EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 2, 2016, 231-239
ISSN 1307-5543 – www.ejpam.com

(-1)-Weak Amenability of Unitized Banach Algebras

S. Alireza Hosseinioun[1], Arezou Valadkhani [2,*]

[1] *University of Arkansas, Department of Mathematical sciences, Fayetteville, AR 72703, USA*
[2] *Department of Mathematics, Shahid Beheshti University, Tehran, Iran*

Abstract. For a Banach algebra A, its second dual A'' is (-1)-weakly amenable if A' is a Banach A''-bimodule and the first cohomology group of A'' with coefficients in A' is zero i.e. $H^1(A'',A') = \{0\}$. We first show that under certain conditions A' is a Banach A''-bimodule. We then consider the relationships between (-1)-weak amenability of A and $A^{\#}$, where $A^{\#}$ is the unitization of A.

2010 Mathematics Subject Classifications: 46H25
Key Words and Phrases: Banach algebra, (-1)-Weak amenability, Arens products Unitization.

1. Introduction

Let A be a Banach algebra and E be a Banach A-bimodule, then a bounded derivation from A into E is a bounded linear mapping $D : A \longrightarrow E$ such that $D(a \cdot b) = Da \cdot b + a \cdot Db$, for each $a, b \in A$. For example let $x \in X$ and define $\delta_x : A \longrightarrow E$ by $\delta_x a = a \cdot x - x \cdot a$, then δ_x is a bounded derivation which is called an inner derivation. Let $Z^1(A,E)$ be the space of all bounded derivations from A into E, $N^1(A,E)$ be the space of all inner derivations from A into E and the first cohomology group of A with coefficients in E be the quotient space $H^1(A,X) = Z^1(A,X)/N^1(A,X)$.

A Banach algebra A is amenable if $H^1(A,E') = \{0\}$ for each Banach A-bimodule E, this concept was introduced by B. E. Johnson in [8].

The notion of weak amenability for commutative Banach algebras was introduced by W. G. Bade, P. C. Curtis and H. G. Dales in [2]. Later Johnson defined weak amenability for arbitrary Banach algebras in [9], in fact a Banach algebra A is weakly amenable if $H^1(A,A') = \{0\}$.

In [10], A. Medghalchi and T. Yazdanpanah introduced the notion of (-1)-weak amenability. A Banach algebra A is (-1)-weakly amenable if A' is a Banach A''-bimodule and $H^1(A'',A') = \{0\}$.

There are some examples of non (-1)-weakly amenable Banach algebras. For instance, in [7] we proved that $(Lip_\alpha K)''$ for $\alpha \in (0,1)$ and infinite compact metric space K is not

*Corresponding author.

Email addresses: ahosseinioun@yahoo.com (S. Hosseinioun), arezou.valadkhani@yahoo.com (A. Valadkhani)

 © 2016 EJPAM All rights reserved.

(-1)-weakly amenable. The space l^p for $1 < p < \infty$ is reflexive and weakly amenable, so is (-1)-weakly amenable which is not amenable since it doesn't factor. Furthermore, the second dual of a C^*-algebra is (-1)-weakly amenable and in the case A'' is a non-nuclear C^*-algebra, we can conclude that A'' is (-1)-weakly amenable which is not amenable. Therefore, the notion of (-1)-weak amenability is different from amenability. For more examples see [7] and [9].

Although there are some main theorems and examples which may suggest that the notion of (-1)-weak amenability is close to the notion of weak amenability, there are some examples which prove that these two notions are different, see [8].

Let A be a Banach algebra and A'' be its second dual, for each $a, b \in A$, $f \in A'$ and $F, G \in A''$ we define $f \cdot a$, $a \cdot f$ and $F \cdot f$, $f \cdot F \in A'$ by

$$f \cdot a(b) = f(a \cdot b), \quad a \cdot f(b) = f(b \cdot a)$$
$$F \cdot f(a) = F(f \cdot a), \quad f \cdot F(a) = F(a \cdot f).$$

Now we define $F \cdot G$, $F \times G \in A''$ as follows

$$F \cdot G(f) = F(G \cdot f), \quad F \times G(f) = G(f \cdot F).$$

Then A'' is a Banach algebra with respect to either of the products $\cdot$ and $\times$. These products are called the first and the second Arens products on A'', respectively. A is called Arens regular if $F \cdot G = F \times G$, for all $F, G \in A''$.

