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Fractional Generalization of Rodrigues-type Formulas For Certain Class Of Special Functions

Maged G. Bin-Saad

Department of Mathematics, University of Aden, Khormaksar P. O. Box 6014, Aden, Yemen

Abstract. This paper refers to some generalizations of certain classical Rodrigues formulas. By means of the Riemann - Liouville operator of fractional calculus general Rodrigues-type representation formulas of fractional order are derived and some of their properties are given and compared with the corresponding properties of known cases.

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1. Preliminaries and Definitions

The subject of fractional calculus is one of the most intensively developing areas of mathematical analysis, mainly due to its fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics (see [8, 10, 12, 14]). Indeed, on behalf of the nature of their definitions the fractional derivatives and integrals provide an excellent instrument for the modeling of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical methods [1]. In this work, based upon Riemann - Liouville fractional derivative and integral operators we introduce a new generalized Rodrigues-type representation for a certain class of special functions involving Laguerre, Hermite , Bessel and Humbert polynomials, which provide further generalization of a number of known Rodrigues - type formulas and new fractional Rodirgues-type formulas (see [2–4, 9, 13]).

Let $L_1(I)$ be a class of Lebesque integrable functions on the interval $I=[a,b]$ where $0 \le a < b < \infty$, and let $\Gamma(\cdot)$ be the gamma function. According to the Riemann-Liouville approach to fractional calculus the fractional derivative D^{α} of order $\alpha \in (n-1,n)$, $(n=1,2,3,\ldots)$

Email address: mgbinsaad@yahoo.com

of the function $f(x)$ is given by (see[8, 12])

$$D_a^{\alpha} f(x) = I_a^{n-\alpha} D^n f(x), D = \frac{d}{dx}, \tag{1}$$

and the fractional integral of the function $f(t)$ of order β is defined by (see [6–13])

$$I_t^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-s)^{\beta-1} f(s) ds. \tag{2}$$

In comparison to the classical calculus let us mention that, for example, if $\mu \geq 0$, $t > 0$ and $\alpha > -1$, then the fractional derivative of the power function x^{α} is given by

$$D^{\mu} x^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} x^{\alpha-\mu}. \tag{3}$$

Definition 1. *Let $\nu, \gamma \in (n-1, n)$, $n = 1, 2, 3, \ldots$, $a, b, \beta \in \Re$ and $k = 1, 2, 3, \ldots$. We define the generalized fractional Rodrigues formula by the two functions*

$$F_{\nu}^{(\beta,\gamma)}(a, b, k; x) = \frac{x^{-\beta} e^{ax^k}}{\Gamma(\nu+1)} Y_{\nu}^{(\beta,\gamma)}(a, b, k; x), \tag{4}$$

$$F_{-\nu}^{(\beta,-\gamma)}(a, b, k; x) = \frac{x^{-\beta} e^{ax^k}}{\Gamma(1-\nu)} Y_{-\nu}^{(\beta,-\gamma)}(a, b, k; x), \tag{5}$$

where

$$Y_{\nu}^{(\beta,\gamma)}(a, b, k; x) = D^{\nu} x^{\beta+b\gamma} e^{-ax^k}, \quad \beta + b\gamma > -1, \tag{6}$$

and

$$Y_{-\nu}^{(\beta,-\gamma)}(a, b, k; x) = I^{\nu} x^{\beta-b\gamma} e^{-ax^k}, \quad \beta - b\gamma > -1. \tag{7}$$

It is important to note that the Laguerre polynomials $L_{\nu}^{\beta}(x)$ and $L_{-\nu}^{\beta}(x)$ due to El-Sayed [2, p.10, (5) and (6)], the Rodirgues formulas $L_{\nu}^{\beta}(\gamma, a; x)$ and $L_{-\nu}^{\beta}(-\gamma, a; x)$ due to Rida and El-Sayed [13, p.30, (3) and (4)] and the Laguerre polynomials $L_{\nu}^{(\alpha)}(x)$ introduced recently in [8] and used by El-Sayed [3, p.10, (5)and (6)] and Mirevski (see [9, p.1273, (15)]) are special cases of our formulas (4) and (5) as given below:

$$\frac{\Gamma(\nu+1)}{n!} F_{\nu}^{(\beta,\gamma)}(1, \frac{n}{\gamma}, 1; x) = \frac{x^{-\beta} e^{x}}{n!} D^{\nu} x^{\beta+n} e^{-x} = L_{\nu}^{\beta}(x), \tag{8}$$

