EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 8, No. 1, 2015, 118-125
ISSN 1307-5543 – www.ejpam.com

Left invariant Finsler manifolds are generalized Berwald

Bernadett Aradi

MTA-DE Research Group "Equations, Functions and Curves", Hungarian Academy of Sciences and Institute of Mathematics, University of Debrecen, Debrecen, Hungary

---

Abstract. In this note we show that a Lie group endowed with a left invariant Finsler function is a generalized Berwald manifold. This observation makes it possible to construct a whole class of generalized Berwald manifolds, thus satisfying a request of Hashiguchi [6]: *'...find much more interesting examples'*. In particular, we show that the Randers Lie group constructed by Libing and Mo in an unpublished manuscript is in fact a proper gerenalized Berwald manifold. We also have a look at the more specific bi-invariant case, and review some essential results concerning bi-invariant Finsler functions with (at least partly) new and conceptual proofs.

2010 Mathematics Subject Classifications: 53C60, 58B20, 22E99

Key Words and Phrases: Generalized Berwald manifold, Lie group, left invariant Finsler function, averaging process, geodesics

---

1. Preliminaries

1.1. Let M be an n-dimensional (Hausdorff, second countable, connected, smooth) *manifold* ($n \geq 2$). The real Lie algebra of smooth vector fields on M is denoted by $\mathfrak{X}(M)$. If $\varphi: M \to M$ is a smooth mapping, then $\varphi_*: TM \to TM$ is its derivative. We use the notions *Finsler function* and *Finsler manifold* as in [2] or [14]. By a *geodesic* of a Finsler function $F: TM \to \mathbb{R}$ we mean a smooth curve in M whose velocity field is an integral curve of the *canonical spray* of (M, F) [14, Lemma and Definition 9.2.17].

In the near one hundred years of history of Finsler geometry many special classes of Finsler manifolds have been introduced and studied in detail. The main actors in our considerations are Berwald manifolds and generalized Berwald manifolds. A possible definition of them sounds as follows:

Definition 1 ([12, Definition 4.1 and Proposition 4.3]). *A Finsler manifold (M, F) is a* generalized Berwald manifold *if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F.*

If the covariant derivative ∇ is also torsion-free, then (M, F) is called a Berwald manifold.

Email address: `bernadett.aradi@science.unideb.hu`

In the case of Berwald manifolds the torsion-free covariant derivative mentioned in the definition above is unique. For a recent introduction to Berwald manifolds we refer to [13].

1.2. Throughout the paper G denotes a *connected Lie group* with identity element e, while λ_g and ρ_g stand for the *left* and *right translation* by $g \in G$, respectively. The Lie algebra of left invariant vector fields is denoted by $\mathfrak{X}_L(G)$. As a real vector space, $\mathfrak{X}_L(G)$ is canonically isomorphic to the tangent space T_eG. For basic Lie theory the reader is referred to [7, Chapter 9].

A Finsler function F on G is *left invariant* if $F \circ (\lambda_g)_* = F$ for any $g \in G$, *right invariant* if $F \circ (\rho_g)_* = F$ for any $g \in G$, and *bi-invariant* if it is both left and right invariant.

2. Bi-invariant Finsler Functions

The following lemma and its proof can be found in the manuscript of Libing and Mo mentioned above. (We found this manuscript on the website of the School of Mathematical Sciences, Peking University, however, it is now unavailable.) We find it useful to recall this result with our proof, which seems to be simpler than the original one, and is based on important ideas.

Lemma 1. *Let F be a bi-invariant Finsler function on a Lie group G. Then the geodesics of F starting at e are the one-parameter subgroups of G. Furthermore, all other geodesics are left translations of the one-parameter subgroups.*

Proof. Let $\alpha\colon \mathbb{R} \to G$ denote a one-parameter subgroup of G with $v := \dot{\alpha}(0) \in T_eG$, and consider the left invariant vector field X corresponding to v, that is, $X(e) = v = \dot{\alpha}(0)$. Then α is an integral curve of X. Furthermore, let $\gamma\colon \mathbb{R} \to G$ be the geodesic of F satisfying $\gamma(0) = e$ and $\dot{\gamma}(0) = v$. Our aim is to show that γ is also an integral curve of X, whence by the uniqueness of integral curves with given initial velocity yields $\alpha = \gamma$, and the first assertion of the lemma will be thus proved.