Let E be a Banach A-bimodule, then the iterated conjugates of E, denoted by $E', E'', E''', \ldots$ are Banach A-bimodules, and the map $\rho : E''' \longrightarrow E'$ with $\rho(\Gamma) = \Gamma \mid_{\hat{A}}$ is an A-bimodule homomorphism which is called natural projection.

All concepts and definitions which are not defined in this paper may be found in [4].

2. When A' is a Banach A''-bimodule?

In the notion of (-1)-weak amenability, a necessary condition is that "A' is a Banach A''-bimodule". Throughout this paper, we shall consider the second dual A'' with the first Arens product. For the relations between (-1)-weak amenability of $(A'', \cdot)$ and $(A'', \times)$, see [9].

Theorem 1. *Let A be a Banach algebra. Then in each of the following cases, A' is a Banach A''-bimodule:*

(1) A is Arens regular;

(2) $\hat{A}$ is a left ideal in A'';

(3) $\hat{A}$ is a right ideal in A'' and $A'' = A'' \cdot A$.

Proof. (1) and (2) are proved in [6].

(3) Let $\hat{A}$ be a right ideal in A'' and $A'' \cdot A = A''$. Let $a \in A$, $F, G \in A''$ and $f \in A'$, then there exist $F_1 \in A''$ and $b, c \in A$ such that $F = F_1 \cdot b$ and $b \cdot G = \hat{c}$, so we have

$$(f \cdot F) \cdot G(a) = \big(f \cdot (F_1 \cdot b)\big) \cdot G(a) = (f \cdot F_1) \cdot (b \cdot G)(a) = (f \cdot F_1) \cdot \hat{c}(a)$$

$$=\hat{c}(a\cdot(f\cdot F_1))=f\cdot F_1(c\cdot a)=F_1\cdot c(a\cdot f)$$
$$=\big(F_1\cdot(b\cdot G)\big)(a\cdot f)=F\cdot G(a\cdot f)=f\cdot(F\cdot G)(a)$$

So A' is a right A''-module.

On the other hand, there exists $d\in A$ such that $a\cdot F=\hat{d}$ and we have

$$\big(F\cdot f\big)\cdot G\big(a\big)=G(a\cdot(F\cdot f))=G((a\cdot F)\cdot f)=G(d\cdot f)$$
$$=\hat{d}(f\cdot G)=(a\cdot F)(f\cdot G)=F\cdot(f\cdot G)(a).$$

Therefore A' is a Banach A''-bimodule. □

Remark 1. *Dales, Rodrigues-palacios and Velasco in [5] proved that for a Banach algebra A, A' is an A''-submodule of A''' if and only if A is Arens regular. So under the condition "A' is a Banach A''-bimodule" we can consider a larger class of Banach algebras.*

Example 1. *In each of the following cases by using Theorem 1, A' is a Banach A''-bimodule.*

(1) Let A be a C^*-algebra, then A is Arens regular and A' is a Banach A''-bimodule [3].

(2) Let $A=l^1(\mathbb{N})$ with product $f\cdot g=f(1)g$. Then A is a Banach algebra with l^1-norm and A is a left ideal in A''. So A' is a Banach A''-bimodule [6].

(3) Let S be an infinite set with product $s\cdot t=t$ for all $s,t\in S$. Then $l^1(S)$ is a left ideal in $(l^1(S))''$ and so $(l^1(S))'$ is a Banach $(l^1(S))''$ -bimodule. But $l^1(S)$ is not a right ideal in $(l^1(S))''$ [6] (so the third condition in Theorem 1 is not a necessary condition).