$$\frac{\Gamma(1-\nu)}{n!} F_{-\nu}^{(\beta,\gamma)}(1, \frac{n}{\gamma}, 1; x) = \frac{x^{-\beta} e^{x}}{n!} I^{\nu} x^{\beta-n} e^{-x} = L_{-\nu}^{\beta}(x), \tag{9}$$

$$F_{\nu}^{(\beta,\gamma)}(a, 1, 1; x) = \frac{x^{-\beta} e^{ax}}{\Gamma(\nu+1)} D^{\nu} x^{\beta+\gamma} e^{-ax} = L_{\nu}^{\beta}(\gamma, a; x), \tag{10}$$

$$F_{-\nu}^{(\beta,\gamma)}(a, 1, 1; x) = \frac{x^{-\beta} e^{ax}}{\Gamma(1-\nu)} I^{\nu} x^{\beta-\gamma} e^{-ax} = L_{-\nu}^{\beta}(-\gamma, a; x), \tag{11}$$

$$F_{\nu}^{(\alpha,\nu)}(1, 1, 1; x) = \frac{x^{-\alpha} e^{x}}{\Gamma(\nu+1)} D^{\nu} x^{\alpha+\nu} e^{-x} = L_{\nu}^{(\alpha)}(x), \tag{12}$$

$$F_{-\nu}^{(\alpha,\nu)}(1,1,1;x)=\frac{x^{-\alpha}e^{x}}{\Gamma(1-\nu)}I^{\nu}x^{\alpha-\nu}e^{-x}=L_{-\nu}^{(\alpha)}(x), \tag{13}$$

where the last formula is new and suggested by the assertion (5).

Next, the introduction of the Rodrigues formulas (4) and (5) leads us to generalization of many well-known Rodrigues formulas up to fractional forms. In this regard the Rodrigues representations (4) and (5), in particular, yield the following new fractional Rodrigues–type representations for the generalized Hermite polynomials $H_n^{(r)}(x,a,b)$ [5], the generalized Laguerre polynomials $L_n^{(\alpha)}(x,k,p)$ [15], Bessel polynomials $y_n(x)$ and Humbert polynomials $h_n(x)$ as follows:

$$(-1)^{\nu}F_{\nu}^{(\beta,\gamma)}(a,0,k;x)=\frac{(-1)^{\nu}x^{-\beta}e^{-ax^{k}}}{\Gamma(\nu+1)}D^{\nu}x^{\beta}e^{-ax^{k}}=H_{\nu}^{(k)}(x,\beta,a), \tag{14}$$

$$(-1)^{\nu}F_{-\nu}^{(\beta,\gamma)}(a,0,k;x)=\frac{(-1)^{\nu}x^{-\beta}e^{-ax^{k}}}{\Gamma(1-\nu)}I^{\nu}x^{\beta}e^{-ax^{k}}=H_{-\nu}^{(k)}(x,\beta,a), \tag{15}$$

$$F_{\nu}^{(\beta,\nu)}(a,1,k;x)=\frac{x^{-\beta}e^{-ax^{k}}}{\Gamma(\nu+1)}D^{\nu}x^{\beta+\nu}e^{-ax^{k}}=L_{\nu}^{\beta}(x,k,a), \tag{16}$$

$$F_{-\nu}^{(\beta,\nu)}(a,1,k;x)=\frac{x^{-\beta}e^{-ax^{k}}}{\Gamma(1-\nu)}I^{\nu}x^{\beta-\nu}e^{-ax^{k}}=L_{-\nu}^{\beta}(x,k,a), \tag{17}$$

$$a^{-\nu}F_{\nu}^{(\beta-2,\nu)}(a,2,-1;x)=\frac{a^{-\nu}x^{-\beta+2}e^{\frac{a}{x}}}{\Gamma(\nu+1)}D^{\nu}x^{\beta+2\nu-2+}e^{\frac{-a}{x}}=y_{\nu}(x,\beta,a), \tag{18}$$

$$a^{-\nu}F_{-\nu}^{(\beta-2,\nu)}(a,2,-1;x)=\frac{a^{-\nu}x^{-\beta+2}e^{\frac{a}{x}}}{\Gamma(1-\nu)}I^{\nu}x^{\beta-2\nu-2+}e^{\frac{-a}{x}}=y_{-\nu}(x,\beta,a), \tag{19}$$

$$F_{\nu}^{(0,\nu)}(1,1,2;x)=\frac{e^{x^{2}}}{\Gamma(\nu+1)}D^{\nu}x^{\nu}e^{-x^{2}}=h_{\nu}(x), \tag{20}$$

$$F_{-\nu}^{(0,\nu)}(1,1,2;x)=\frac{e^{x^{2}}}{\Gamma(1-\nu)}I^{\nu}x^{-\nu}e^{-x^{2}}=h_{-\nu}(x). \tag{21}$$