Notice first, that since geodesics have constant speed,

$$F(\dot{\gamma}(t)) = F(\dot{\gamma}(0)) = F(v) \quad (t \in \mathbb{R}). \tag{1}$$

Also, the left invariance of F and X yield

$$F(X(g)) = F\big(X(\lambda_g(e))\big) = F\big((\lambda_g)_*(X(e))\big) = F(v), \text{ for any } g \in G. \tag{2}$$

Secondly, X is a Killing vector field. Indeed, as X is left invariant, the stages φ_t of the (global) flow φ of X have the form

$$\varphi_t\colon G \to G,\ g \mapsto \varphi_t(g) := \varphi(t,g) = g \cdot \alpha(t) = \rho_{\alpha(t)}(g), \quad (t \in \mathbb{R})$$

(see [7, Lemma 9.2.4]). It follows from the right invariance of F that these mappings are isometries of the Finsler manifold:

$$F \circ (\varphi_t)_* = F \circ (\rho_{\alpha(t)})_* = F. \tag{3}$$

By Proposition 5.2 in [9] equation (3) means that X is a Killing vector field. However, Proposition 7.2 of the cited paper implies that for a Killing vector field X and for a geodesic γ the function

$$t \in \mathbb{R} \mapsto g_{\dot{\gamma}(t)}\big(\dot{\gamma}(t), X(\gamma(t))\big) \in \mathbb{R} \tag{4}$$

is constant, where g is the metric tensor of (G, F).

By combining the results above we obtain that for any $t \in \mathbb{R}$

$$\begin{aligned} g_{\dot{\gamma}(t)}\big(\dot{\gamma}(t), X(\gamma(t))\big) &\overset{(4)}{=} g_{\dot{\gamma}(0)}\big(\dot{\gamma}(0), X(\gamma(0))\big) \\ &= g_v(v,v) = F^2(v) \overset{(1),(2)}{=} F(X(\gamma(t)))F(\dot{\gamma}(t)). \end{aligned}$$

Using the fundamental inequality (see, e.g., [14, Proposition 9.1.37]), we conclude that

$$X(\gamma(t)) = \theta(t)\dot{\gamma}(t), \text{ for some } \theta(t) \geq 0, \ t \in \mathbb{R}.$$

Since we already have $F(X(\gamma(t))) \overset{(2)}{=} F(v) \overset{(1)}{=} F(\dot{\gamma}(t))$, it follows that $\theta(t) = 1$ for all $t \in \mathbb{R}$ and hence $X(\gamma(t)) = \dot{\gamma}(t)$. Thus γ is also an integral curve of X with initial velocity v, which indeed means that the geodesics of F starting at e are the one-parameter subgroups of G.

To show that the other geodesics are left translations of the one-parameter subgroups, choose a geodesic $\widetilde{\gamma}$ of F and let $\dot{\widetilde{\gamma}}(0) = w \in T_pG$. Consider the geodesic γ starting at e (thus, a one-parameter subgroup) with initial velocity

$$\dot{\gamma}(0) = (\lambda_{p^{-1}})_*(w) \in T_eG.$$

Then, since left translations are isometries, $\lambda_p \circ \gamma$ is a geodesic as well such that

$$\lambda_p \circ \gamma(0) = p \text{ and } \dot{\overbrace{\lambda_p \circ \gamma}}(0) = (\lambda_p)_*(\dot{\gamma}(0)) = w,$$

which imply $\widetilde{\gamma} = \lambda_p \circ \gamma$. □

Remark 1. *The Riemannian analogue of Lemma 1 is well known, see, e.g., [10], Proposition 9 in Chapter 11.*

Proposition 1 (Latifi–Razavi [8]). *If F is a bi-invariant Finsler function on a Lie group G, then (G, F) is a Berwald manifold.*

Proof. It has been shown in [5] (Theorem 1.4), that there exists a bi-invariant Riemannian metric $\widehat{g}$ on G as well, and we know from [10] (or we obtain as a special case of Lemma 1), that the geodesics of a bi-invariant Riemannian metric starting at e are the one-parameter subgroups of G.

However, it follows from Lemma 1 as well, that the geodesics of the Finsler manifold (G, F) starting at the identity element are also the one-parameter subgroups of the Lie group, hence (G, F) and the Riemannian manifold $(G, \widehat{g})$ have the same geodesics. Then Proposition 7 of [13] (see condition (B9) in the cited paper) assures that (G, F) is a Berwald manifold. □

The key step in the previous proof is the existence of a bi-invariant Riemannian metric $\widehat{g}$ on G. Below we indicate an explicit way of constructing such a Riemannian metric on G from a given Finsler function F.

