. Since $\{0\}\neq SM^m=M^mS$, therefore

$$\{0\}\neq M^mS\subseteq R^mS=R^{m-1}R\cdot S=SR\cdot R^{m-1}=SR\cdot R^{m-2}R$$

$$=RR^{m-2}\cdot RS\subseteq RR^{m-2}\cdot R=R^{m},$$

and

$$\begin{aligned}R^{m}\subseteq&SR^{m}=SS\cdot RR^{m-1}=R^{m-1}R\cdot S=(R^{m-2}R\cdot R)S\\=&(RR\cdot R^{m-2})S=SR^{m-2}\cdot RR\subseteq SR^{m-2}\cdot R\\=&(SS\cdot R^{m-3}R)R=(RR^{m-3}\cdot SS)R=(RS\cdot R^{m-3}S)R\\\subseteq&(R\cdot R^{m-3}S)R=(R^{m-3}\cdot RS)R\subseteq R^{m-3}R\cdot R=R^{m-1},\end{aligned}$$

therefore $\{0\}\neq M^{m}S\subseteq R^{m}\subseteq R^{m-1}\subseteq\ldots\subseteq R$. It is easy to see that $M^{m}S$ is a right ideal of S. Thus $M^{m}S=R$ since R is 0-minimal. Also

$$\{0\}\neq SM^{n}\subseteq\{0\}\neq SL^{n}=S\cdot L^{n-1}L=L^{n-1}\cdot SL\subseteq L^{n-1}L=L^{n},$$

and

$$\begin{aligned}L^{n}\subseteq&SL^{n}=SS\cdot LL^{n-1}=L^{n-1}L\cdot S=(L^{n-2}L\cdot L)S=SL\cdot L^{n-2}L\\\subseteq&L\cdot L^{n-2}L=L^{n-2}\cdot LL\subseteq L^{n-2}L=L^{n-1}\subseteq\ldots\subseteq L,\end{aligned}$$

therefore $\{0\}\neq SM^{n}\subseteq L^{n}\subseteq L^{n-1}\subseteq\ldots\subseteq L$. It is easy to see that SM^{n} is a left ideal of S. Thus $SM^{n}=L$ since L is 0-minimal. Therefore

$$\begin{aligned}M\subseteq&RL=M^{m}S\cdot SM^{n}=M^{n}S\cdot SM^{m}=(SM^{m}\cdot S)M^{n}\\=&(SM^{m}\cdot SS)M^{n}=(S\cdot M^{m}S)M^{n}=(M^{m}\cdot SS)M^{n}\\=&M^{m}S\cdot M^{n}\subseteq M.\end{aligned}$$

Thus $M=RL$, which means that RL is a 0-minimal (m,n)-ideal of S. □

Theorem 7. *Let S be a unitary $\mathscr{LA}$-semigroup. If R (L) is a 0-minimal right $(left)$ ideal of S, then either $R^{m}L^{n}=\{0\}$ or $R^{m}L^{n}$ is a 0-minimal (m,n)-ideal of S.*

Proof. Assume that $R(L)$ is a 0-minimal right $(left)$ ideal of S such that $R^{m}L^{n}\neq\{0\}$, then $R^{m}\neq\{0\}$ and $L^{n}\neq\{0\}$. Hence $\{0\}\neq R^{m}\subseteq R$ and $\{0\}\neq L^{n}\subseteq L$, which shows that $R^{m}=R$ and $L^{n}=L$ since R (L) is a 0-minimal right $(left)$ ideal of S. Thus by Lemma 7, $R^{m}L^{n}=RL$ is an (m,n)-ideal of S. Now we show that $R^{m}L^{n}$ is a 0-minimal (m,n)-ideal of S. Let $\{0\}\neq M\subseteq R^{m}L^{n}=RL\subseteq R\cap L$ be an (m,n)-ideal of S. Hence

$$\{0\}\neq SM^{2}=MM\cdot SS=MS\cdot MS\subseteq RS\cdot RS\subseteq R$$

and $\{0\}\neq SM\subseteq SL\subseteq L$. Thus $R=SM^{2}=MM\cdot SS=SM\cdot M\subseteq SM$ and $SM=L$ since R (L) is a 0-minimal right $(left)$ ideal of S. Therefore

$$\begin{aligned}M\subseteq&R^{m}L^{n}\subseteq(SM)^{m}(SM)^{n}=S^{m}M^{m}\cdot S^{n}M^{n}=SS\cdot M^{m}M^{n}\\=&M^{n}M^{m}\cdot S=SM^{m}\cdot M^{n}=M^{m}S\cdot M^{n}\subseteq M,\end{aligned}$$