From the properties of the fractional calculus and the definitions (6) and (7), we can easily prove the following lemma:

Lemma 1. *Let* $\nu,\gamma\in(n-1,n), n=1,2,3,\ldots$*;* $a,b,\beta\in\Re$ *and* $k=1,2,3,\ldots$*. Then*

$$D^{\alpha}Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)=Y_{\nu+\alpha}^{(\beta,\gamma)}(a,b,k;x)=D^{\nu}Y_{\alpha}^{(\beta,\gamma)}(a,b,k;x), \tag{22}$$

$$D^{\alpha}Y_{-\nu}^{(\beta,\gamma)}(a,b,k;x)=I^{\nu}Y_{\alpha}^{(\beta,\gamma)}(a,b,k;x), \tag{23}$$

$$D^{\alpha}Y_{-\nu}^{(\beta,-\gamma)}(a,b,k;x)=I^{\nu}Y_{\alpha}^{(\beta,-\gamma)}(a,b,k;x). \tag{24}$$

2. Hypergeometric Series Representations

Taking to account that the generalized hypergeometric function ${}_pF_q$ is defined by [17, p.19, (2)]:

$$ {}_pF_q\left[a_1,\ldots,a_p;b_1,\ldots,b_q;x\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{x^k}{k!}, \tag{25}$$

and $\triangle(k;\lambda)$ abbreviates the array of k parameters $\frac{\lambda}{k},\frac{\lambda+1}{k},\ldots,\frac{\lambda+k-1}{k}, k=1,2,3,\ldots$, we establish the following series representations for the functions $Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)$ and $Y_{-\nu}^{(\beta,-\gamma)}(a,b,k;x)$.

Theorem 1. *Let* $\nu,\gamma\in(n-1,n), n=1,2,3,\ldots$*;* $a,b,\beta\in\Re$ *and* $k=1,2,3,\ldots$*. Then*

$$\begin{aligned} Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)=&\sum_{p}^{n}\binom{n}{p}\frac{\Gamma(\beta+b\gamma+1)\Gamma(\beta+b\gamma-n+1)x^{\beta+b\gamma-\nu}}{\Gamma(p-n+1)\Gamma(\beta+b\gamma-p+1)\Gamma(\beta+b\gamma-\nu+1)}\\ &\times {}_{2k}F_{2k}\left[\triangle(k;1),\triangle(k;\beta+b\gamma-n);\triangle(k;\beta+b\gamma-\nu+1),\triangle(k;p-n+1);-ax^k\right], \end{aligned}\tag{26}$$

$$\begin{aligned} Y_{-\nu}^{(\beta,-\gamma)}(a,b,k;x)=&x^{\nu+\beta-b\gamma}\frac{\Gamma(\beta-b\gamma+1)}{\Gamma(\beta-b\gamma+\nu+1)}\\ &\times {}_kF_k\left[\triangle(k;\beta-b\gamma+1);\triangle(k;\beta-b\gamma+\nu+1);-ax^k\right]. \end{aligned}\tag{27}$$

Proof. From properties of the fractional calculus and the definition of $Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)$, we get

$$\begin{aligned} Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)=&I^{n-\nu}\left\{\sum_{p=0}^{n}\binom{n}{p}\left(D^p x^{\beta+b\gamma}\right)\left(D^{n-p}e^{-ax^k}\right)\right\}\\ =&\sum_{p=0}^{n}\binom{n}{p}\sum_{q=0}^{\infty}\frac{(-a)^q\Gamma(\beta+b\gamma+1)\Gamma(kq+1)}{q!\Gamma(\beta+b\gamma-p+1)\Gamma(kq+p-n+1)}I^{n-\nu}\left(x^{\beta+b\gamma+kq-n}\right). \end{aligned}\tag{28}$$