Lemma 2. *Suppose that a Lie group G admits a bi-invariant Finsler function F. Then G admits a bi-invariant Riemannian metric $\widehat{g}$ as well, given by*

$$\widehat{g}_p(v,w) := \int_{S_p} (u \mapsto (E_p)''(u)(v,w))\omega_p,$$

where

$p \in G$, $v, w \in T_pG$;

E_p *is the energy function associated to* $F_p := F \restriction T_pG$;

$S_p \subset T_pG$ *is the unit sphere determined by* F_p;

ω_p *is the unique volume form on* S_p *such that* $\int_{S_p} \omega_p = 1$.

Sketch of proof. To see that $\widehat{g}$ is indeed a Riemannian metric on G we refer to [13, Proposition 17]. The fact that this metric is bi-invariant follows from Proposition 9.1.53 of [14], since the cited result expresses that the left and right translations are isometries between the Euclidean vector spaces $(T_pG, \widehat{g}_p)$ and $(T_qG, \widehat{g}_q)$, where p and q are any two points in G.

Remark 2. *The metric constructed in the lemma is called an* averaged Riemannian metric*. Averaging process was first applied in Finsler geometry by Z. I. Szabó in [11]. For recent results and applications we refer to [4, 15, 16]. Clearly, if a Lie group is endowed with a left (resp. right) invariant Finsler function, then the averaged Riemannian metric is left (resp. right) invariant.*

3. Left Invariant Finsler Functions

The main result of the paper, as indicated in the title, is the following.

Theorem 1. *A connected Lie group equipped with a left invariant Finsler function is a generalized Berwald manifold.*

Proof. Let G be an n-dimensional connected Lie group and let $F\colon TG \to \mathbb{R}$ be a Finsler function such that

$$F \circ (\lambda_g)_* = F \text{ for all } g \in G. \tag{5}$$

More precisely, $F_{gp} \circ ((\lambda_g)_*)_p = F_p$ for all $p, g \in G$, where $F_p := F \restriction T_pG$. Furthermore, let $(E_i)_{i=1}^n$ be a frame field of G consisting of left invariant vector fields.

Observe first that for any two points p, q in G and any n-tuple $(\nu^1, \dots, \nu^n) \in \mathbb{R}^n$ we have

$$F_q(\nu^i E_i(q)) = F_p(\nu^i E_i(p)) \tag{6}$$

(summation convention in force), that is, parallel vectors have the same Finsler norms. Indeed,

$$\begin{aligned} F_q(\nu^i E_i(q)) =& F_q\big(\nu^i E_i(\lambda_{qp^{-1}}(p))\big) = F_q(\nu^i (\lambda_{qp^{-1}})_* E_i(p)) \\ =& F_q\big((\lambda_{qp^{-1}})_*(\nu^i E_i(p))\big) \overset{(5)}{=} F_p(\nu^i E_i(p)). \end{aligned}$$

Now consider the covariant derivative ∇ on G specified by

$$\nabla_{E_j} E_k = 0; \quad j,k \in \{1,\dots,n\}. \tag{7}$$

Given a smooth curve $\gamma\colon I \to G$ (where $0 \in I$), let ∇_γ be the induced covariant derivative along γ, and let $(P_\gamma)_0^t\colon T_{\gamma(0)}G \to T_{\gamma(t)}G$ be the corresponding parallel translation from $\gamma(0)$ to $\gamma(t)$ $(t \in I)$. For any tangent vector $v \in T_{\gamma(0)}G$, there exists a unique parallel vector field X along γ such that $X(0) = v$. It can be expressed in the form

$$X = X^i(E_i \circ \gamma),$$

where $X^i\colon I \to \mathbb{R}$, $i \in \{1,\dots,n\}$ are smooth functions. Thus we have

$$0 = \nabla_\gamma X = \nabla_\gamma(X^i(E_i \circ \gamma)) \overset{(7)}{=} X^{i\prime}(E_i \circ \gamma),$$

which implies that the functions X^i are constant. So there is an n-tuple $(v^1,\dots,v^n) \in \mathbb{R}^n$ such that

$$X = v^i(E_i \circ \gamma).$$

Then $v = X(0) = v^i E_i(\gamma(0))$. Now, for every $t \in I$,

$$(P_\gamma)_0^t(v) = X(t) = v^i E_i(\gamma(t)), \tag{8}$$

and hence

$$F_{\gamma(t)} \circ (P_\gamma)_0^t(v) \overset{(8)}{=} F_{\gamma(t)}(v^i E_i(\gamma(t))) \overset{(6)}{=} F_{\gamma(0)}\big(v^i E_i(\gamma(0))\big) = F_{\gamma(0)}(v).$$