Thus $M=R^{m}L^{n}$, which shows that $R^{m}L^{n}$ is a 0-minimal (m,n)-ideal of S. □

Theorem 8. *Let S be a unitary $\mathscr{LA}$-semigroup with zero. Assume that A is an (m,n)-ideal of S and B is an (m,n)-ideal of A such that B is idempotent. Then B is an (m,n)-ideal of S.*

Proof. It is trivial that B is an $\mathscr{LA}$-subsemigroup S. Secondly, since $A^mS\cdot A^n\subseteq A$ and $B^mA\cdot B^n\subseteq B$, then

$$\begin{aligned}B^mS\cdot B^n =&(B^mB^m\cdot S)(B^nB^n)=(B^nB^n)(S\cdot B^mB^m)\\ =&[(S\cdot B^mB^m)B^n]B^n=[(B^n\cdot B^mB^m)(SS)]B^n\\ =&[(B^m\cdot B^nB^m)(SS)]B^n=[S(B^nB^m\cdot B^m)]B^n\\ =&[S(B^nB^m\cdot B^{m-1}B)]B^n=[S(BB^{m-1}\cdot B^mB^n)]B^n\\ =&[S(B^m\cdot B^mB^n)]B^n=[B^m(SS\cdot B^mB^n)]B^n\\ =&[B^m(B^nB^m\cdot SS)]B^n=[B^m(SB^m\cdot B^n)]B^n\\ =&[B^m\{(SS\cdot B^{m-1}B)B^n\}]B^n=[B^m(B^mS\cdot B^n)]B^n\\ \subseteq&[B^m(A^mS\cdot A^n)]B^n\subseteq B^mA\cdot B^n\subseteq B,\end{aligned}$$

which shows that B is an (m,n)-ideal of S. □

Lemma 8. *Let $\langle a\rangle_{(m,n)}=a^mS\cdot a^n$, then $\langle a\rangle_{(m,n)}$ is an (m,n)-ideal of a unitary $\mathscr{LA}$-semigroup S.*

Proof. Assume that S is a unitary $\mathscr{LA}$-semigroup and m,n are non-negative integers, then

$$\begin{aligned}\left(\{\langle a\rangle_{(m,n)}\}^mH\right)\{\langle a\rangle_{(m,n)}\}^n =&[\{((a^mH)a^n)\}^mH]\{(a^mH)a^n\}^n\\ =&[\{(a^{mm}H^m)a^{mn}\}H]\{(a^{mn}H^n)a^{nn}\}\\ =&[a^{nn}(a^{mn}H^n)][H\{(a^{mm}H^m)a^{mn}\}]\\ =&[[H\{(a^{mm}H^m)a^{mn}\}](a^{mn}H^n)]a^{nn}\\ =&[a^{mn}[[H\{(a^{mm}H^m)a^{mn}\}]H^n]]a^{nn}\\ \subseteq&(a^{mn}H)a^{nn}=(a^{mn}H^n)a^{nn}\\ =&\{(a^mH)a^n\}^n\subseteq\left(\langle a\rangle_{(m,n)}\right)^n\subseteq\langle a\rangle_{(m,n)},\end{aligned}$$

and similarly we can show that $\left(\langle a\rangle_{(m,n)}\right)^2\subseteq\langle a\rangle_{(m,n)}$. □

Theorem 9. *Let S be a unitary $\mathscr{LA}$-semigroup and $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of S. Then the following statements hold:*

(i) $\left(\langle a\rangle_{(1,0)}\right)^mS=a^mS$;

(ii) $S\left(\langle a\rangle_{(0,1)}\right)^n=Sa^n$;

(iii) $\left(\langle a\rangle_{(1,0)}\right)^mS\cdot\left(\langle a\rangle_{(0,1)}\right)^n=(a^mS)a^n$.