(4) We know that for each semisimple annihilator Banach algebra A, A is an ideal in A'' [13]. So A' is a Banach A''-bimodule and we have the following assertion:

- *Let G be an infinite compact group, then $L^1(G)$ is not Arens regular but $L^1(G)$ is an ideal in $\big(L^1(G)\big)''$. So $\big(L^1(G)\big)'$ is a Banach $\big(L^1(G)\big)''$-bimodule, whereas $L^1(G)$ is not Arens regular (so the first condition in Theorem 1 is not a necessary condition).*
- *Let G be an finite group then $M(G)$ is an ideal in $M(G)''$. So $M(G)'$ is a Banach $M(G)''$-bimodule [11] and [12].*

(5) Let X be a reflexive Banach space and $KL(X)$ be the algebra of compact operators on X. Then $KL(X)$ is an ideal in $KL(X)''$ and so $(KL(X))'$ is a Banach $(KL(X))''$-bimodule. Note that in the case X has not approximation property $KL(X)$ is not an annihilator algebra [1].

Now we give an example of a Banach algebra A for which A' is not a Banach A''-bimodule.

Example 2. *Consider $A=(l^1,*)$ for $n,m\in\mathbb{N}$. Set $a_n=\delta_{2^{2n}}$, $b_m=\delta_{2^{2m+1}-1}$ and $x=\delta_1$ that $(a_n)_n$, $(b_m)_m$ are bounded sequences in l^1. There are $F,G\in A''$ for which $F=w^*-\lim_n\hat{a}_n$, $G=w^*-\lim_m\hat{b}_m$. Now, let*

$$S=\big\{2^{2n}+2^{2m+1}:n,m\in\mathbb{N},n<m\big\}$$

and set $\lambda = \chi_S$, where χ_S is characteristic function on S. So $(b_m * x) * a_n = \delta_{2^{2n}+2^{2m+1}}$ and we have

$$\lim_{n\to\infty} \lambda(b_m * x * a_n) = 0, \quad \lim_{m\to\infty} \lambda(b_m * x * a_n) = 1.$$

So,

$$\begin{aligned}(F \cdot \lambda) \cdot G(x) =& G(x \cdot (F \cdot \lambda)) = \lim_m F(\lambda \cdot (b_m * x)) \\ =& \lim_m \lim_n \lambda(b_m * x * a_n) = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}F \cdot (\lambda \cdot G)(x) =& F((\lambda \cdot G) \cdot x) = \lim_n \lambda \cdot G(x * a_n) \\ =& \lim_n \lim_m \lambda(b_m * x * a_n) = 1.\end{aligned}$$

Therefore A' is not a Banach A''-bimodule and so A'' is not (-1)-weakly amenable.

Question. Is there any Banach algebra A such that A is amenable but A' is not A''-bimodule?

3. Unitization

Let A has not unit element and $A^{\#} = A \oplus \mathbb{C}e$ be the unitization of A.

For $e \in A^{\#}$, by Hahn-Banach Theorem there exists $e' \in A^{\#\prime}$ such that $e'(e) = 1$ and $e'(a) = 0$ for each $a \in A$, and we can extend $\lambda \in A'$ to an element of $A^{\#\prime}$ with $\lambda(e) = 1$. So $A^{\#\prime} = \mathbb{C}e' \oplus_\infty A'$ and $\|\alpha e' + \lambda\| = \max\{|\alpha|, \|\lambda\|\}$ for $\alpha \in \mathbb{C}$ and $\lambda \in A'$. Moreover, $A^{\#\prime}$ is a Banach space and is a Banach $A^{\#}$-bimodule by module multiplications

$$\begin{aligned}(\alpha e + a) \cdot (\gamma e' + \lambda) =& (\alpha\gamma + \lambda(a))e' + \alpha\lambda + a \cdot \lambda \\ (\gamma e' + \lambda) \cdot (\alpha e + a) =& (\alpha\gamma + \lambda(a))e' + \alpha\lambda + \lambda \cdot a\end{aligned}$$

where $\alpha, \gamma \in \mathbb{C}$, $a \in A$ and $\lambda \in A'$.

Let $\hat{e} \in A''$ with $\hat{e}(\lambda) = \lambda(e)$, then $(A^{\#})'' = A'' \oplus \mathbb{C}\hat{e}$. For more details see [4].