Thus the parallel translations determined by ∇ preserve the Finsler norms of the tangent vectors to G, as was to be shown. □

Remark 3. *The "compatible"and "covariant "derivative constructed in the proof of Theorem 1 is a special case of a covariant derivative appearing in [1] (Corollary 4.3, see also Theorem 4.1). This result characterizes generalized Berwald manifolds by using the concept of a* covering parallelism, *which is compatible with the Finsler function. Namely, it is shown that* a Finsler manifold is a generalized Berwald manifold if, and only if, the Finsler function is compatible with a covering parallelism.

4. An Example

Theorem 1 makes it possible to obtain numerous examples of generalized Berwald manifolds. Here we show that the Finsler function constructed by Libing and Mo as an example of a left invariant Finsler function not all of whose geodesics are left translations of one-parameter subgroups, yields a proper generalized Berwald manifold.

Let $G = \{(g^1,g^2) \in \mathbb{R}^2 \mid g^2 > 0\}$ and define a multiplication on it by

$$(g^1,g^2) \times (h^1,h^2) := (h^1 g^2 + g^1, g^2 h^2), \quad (g^1,g^2),(h^1,h^2) \in G.$$

Then G is a Lie group with identity element $e = (0,1)$; the inverse of an element (g^1, g^2) is $(g^1, g^2)^{-1} = \left(-\frac{g^1}{g^2}, \frac{1}{g^2}\right)$.

First we calculate the derivative of a left translation λ_g, where $g := (a,b)$. Consider a tangent vector $v \in T_pG$ $(p \in G)$ and let I be an open interval containing 0. Choose a curve $\gamma = (\gamma^1, \gamma^2)\colon I \to G$ such that $\dot{\gamma}(0) = v$. Then for some $t \in I$ we have

$$\lambda_g(\gamma(t)) = g \times (\gamma^1(t), \gamma^2(t)) = (b\gamma^1(t) + a, b\gamma^2(t)),$$

thus

$$(\lambda_g)_*(v) = (\lambda_g)_*(\dot{\gamma}(0)) = \big(t \mapsto \lambda_g(\gamma(t))\big)'(0) = (b\gamma^{1\prime}(0), b\gamma^{2\prime}(0)) = bv. \tag{9}$$

Now we define an appropriate Finsler function F on TG. Let

$$\alpha_p(v,w) := \frac{1}{(p^2)^2}(v^1 v^2)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix},$$
$$\beta_p(v) := \frac{1}{p^2}(v^1 + v^2),$$

and

$$F(v) := \sqrt{\alpha_p(v,v)} + \beta_p(v) = \frac{1}{p^2}\left(\sqrt{2(v^1)^2 + 2v^1v^2 + 2(v^2)^2} + v^1 + v^2\right),$$

where $v = (v^1, v^2), w = (w^1, w^2) \in T_pG$ and $p = (p^1, p^2) \in G$. Since

$$\begin{aligned}\|\beta\|_\alpha(p) &= \sqrt{\alpha^{ij}(p)\beta_i(p)\beta_j(p)} \\ &= \sqrt{\frac{(p^2)^2}{3}\left(2\cdot\frac{1}{(p^2)^2} - \frac{1}{(p^2)^2} - \frac{1}{(p^2)^2} + 2\cdot\frac{1}{(p^2)^2}\right)} = \sqrt{\frac{2}{3}} < 1,\end{aligned}$$

F is in fact a Randers function [14, Lemma and Definition 9.6.2].

This Randers function F is left invariant. To see this let $g = (a,b) \in G$ and $v \in T_pG$. Then

$$\begin{aligned}F((\lambda_g)_*(v)) &= F_{g\times p}((\lambda_{(a,b)})_*(v)) \overset{(9)}{=} F_{(p^1b+a, bp^2)}(bv) \\ &= \frac{1}{bp^2}\left(\sqrt{2b^2(v^1)^2 + 2b^2v^1v^2 + 2b^2(v^2)^2} + bv^1 + bv^2\right) \\ &= \frac{1}{p^2}\left(\sqrt{2(v^1)^2 + 2v^1v^2 + 2(v^2)^2} + v^1 + v^2\right) \\ &= F(v),\end{aligned}$$

as we claimed. Thus (G,F) is a generalized Berwald manifold by Theorem 1.