Proof. (i). As $\langle a\rangle_{(1,0)} = aS$, we have

$$\begin{aligned}\left(\langle a\rangle_{(1,0)}\right)^m S =&(aS)^m S = (aS)^{m-1}(aS)\cdot S = S(aS)\cdot(aS)^{m-1}\\ =&(aS)(aS)^{m-1} = (aS)[(aS)^{m-2}(aS)]\\ =&(aS)^{m-2}(aS\cdot aS) = (aS)^{m-2}(a^2S)\\ =&\ldots = \begin{cases}(aS)^{m-(m-1)}(a^{m-1}S) & \text{if } m \text{ is odd}\\ (a^{m-1}S)(aS)^{m-(m-1)} & \text{if } m \text{ is even}\end{cases}\\ =&a^m S.\end{aligned}$$

Analogously, we can prove (ii) and (iii) is simple. □

Corollary 5. *Let S be a unitary $\mathscr{L}\mathscr{A}$-semigroup and let $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of S. Then the following statements hold:*

(i) $\left(\langle a\rangle_{(1,0)}\right)^m S = Sa^m$;

(ii) $S\left(\langle a\rangle_{(0,1)}\right)^n = a^n S$;

(iii) $\left(\langle a\rangle_{(1,0)}\right)^m S\cdot\left(\langle a\rangle_{(0,1)}\right)^n = (Sa^m)(a^n S)$.

Let $\mathfrak{L}_{(0,n)}$, $\mathfrak{R}_{(m,0)}$ and $\mathfrak{A}_{(m,n)}$ denote the sets of $(0,n)$-ideals, $(m,0)$-ideals and (m,n)-ideals of an $\mathscr{L}\mathscr{A}$-semigroup S respectively.

Theorem 10. *If S is a unitary $\mathscr{L}\mathscr{A}$-semigroup, then the following statements hold:*

(i) S *is* $(0,1)$*-regular if and only if* $\forall L\in\mathfrak{L}_{(0,1)}$, $L = SL$;

(ii) S *is* $(2,0)$*-regular if and only if* $\forall R\in\mathfrak{R}_{(2,0)}$, $R = R^2S$ *such that every* R *is semiprime;*

(iii) S *is* $(0,2)$*-regular if and only if* $\forall U\in\mathfrak{A}_{(0,2)}$, $U = U^2S$ *such that every* U *is semiprime.*

Proof. (i). Let S be $(0,1)$-regular, then for $a\in S$ there exists $x\in S$ such that $a = xa$. Since L is $(0,1)$-ideal, therefore $SL\subseteq L$. Let $a\in L$, then $a = xa\in SL\subseteq L$. Hence $L = SL$. Converse is simple.

(ii). Let S be $(2,0)$-regular and R be $(2,0)$-ideal of S, then it is easy to see that $R = R^2S$. Now for $a\in S$ there exists $x\in S$ such that $a = a^2x$. Let $a^2\in R$, then

$$a = a^2x\in RS = R^2S\cdot S = SS\cdot R^2 = R^2S = R,$$

which shows that every $(2,0)$-ideal is semiprime.

Conversely, let $R = R^2S$ for every $R\in\mathfrak{R}_{(2,0)}$. Since Sa^2 is a $(2,0)$-ideal of S such that $a^2\in Sa^2$, therefore $a\in Sa^2$. Thus

$$\begin{aligned}a\in&Sa^2 = (Sa^2)^2S = (Sa^2\cdot Sa^2)S = (a^2S\cdot a^2S)S = [a^2(a^2S\cdot S)]S\\ =&(a^2\cdot Sa^2)S = (S\cdot Sa^2)a^2\subseteq Sa^2 = a^2S,\end{aligned}$$

which implies that S is $(2,0)$-regular.

Analogously, we can prove (iii). □

Lemma 9. *If S is a unitary $\mathscr{L}\mathscr{A}$-semigroup, then the following statements hold:*

(i) If S is $(0,n)$-regular, then $\forall L \in \mathfrak{L}_{(0,n)}$, $L = SL^n$;

(ii) If S is $(m,0)$-regular, then $\forall R \in \mathfrak{R}_{(m,0)}$, $R = R^m S$;

(iii) If S is (m,n)-regular, then $\forall U \in \mathfrak{A}_{(m,n)}$, $U = (U^m S)U^n$.