Lemma 1. *Let A be an Arens regular Banach algebra. Then A' and $A^{\#\prime}$ are Banach $A^{\#\prime\prime}$-bimodule.*

Proof. The proof is straightforward. □

Theorem 2. *Let A be an Arens regular Banach algebra and $\overline{A''^2} = A''$. If A'' is (-1)-weakly amenable, Then $A''^{\#}$ is (-1)-weakly amenable.*

Proof. Suppose that A has not unit element and $A^{\#} = A \oplus \mathbb{C}e$ be its unitization. By the previous Lemma, A' is a Banach A''-bimodule and $A^{\#\prime}$ is a Banach $A^{\#\prime\prime}$-bimodule.

Since $A''^{\#}$ is a unital Banach algebra and $A'^{\#}$ is a unital $A''^{\#}$-bimodule and A'' is a maximal ideal of codimension one in $A''^{\#}$, by 2.8.23 (iii) in [4] we can conclude that $H^1(A^{\#\prime\prime}, A^{\#\prime}) = H^1(A'', A^{\#\prime})$. Let $D : A'' \longrightarrow A^{\#\prime}$ be a bounded derivation. We define

$\overline{D} : A'' \longrightarrow A'$ by $\overline{D}(F) = DF|_{A\times\{0\}}$, for each $F \in A''$. Then $\overline{D}$ is a bounded derivation (Note that $\overline{D}F(a) = DF(a+0e)$. So $\overline{D}F \in A'$). By (-1)-weakly amenability of A'', there exists $f_0 \in A'$ such that for each $F \in A''$, $\overline{D}F = \delta_{f_0}F$. Let $D_1 = D - \overline{D}$, then D_1 is a bounded derivation. Now we show that $D_1 = 0$ (Consider DF as an element in A' with its extension).

For $F, G \in A''$, there is $(b_j)_j$ in A with $\hat{b}_j \xrightarrow{w^*} G$, then

$$e' \cdot G(a + \alpha e) = G((a + \alpha e)(0 + e')) = G(\alpha e') = \lim_j \hat{b}_j(\alpha e') = \lim_j \alpha e'(b_j) = 0.$$

On the other hand, since $D : A'' \longrightarrow A^{\#\prime}$ and $A^{\#\prime} = A' \oplus \mathbb{C}e'$, for each $F \in A''$ there are unique elements, $\lambda_F \in A'$ and $\alpha_F \in \mathbb{C}$ such that $DF = \lambda_F + \alpha_F e'$. Since $\overline{D}F = DF|_{A\times\{0\}}$, and $\overline{D}F = \lambda_F$ then $D_1F = \alpha_F e'$. So we have

$$D_1(F \cdot G) = D_1F \cdot G + F \cdot D_1G = \alpha_F(e' \cdot G) + \alpha_G(F \cdot e') = 0.$$

Since D_1 is bounded then $D_1|_{\overline{A''^2}} = 0$. So by the essentiality of A'', $D_1 = 0$, so $D = \delta_{f_0}$ where $f_0 = f_0 + 0e' \in A' \oplus \mathbb{C}e' = A^{\#\prime}$. Therefore $H^1(A^{\#\prime\prime}, A^{\#\prime}) = H^1(A'', A^{\#\prime}) = \{0\}$. □

Theorem 3. *Let A be an Arens regular Banach algebra, $A^{\#\prime\prime}$ be (-1)-weakly amenable and $H^2(A'', \mathbb{C}_0) = (0)$. Then A'' is (-1)-weakly amenable.*

Proof. We may suppose that A has not unit element and $A^{\#} = A \oplus \mathbb{C}_0 e$. Then

$$\Sigma : 0 \longrightarrow A \longrightarrow A^{\#} \longrightarrow \mathbb{C}_0 \longrightarrow 0$$

is an admissible short exact sequence and hence so is its dual,

$$\Sigma' : 0 \longrightarrow \mathbb{C}_0 \longrightarrow A^{\#\prime} \longrightarrow A' \longrightarrow 0.$$

Using 2.8.25 in [4] we have exact sequence

$$S : \ldots \longrightarrow H^1(A'', \mathbb{C}_0) \longrightarrow H^1(A'', A^{\#\prime}) \longrightarrow H^1(A'', A') \longrightarrow H^2(A'', \mathbb{C}_0) \longrightarrow \ldots,$$

from 2.8.23 (iii) in [4], $H^1(A'', A^{\#\prime}) = H^1(A''^{\#}, A^{\#\prime}) = (0)$, since $A''^{\#}$ is (-1)-weakly amenable. Moreover $H^2(A'', \mathbb{C}_0) = (0)$, so in the exact sequence S, $H^1(A'', A^{\#\prime}) = H^2(A'', \mathbb{C}_0) = (0)$ then $H^1(A'', A') = (0)$. □