On putting $n-\nu=\alpha$, we get

$$I^{\alpha}\left(x^{\beta+b\gamma+kq-n}\right)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}s^{\beta+b\gamma+kq-n}ds,$$

which on putting $x-s=xt$, gives us

$$I^{\alpha}\left(x^{\beta+b\gamma+kq-n}\right)=\frac{x^{\alpha+\beta+b\gamma+kq-n}}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta+b\gamma+kq-n}dt.$$

Hence

$$I^{n-\nu}=x^{\beta+b\gamma+kq-\nu}\frac{\Gamma(\beta+b\gamma+kq-n+1)}{\Gamma(\beta+b\gamma+kq-\nu+1)}. \tag{29}$$

Now, substituting from (29) into (28), we get

$$Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)=x^{\beta+b\gamma-\nu}\sum_{p=0}^{n}\binom{n}{p}$$

$$\times \sum_{q=0}^{\infty} \frac{\Gamma(\beta+b\gamma+1)\Gamma(kq+1)\Gamma(\beta+b\gamma+kq-n+1)}{\Gamma(\beta+b\gamma-p+1)\Gamma(kq+p-n+1)\Gamma(\beta+b\gamma+kq-\nu+1)} \frac{(-ax^k)^q}{q!}, \tag{30}$$

which on applying the results (see[17, p.16-17]):

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \tag{31}$$

and

$$(\lambda)_{mn} = m^{mn}\Pi_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n, n=0,1,2,3,\dots. \tag{32}$$

yields the assertion (26). Similarly, we have

$$\begin{aligned} Y_{-\nu}^{(\beta,-\gamma)}(a,b,k;x) =& \sum_{q=0}^{\infty}\frac{(-a)^q}{q!} I^{\nu}\left(x^{\beta-b\gamma+kq}\right) \\ =& \sum_{q=0}^{\infty}\frac{(-a)^q}{q!}\frac{1}{\Gamma(\nu)}\int_0^x (x-s)^{\nu-1}s^{\beta-b\gamma+kq}ds \\ =& x^{\beta-b\gamma+\nu}\sum_{q=0}^{\infty}\frac{(-ax^k)^q}{q!}\frac{1}{\Gamma(\nu)}\int_0^1 t^{\nu-1}(1-t)^{\beta-b\gamma+kq}dt. \end{aligned}$$

Thus

$$Y_{-\nu}^{(\beta,\gamma)}(a,b,k;x) = x^{\beta-b\gamma+\nu}\sum_{q=0}^{\infty}\frac{(-ax^k)^q}{q!}\frac{\Gamma(\beta-b\gamma+kq+1)}{\Gamma(\beta-b\gamma+kq+\nu+1)}, \tag{33}$$

which on using (31) and (32) gives us the assertion (27) and this complete the proof of the Theorem 1. □

In the same manner one can easily prove the following useful result.

Theorem 2. *Let* $\nu,\gamma \in (n-1,n), n=1,2,3,\dots,$ $a,b,\beta \in \Re$ *and* $k=1,2,3,\dots$. *Then*

$$\begin{aligned} Y_{\nu}^{(\beta,\gamma)}(a,b,k;x) =& x^{\beta+b\gamma-\nu}\frac{\Gamma(\beta+b\gamma+1)}{\Gamma(\beta+b\gamma-\nu+1)} \\ & \times {}_kF_k\left[\triangle(k;\beta+b\gamma+1);\triangle(k;\beta+b\gamma-\nu+1);-ax^k\right]. \end{aligned} \tag{34}$$

Proof. We infer to the proof of Theorem 1. □

The following results are an immediate consequence of Theorems 1 and 2, respectively.

Corollary 1. *Let* $\nu,\gamma \in (n-1,n), n=1,2,3,\dots,$ $a,b,\beta \in \Re$ *and* $k=1,2,3,\dots$. *Then*

$$F_{\nu}^{(\beta,\gamma)}(a,b,k;x) = \frac{e^{ax^k}}{\Gamma(\nu+1)}\sum_{p}^{n}\binom{n}{p}\frac{\Gamma(\beta+b\gamma+1)\Gamma(\beta+b\gamma-n+1)x^{b\gamma-\nu}}{\Gamma(p-n+1)\Gamma(\beta+b\gamma-p+1)\Gamma(\beta+b\gamma-\nu+1)}$$

$$\times {}_{2k}F_{2k}\left[\triangle(k;1),\triangle(k;\beta+b\gamma-n);\triangle(k;\beta+b\gamma-\nu+1),\triangle(k;p-n+1);-ax^k\right], \tag{35}$$

$$F_{-\nu}^{(\beta,-\gamma)}(a,b,k;x)=\frac{e^{ax^k}}{\Gamma(1-\nu)}x^{\nu-b\gamma}\frac{\Gamma(\beta-b\gamma+1)}{\Gamma(\beta-b\gamma+\nu+1)} \times {}_kF_k\left[\triangle(k;\beta-b\gamma+1);\triangle(k;\beta-b\gamma+\nu+1);-ax^k\right]. \tag{36}$$