Proof. It is simple. □

Corollary 6. *If S is a unitary $\mathscr{L}\mathscr{A}$-semigroup, then the following statements hold:*

(i) If S is $(0,n)$-regular, then $\forall L \in \mathfrak{L}_{(0,n)}$, $L = L^n S$;

(ii) If S is $(m,0)$-regular, then $\forall R \in \mathfrak{R}_{(m,0)}$, $R = SR^m$;

(iii) If S is (m,n)-regular, then $\forall U \in \mathfrak{A}_{(m,n)}$, $U = U^{m+n}S = SU^{m+n}$.

Theorem 11. *Let S be a unitary (m,n)-regular $\mathscr{L}\mathscr{A}$-semigroup such that $m = n$. Then for every $R \in \mathfrak{R}_{(m,0)}$ and $L \in \mathfrak{L}_{(0,n)}$, $R \cap L = R^m L \cap RL^n$.*

Proof. It is simple. □

Theorem 12. *Let S be a unitary (m,n)-regular $\mathscr{L}\mathscr{A}$-semigroup. If M (N) is a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S such that $MN \subseteq M \cap N$, then either $MN = \{0\}$ or MN is a 0-minimal (m,n)-ideal of S.*

Proof. Let M (N) be a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S. Let $O = MN$, then clearly $O^2 \subseteq O$. Moreover

$$\begin{aligned} O^m S \cdot O^n =&(MN)^m S \cdot (MN)^n = (M^m N^m)S \cdot M^n N^n \subseteq (M^m S)S \cdot SN^n \\ =&SM^m \cdot SN^n = M^m S \cdot SN^n \subseteq MN = O, \end{aligned}$$

which shows that O is an (m,n)-ideal of S. Let $\{0\} \neq P \subseteq O$ be a non-zero (m,n)-ideal of S. Since S is (m,n)-regular, therefore by using Lemma 9, we have

$$\begin{aligned} \{0\} \neq& P = P^m S \cdot P^n = (P^m \cdot SS)P^n = (S \cdot P^m S)P^n = (P^n \cdot P^m S)(SS) \\ =&(P^n S)(P^m S \cdot S) = P^n S \cdot SP^m = P^m S \cdot SP^n. \end{aligned}$$

Hence $P^m S \neq \{0\}$ and $P^m S \neq \{0\}$. Further $P \subseteq O = MN \subseteq M \cap N$ implies that $P \subseteq M$ and $P \subseteq N$. Therefore $\{0\} \neq P^m S \subseteq M^m S \subseteq M$ which shows that $P^m S = M$ since M is 0-minimal. Likewise, we can show that $SP^n = N$. Thus we have

$$\begin{aligned} P \subseteq& O = MN = P^m S \cdot SP^n = P^n S \cdot SP^m = (SP^m \cdot SS)P^n \\ =&(S \cdot P^m S)P^n = P^m S \cdot P^n \subseteq P. \end{aligned}$$

This means that $P = MN$ and hence MN is 0-minimal. □

Theorem 13. *Let S be a unitary (m,n)-regular $\mathscr{LA}$ -semigroup. If M (N) is a* 0*-minimal $(m,0)$-ideal ($(0,n)$-ideal) of S, then either $M\cap N=\{0\}$ or $M\cap N$ is a* 0*-minimal (m,n)-ideal of S.*

Proof. Once we prove that $M\cap N$ is an (m,n)-ideal of S, the rest of the proof is same as in Theorem 11. Let $O=M\cap N$, then it is easy to see that $O^2\subseteq O$. Moreover $O^mS\cdot O^n\subseteq M^mS\cdot N^n\subseteq MN^n\subseteq SN^n\subseteq N$. But, we also have

$$\begin{aligned}O^mS\cdot O^n\subseteq & M^mS\cdot N^n=(M^m\cdot SS)N^n=(S\cdot M^mS)N^n=(N^n\cdot M^mS)S\\ =&(M^m\cdot N^nS)(SS)=(M^mS)(N^nS\cdot S)=M^mS\cdot SN^n\\ =&M^mS\cdot N^nS=N^n(M^mS\cdot S)=N^n\cdot SM^m=N^n\cdot M^mS\\ =&M^m\cdot N^nS=M^m\cdot SN^n\subseteq M^mN\subseteq M^mS\subseteq M.\end{aligned}$$

Thus $O^mS\cdot O^n\subseteq M\cap N=O$ and therefore O is an (m,n)-ideal of S. □

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