Remark 2. *The condition $H^2(A'', \mathbb{C}_0) = (0)$ in Theorem 3 is not trivial. To this end, let $B = \left\{f \in A(\overline{\mathbb{D}}) : f(0) = f'(0) = 0\right\}$ then B is a closed subalgebra of the disc algebra $A(\overline{\mathbb{D}})$. Consider $\mathbb{C}_0$ as the annihilator B-module i.e. B acts trivially on the left and right on $\mathbb{C}_0$. Now we define $\mu : B \times B \longrightarrow \mathbb{C}_0$, by $(f, g) \mapsto f'''(0)g'''(0)$. Then μ is a continuous functional for which $\mu(f, g) = \mu(g, f)$. If $H^2(B, \mathbb{C}_0) = \{0\}$, then for some $\lambda \in B'$ we have $\mu = \delta^1(\lambda)$ where*

$$\delta^1(\lambda)(f, g) = f \cdot \lambda g - \lambda(f \cdot g) + \lambda f \cdot g. \tag{1}$$

If for $z \in \overline{\mathbb{D}}$ we define $f, g, h \in B$ by $f(z) = z^2$, $g(z) = z^4$ and $h(z) = z^3$ then $f'''(z) = 0$, $g'''(z) = 24z$, $h'''(z) = 6$ and we have $\mu(f, g) = f'''(0)g'''(0) = 0$ and $\mu(h, h) = 36$. Since $\mathbb{C}$ is an annihilator B-module then $f \cdot \lambda g = \lambda f \cdot g = 0$. On the other hand by (1) we have

$$\mu(f,g) = \delta^1(\lambda)(f,g) = -\lambda(f \cdot g),$$
$$\mu(h,h) = \delta^1(\lambda)(h,h) = -\lambda(h \cdot h).$$

So $\lambda(f \cdot g) = 0$ and $\lambda(h \cdot h) = -36$. But $f \cdot g(z) = z^2 \cdot z^4 = z^6 = h \cdot h(z)$, which is a contradiction. So $H^2(B, \mathbb{C}_0) \neq \{0\}$.

Now consider $\mu'' : B'' \times B'' \longrightarrow \mathbb{C}_0$, and suppose that for some $\Lambda \in B'''$ we have $\mu'' = \delta^1(\Lambda)$ and so $\Lambda(\widehat{f \cdot g}) = 0$ and $\Lambda(h \hat{\cdot} h) = -36$, but $f \cdot g = h \cdot h$. So there is no $\Lambda \in B'''$ with $\mu'' = \delta^1(\Lambda)$. Therefore $H^2(B'', \mathbb{C}_0) \neq \{0\}$.

The ext example shows that the converse of Theorem 1 is not true.

Example 3. *By 4.1.42 in [4], $H^2(l^p, \mathbb{C}_0) \neq \{0\}$ for $p > 1$ and l^p is weakly amenable and reflexive. So l^p is (-1)-weakly amenable.*

Since l^p has an approximate identity, then $l^p = \overline{(l^p)^2}$ and by Theorem 2, $(l^p)^\#$ is (-1)-weakly amenable (note that $(l^p)^{\#''} = (l^p)''^\# \simeq l^{p\#}$).

A normed algebra A has π-property if there is a constant $c > 0$ with $|||a|||_\pi \leq c\|a\|$, for $a \in A^2$, where $|||a|||_\pi = \inf\left\{\sum_{j=1}^{\infty} \|a_j\|\|b_j\| : a = \sum_{j=1}^{\infty} a_j b_j\right\}$, for more details see [4]. By 2.8.21 in [4], a Banach algebra with $H^2(A, \mathbb{C}_0) = \{0\}$ has π-property. Now, by using Theorem 3 we have the following corollary.