Corollary 2. *Let* $\nu,\gamma\in(n-1,n), n=1,2,3,\ldots,\ a,b,\beta\in\Re$ *and* $k=1,2,3,\ldots$. *Then*

$$F_{\nu}^{(\beta,\gamma)}(a,b,k;x)=\frac{e^{ax^k}x^{b\gamma-\nu}}{\Gamma(\nu+1)}\frac{\Gamma(\beta+b\gamma+1)}{\Gamma(\beta+b\gamma-\nu+1)} \times {}_kF_k\left[\triangle(k;\beta+b\gamma+1);\triangle(k;\beta+b\gamma-\nu+1);-ax^k\right]. \tag{37}$$

3. Recurrence and Fractional Operational Relations

First, we establish the following pure recurrence relations.

Theorem 3. *Let* $\nu,\gamma\in(n-1,n), n=1,2,3,\ldots,\ a,b,\beta\in\Re$ *and* $k=1,2,3,\ldots$. *Then*

$$Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)=(\beta+b\gamma)Y_{\nu-1}^{(\beta-1,\gamma)}(a,b,k;x)-akY_{\nu-1}^{(\beta+k-1,\gamma)}(a,b,k;x), \tag{38}$$

$$Y_{\nu+1}^{(\beta,\gamma)}(a,b,k;x)=DY_{\nu}^{(\beta,\gamma)}(a,b,k;x) =(\beta+b\gamma)Y_{\nu}^{(\beta-1,\gamma)}(a,b,k;x)-akY_{\nu-1}^{(\beta+k-1,\gamma)}(a,b,k;x), \tag{39}$$

$$xY_{\nu}^{(\beta,\gamma)}(a,b,k;x)=Y_{\nu}^{(\beta+1,\gamma)}(a,b,k;x)-\nu Y_{\nu-1}^{(\beta,\gamma)}(a,b,k;x), \tag{40}$$

$$xY_{\nu}^{(\beta,\gamma)}(a,b,k;x)=(\beta+b\gamma-\nu+1)Y_{\nu-1}^{(\beta,\gamma)}(a,b,k;x)-akY_{\nu-1}^{(\beta+k,\gamma)}(a,b,k;x). \tag{41}$$

Proof.

- From (6), we have

$$Y_{\nu}^{(\beta,\gamma)}(a,b,k;x)=D^{\nu-1}\left[(\beta+b\gamma)x^{\beta+b\gamma-1}e^{-ax^k}-akx^{\beta+b\gamma+k-1}e^{-ax^k}\right], =(\beta+b\gamma)D^{\nu-1}\left(x^{\beta+b\gamma-1}e^{-ax^k}\right)-akD^{\nu-1}\left(x^{\beta+b\gamma+k-1}e^{-ax^k}\right),$$

  which on using (6) gives us (38).

- Again, from (6), we have

$$DY_{\nu}^{(\beta,\gamma)}(a,b,k;x)=Y_{\nu+1}^{(\beta,\gamma)}(a,b,k;x)=D^{\nu}\left[(\beta+b\gamma)x^{\beta+b\gamma-1}e^{-ax^k}-akx^{\beta+b\gamma+k-1}e^{-ax^k}\right], =(\beta+b\gamma)D^{\nu}\left(x^{\beta+b\gamma-1}e^{-ax^k}\right)-akD^{\nu}\left(x^{\beta+b\gamma+k-1}e^{-ax^k}\right),$$

  which on using (6) gives us (39).

In view of the definition (6), if we expand the exponential function e^{-ax^k} in power series and use (3), we obtain

$$Y_\nu^{(\beta,\gamma)}(a,b,k;x) = \sum_{q=0}^{\infty} \frac{(-a)^q}{q!} \frac{\Gamma(\beta+b\gamma+kq+1)}{\Gamma(\beta+b\gamma+kq-\nu+1)} x^{\beta+b\gamma-\nu+kq}, \tag{42}$$

from which, we can prove the assertions (40) and (41).