Finally, we show that $\nabla\beta \neq 0$, where ∇ is the Levi-Civita derivative of α, and hence, by a nice theorem of M. Crampin [3], (G,F) is surely not a Berwald manifold. Indeed, the Christoffel symbols of the Riemannian metric α are

$$\Gamma^1_{11} = -\frac{2}{3}\frac{1}{e^2},\ \Gamma^1_{12} = -\frac{4}{3}\frac{1}{e^2},\ \Gamma^1_{22} = -\frac{2}{3}\frac{1}{e^2},\ \Gamma^2_{11} = \frac{4}{3}\frac{1}{e^2},\ \Gamma^2_{12} = \frac{2}{3}\frac{1}{e^2},\ \Gamma^2_{22} = -\frac{2}{3}\frac{1}{e^2},$$

where (e^1, e^2) is the dual of the canonical basis of $\mathbb{R}^2$. Thus, for example,

$$\nabla\beta\left(\frac{\partial}{\partial e^1},\frac{\partial}{\partial e^1}\right)=\frac{\partial}{\partial e^1}\left(\beta\left(\frac{\partial}{\partial e^1}\right)\right)-\beta\left(\Gamma^1_{11}\frac{\partial}{\partial e^1}\right)-\beta\left(\Gamma^2_{11}\frac{\partial}{\partial e^2}\right)$$
$$=\frac{\partial}{\partial e^1}\left(\frac{1}{e^2}\right)-\Gamma^1_{11}\frac{1}{e^2}-\Gamma^2_{11}\frac{1}{e^2}=-\frac{2}{3}\frac{1}{(e^2)^2}\neq 0.$$

ACKNOWLEDGEMENTS. I am grateful to my supervisor, József Szilasi for his valuable guidance throughout the work. I would also like to thank Dávid Csaba Kertész for his useful comments.

This research was supported by the Hungarian Academy of Sciences, furthermore, by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11/1-2012-0001 "National Excellence Program".

References

[1] B. Aradi and D. Cs. Kertész. A characterization of holonomy invariant functions on tangent bundles. *Balkan Journal of Geometry and Its Applications*, 19 (2):1–10, 2014.

[2] D. Bao, S. S. Chern, and Z. Shen. *An Introduction to Riemann–Finsler Geometry.* Springer-Verlag, Berlin and New York, 2000.

[3] M. Crampin. Randers spaces with reversible geodesics. *Publicationes Mathematicae Debrecen*, 67 (3-4):401–409, 2005.

[4] M. Crampin. On the construction of riemannian metrics for berwald spaces by averaging. *Houston Journal of Mathematics*, 40 (3):737–750, 2014.

[5] S. Deng and Z. Hou. Invariant Finsler metrics on homogeneous manifolds. *Journal of Physics A: Mathematical and General*, 37:8245–8253, 2004.

[6] M. Hashiguchi. Some problems on generalized Berwald spaces. *Reports of the Faculty of Science. Kagoshima University. (Mathematics, Physics and Chemistry)*, 16:59–63, 1983.

[7] J. Hilgert and K. H. Neeb. *Structure and Geometry of Lie Groups.* Springer, 2012.

[8] D. Latifi and A. Razavi. Bi-invariant Finsler Metrics on Lie Groups. *Australian Journal of Basic and Applied Sciences*, 5 (12):507–511, 2011.

[9] R. L. Lovas. On the Killing vector fields of generalized metrics. *SUT Journal of Mathematics*, 40 (2):133–156, 2004.

[10] B. O'Neill. *Semi-Riemannian Geometry.* Academic Press, New York, 1983.

[11] Z. I. Szabó. Positive definite Berwald spaces (Structure Theorems on Berwald spaces). *Tensor New Series*, 35:25–39, 1981.

[12] Sz. Szakál and J. Szilasi. A new approach to generalized Berwald manifolds I. *SUT Journal of Mathematics*, 37:19–41, 2001.

[13] J. Szilasi, R. L. Lovas, and D. Cs. Kertész. Several ways to a Berwald manifold – and some steps beyond. *Extracta Mathematicae*, 26:89–130, 2011.

[14] J. Szilasi, R. L. Lovas, and D. Cs. Kertész. *Connections, Sprays and Finsler structures.* World Scientific, 2014.

[15] Cs. Vincze. Average methods and their applications in Differential Geometry I. *arXiv:1309.0827v2*, 2013.

[16] Cs. Vincze. On generalized conics' theory and averaged Riemannian metrics in Finsler geometry. *Tensor New Series*, 74 (1):101–116, 2013.