□

Next, according to the formulas (4) to (7) it may of interest to point out that the polynomials $F_\nu^{(\beta,\gamma)}(a,b,k;x)$ and $F_{-\nu}^{(\beta,-\gamma)}(a,b,k;x)$ have the following basic properties.

Theorem 4. *Let* $\nu,\gamma \in (n-1,n), n = 1,2,3,\dots; a,b,\beta \in \Re$ *and* $k = 1,2,3,\dots$. *Then*

$$DF_\nu^{(\beta,\gamma)}(a,b,k;x) =(\nu+1)F_{\nu+1}^{(\beta,\gamma)}(a,b,k;x) + \left(\frac{akx^k-\beta}{x}\right)F_\nu^{(\beta,\gamma)}(a,b,k;x), \tag{43}$$

$$(\nu+1)F_{\nu+1}^{(\beta,\gamma)}(a,b,k;x) = \left(\beta+b\gamma\right)F_\nu^{(\beta-1,\gamma)}(a,b,k;x) - akF_\nu^{(\beta+k-1,\gamma)}(a,b,k;x), \tag{44}$$

$$\nu F_\nu^{(\beta,\gamma)}(a,b,k;x) = \left(\beta+b\gamma\right)F_{\nu-1}^{(\beta-1,\gamma)}(a,b,k;x) - akF_{\nu-1}^{(\beta+k-1,\gamma)}(a,b,k;x), \tag{45}$$

$$(1-\nu)F_{1-\nu}^{(\beta,\gamma)}(a,b,k;x) = \left(\beta+b\gamma\right)F_{-\nu}^{(\beta-1,\gamma)}(a,b,k;x) - akF_{-\nu}^{(\beta+k-1,\gamma)}(a,b,k;x), \tag{46}$$

$$(1-\nu)F_{1-\nu}^{(\beta,\gamma)}(a,b,k;x) =(\beta+b\gamma)F_{-\nu}^{(\beta,\gamma-\frac{1}{b})}(a,b,k;x) - akF_{-\nu}^{(\beta+k-1,\gamma)}(a,b,k;x), \tag{47}$$

$$(1-\nu)F_{1-\nu}^{(\beta,\gamma)}(a,b,k;x) =DF_{-\nu}^{(\beta,\gamma)}(a,b,k;x) + \left(\frac{akx^k-\beta}{x}\right)F_{-\nu}^{(\beta,\gamma)}(a,b,k;x), \tag{48}$$

$$F_\nu^{(\beta,\gamma+1)}(a,b,k;x) =x^bF_\nu^{(\beta+b,\gamma)}(a,b,k;x) = x^{-b}F_\nu^{(\beta-b,\gamma+2)}(a,b,k;x), \tag{49}$$

$$F_\nu^{(\beta,1-\gamma)}(a,b,k;x) =x^bF_\nu^{(\beta+b,-\gamma)}(a,b,k;x) = x^{-b}F_\nu^{(\beta-b,2-\gamma)}(a,b,k;x). \tag{50}$$

Proof.

- From (4), we have

$$DF_\nu^{(\beta,\gamma)}(a,b,k;x) =\frac{-\beta}{x}\left[\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}D^\nu\left(x^{\beta+b\gamma}e^{-ax^k}\right)\right] + akx^{k-1}\left[\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}D^\nu\left(x^{\beta+b\gamma}e^{-ax^k}\right)\right] + (\nu+1)\left[\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+2)}D^{\nu+1}\left(x^{\beta+b\gamma}e^{-ax^k}\right)\right],$$

which gives us (43).

- From (4), we have

$$F_{\nu+1}^{(\beta,\gamma)}(a,b,k;x) = \frac{x^{-\beta}e^{ax^k}}{(\nu+1)\Gamma(\nu+1)}D^\nu\left[(\beta+b\gamma)x^{\beta+b\gamma-1}e^{-ax^k} - akx^{\beta+b\gamma+k-1}e^{-ax^k}\right],$$

which on using (4) gives us (44).

- We have

$$F_{\nu}^{(\beta,\gamma)}(a,b,k;x)=\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}D^{\nu-1}\left[(\beta+b\gamma)x^{\beta+b\gamma-1}e^{-ax^k}-akx^{\beta+b\gamma+k-1}e^{-ax^k}\right],$$

which on using (4) gives us (45).

- We have

$$\begin{aligned}F_{1-\nu}^{(\beta,\gamma)}(a,b,k;x)=&\frac{x^{-\beta}e^{ax^k}}{\Gamma(2-\nu)}D^{1-\nu}\left(x^{\beta+b\gamma}e^{-ax^k}\right)\\=&\frac{x^{-\beta}e^{ax^k}}{\Gamma(2-\nu)}I^{\nu}\left[(\beta+b\gamma)x^{\beta+b\gamma-1}e^{-ax^k}-akx^{\beta+b\gamma+k-1}e^{-ax^k}\right],\end{aligned}$$

which gives us (46).

- From (31) and the fact that

$$F_{-\nu}^{(\beta-1,\gamma)}(a,b,k;x)=F_{-\nu}^{(\beta,\gamma-\frac{1}{b})}(a,b,k;x),$$

we get (47).

The proofs of the assertions (48) to (50) are similar to that of (43) to (47) , then we skip the details. □

Theorem 5. *Let* $\nu,\gamma\in(n-1,n), n=2,3,\ldots, a,b,\beta\in\Re$ *and* $k=1,2,3,\ldots$. *Then*

$$\begin{aligned}&x^{\beta}e^{-ax^k}\sum_{s=0}^{\infty}\binom{\nu}{s}\Gamma(\nu-s+1)F_{\nu-s}^{(\beta,\gamma)}(a,b,k;x)\left(D^s f(x)\right)\\&=I^{n-\nu}\left\{x^{\beta+b\gamma}e^{-ax^k}\left[D+\frac{\beta+b\gamma-akx^k}{x}\right]^n f(x)\right\},\end{aligned}\tag{51}$$

$$\begin{aligned}&x^{\beta}e^{-ax^k}\sum_{s=0}^{\infty}\binom{\nu}{s}\Gamma(\nu-s+1)F_{\nu-s}^{(\beta,\gamma)}(a,b,k;x)\left(D^s f(x)\right)\\&=I^{n-\nu}\left\{\Omega_{\beta,\gamma,b}^{n,m}(x)\left[D-akx^{k-1}\right]^m f(x)\right\},\end{aligned}\tag{52}$$

where

$$\Omega_{\beta,\gamma,b}^{n,m}(x)=\sum_{m=0}^{n}\binom{n}{n-m}\binom{\beta+b\gamma}{n-m}(n-m)!e^{ax^k}x^{\beta+b\gamma+m-n},$$

and

$$\begin{aligned}&x^{\beta}e^{-ax^k}\sum_{s=0}^{\infty}\binom{\nu}{s}\Gamma(\nu-s+1)F_{\nu-s}^{(\beta,\gamma)}(a,b,k;x)\left(D^s f(x)\right)\\&=I^{n-\nu}\left\{x^{\beta+b\gamma-n}e^{-ax^k}\prod_{j=0}^{n-1}\left(xD+\beta+b\gamma-j-akx^k\right)f(x)\right\}.\end{aligned}\tag{53}$$

Proof. By the generalized Leibnitz rule for fractional derivative [8, p.90, (4.3)], we find

$$D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)=\sum_{s=0}^{\infty}\binom{\nu}{s}D^{\nu-s}\left(x^{\beta+b\gamma}e^{-ax^k}\right)D^s\left(f(x)\right),$$

which in view of (4), gives us

$$D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}\right)=\sum_{s=0}^{\infty}\binom{\nu}{s}\Gamma(\nu-s+1)x^{\beta}e^{-ax^k}F_{\nu-s}^{(\beta,\gamma)}(a,b,k;x)D^s\left(f(x)\right). \tag{54}$$

On other hand, we obtain

$$D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)=I^{n-\nu}\left[D^n\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\right],$$

which on using the shift relation [16]

$$D^n\left[e^{\phi(x)}f(x)\right)=e^{\phi(x)}\left[D+D\phi(x)\right]f(x),$$

gives us

$$D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)=I^{n-\nu}\left\{x^{\beta+b\gamma}e^{-ax^x}\left[D+\frac{\beta+b\gamma-akx^x}{x}\right]^n f(x)\right\}. \tag{55}$$

Hence from (54) and (55), we get (51). Similarly, since

$$\begin{aligned}D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)&=I^{n-\nu}\left[D^n\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\right]\\ &=I^{\nu-n}\left\{\sum_{s=0}^{n}\binom{n}{s}D^s\left(x^{\beta+b\gamma}\right)D^{n-s}\left(e^{-ax^k}f(x)\right)\right\},\end{aligned}$$

we find that

$$\begin{aligned}&D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\\ &=I^{n-\nu}\left\{\sum_{m=0}^{n}\binom{n}{n-m}\binom{\beta+b\gamma}{n-m}(n-m)!e^{ax^k}x^{\beta+b\gamma+m-n}\left[D-akx^{k-1}\right]^m f(x)\right\}.\end{aligned} \tag{56}$$

Hence, from (54) and (56), we get (52). Next,we have

$$D^{\nu}\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)=I^{n-\nu}\left[D^{n-1}\left(D\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\right)\right]. \tag{57}$$

Now

$$D^{n-1}\left(D\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\right)=D^{n-1}\left(x^{\beta+b\gamma}e^{-ax^k}\left(xD+\beta+b\gamma-akx^k\right)f(x)\right).$$

Let $\left(xD+\beta+b\gamma-akx^k\right)f(x)=f_1(x)$, then

$$D^{n-1}\left(x^{\beta+b\gamma}e^{-ax^k}f_1(x)\right)=D^{n-2}\left(x^{\beta+b\gamma}e^{-ax^k}\left(xD+\beta+b\gamma-1-akx^k\right)f_1(x)\right)$$

$$=D^{n-2}\left(x^{\beta+b\gamma}e^{-ax^k}\left(xD+\beta+b\gamma-1-akx^k\right)\left(xD+\beta+b\gamma-akx^k\right)f(x)\right),$$

which on repetition of the process gives

$$D^{n-1}\left(D\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\right)=\left(x^{\beta+b\gamma-n}e^{-ax^k}\prod_{j=1}^{n}\left(xD+\beta+b\gamma+j-n+1-akx^k\right)f(x)\right), \tag{58}$$

where the product has been taken in the operative sense. Since the operators involved are commutative (60) can be written in the form

$$D^{n-1}\left(D\left(x^{\beta+b\gamma}e^{-ax^k}f(x)\right)\right)=\left(x^{\beta+b\gamma-n}e^{-ax^k}\prod_{j=0}^{n-1}\left(xD+\beta+b\gamma-j-akx^k\right)f(x)\right)$$

which with the help of (54) yields (53). □

When $f(x)=1$, assertions (51) to (53) reduce to the interesting fractional relations

$$F_{\nu}^{(\beta,\gamma)}(a,b,k;x)=\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\beta+b\gamma}e^{-ax^k}\left[D+\frac{\beta+b\gamma-akx^k}{x}\right]^n\right\}. \tag{59}$$

$$F_{\nu}^{(\beta,\gamma)}(a,b,k;x)=\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}I^{n-\nu}\left\{\Omega^{n,m}_{\beta,\gamma,b}(x)\left[D-akx^{k-1}\right]^m\right\}, \tag{60}$$

and

$$F_{\nu}^{(\beta,\gamma)}(a,b,k;x)=\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\beta+b\gamma-n}e^{-ax^k}\prod_{j=0}^{n-1}\left(xD+\beta+b\gamma-j-akx^k\right)\right\}, \tag{61}$$

respectively.

These are three fractional formulas which happens to give many new fractional representations for the special functions mentioned in the first section of this work as particular cases. For example

$$L_{\nu}^{\beta}(\gamma,a;x)=\frac{x^{-\beta}e^{ax}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\beta+\gamma}e^{-ax}\left[D+\frac{\beta+\gamma-ax}{x}\right]^n\right\}, \tag{62}$$

$$L_{\nu}^{\beta}(\gamma,a;x)=\frac{x^{-\beta}e^{ax}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\beta+b\gamma-n}e^{-ax}\prod_{j=0}^{n-1}\left(xD+\beta+b\gamma-j-ax\right)\right\}, \tag{63}$$

$$H_{\nu}^{(k)}(x,\beta,a)=\frac{(-1)^{\nu}x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\beta}e^{-ax^k}\left[D+\frac{\beta-akx^k}{x}\right]^n\right\}, \tag{64}$$

$$L_{\nu}^{\beta}(x,k,a)=\frac{x^{-\beta}e^{ax^k}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\beta+\nu}e^{-ax^k}\left[D+\frac{\beta+\nu-akx^k}{x}\right]^n\right\}, \tag{65}$$

and

$$h_{\nu}(x)=\frac{e^{x^2}}{\Gamma(\nu+1)}I^{n-\nu}\left\{x^{\nu}e^{-x^2}\left[D+\frac{\nu-a2x^2}{x}\right]^n\right\}. \tag{66}$$

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