Corollary 1. *Let A be a Banach algebra for which A'' has π-property. If $A^{\#''}$ is (-1)-weakly amenable, then A'' is (-1)-weakly amenable.*

Theorem 4. *Let A be an Arens regular Banach algebra and $A''^\#$ is (-1)-weakly amenable. If $G(DF) = -F(DG)$, for each $D \in Z^1(A'', A')$ and each $F, G \in A''$. Then A'' is (-1)-weakly amenable.*

Proof. Let $D \in Z^1(A'', A')$. We define

$$D^\# : A'' \longrightarrow A^{\#\prime}$$
$$D^\#(F)(\alpha e + a) := D(F)(a).$$

We prove $D^\#$ is a derivation. Let $F, G \in A''$, then there are nets $(a_i)_i$ and $(b_j)_j$ in A such that $a_i \xrightarrow{w^*} F$ and $b_j \xrightarrow{w^*} G$ and for each $\alpha \in \mathbb{C}, a \in A$ we have

$$D^\# F \cdot G(\alpha e + a) = G((\alpha e + a) \cdot D^\# F) = \lim_j \big((\alpha e + a) \cdot D^\# F\big)(b_j)$$
$$= \lim_j DF(\alpha b_j + b_j a) = \lim_j \alpha \cdot \hat{b}_j(DF) + \lim_j \hat{b}_j(a \cdot DF).$$

So $\big(D^\# F \cdot G\big)\big(\alpha e + a) = \alpha G(DF) + G(a \cdot DF)$ and similarly

$$\big(F \cdot D^\# G\big)(\alpha e + a) = \alpha F(DG) + F(DG \cdot a).$$

Then we have

$$\begin{aligned}\left(D^{\#}F\cdot G+F\cdot D^{\#}G\right)(\alpha e+a)=&\alpha G(DF)+G(a\cdot DF)+\alpha F(DG)+F(DG\cdot a)\\=&\alpha\left(G(DF)+F(DG)\right)+F\cdot DG(a)+DF\cdot G(a)\\=&(F\cdot DG+DF\cdot G)(a)=D^{\#}(F\cdot G)(a+\alpha e).\end{aligned}$$

Therefore $D^{\#}$ is a bounded derivation and there exists $\lambda_1=\lambda_0+\alpha_0 e'\in A^{\#\prime}$ such that $D^{\#}(F)=\delta_{\lambda_1}(F)$, for $F\in A''$ (Note that since $A^{\#\prime}=A'\oplus\mathbb{C}e'$, $\lambda_0\in A'$ and $\alpha_0\in\mathbb{C}$ are unique and $H^1(A'',A^{\#\prime})=H^1(A''^{\#},A^{\#\prime})=(0)$). We show that $D=\delta_{\lambda_0}$. Toward this end, let $F\in A''$ and $a\in A$, we have

$$\begin{aligned}(DF)(a)=&D^{\#}F(a+0e)=\delta_{\lambda_1}(F)(a+0e)=(F\cdot\lambda_1-\lambda_1\cdot F)(a+0e)\\=&F(\lambda_1\cdot(a+0e)-(a+0e)\cdot\lambda_1)\\=&F\left((\lambda_0+\alpha_0 e')(a+0e)-(a+0e)(\lambda_0+\alpha_0 e')\right)\\=&F(\lambda_0\cdot a-a\cdot\lambda_0)=(F\cdot\lambda_0-\lambda_0\cdot F)(a)=\delta_{\lambda_0}(F)(a).\end{aligned}$$

So $D=\delta_{\lambda_0}$. Therefore A'' is (-1)-weakly amenable. □

The following example shows that the condition in Theorem 4, is not trivial.

Example 4. *Let $\mathbb{T}$ be the unit circle and $A=lip_\alpha\mathbb{T}$. Let $(\hat{F}(n))_{n\in\mathbb{Z}}$ and $(\hat{g}(n))_{n\in\mathbb{Z}}$ are the Fourier coefficients of $F\in Lip_\alpha\mathbb{T}$ and $g\in lip_\alpha\mathbb{T}$. We define D as follows*

$$\begin{aligned}D:&A''\to A'\\DF(g)=&\sum_{n=-\infty}^{+\infty}n\hat{g}(n)\hat{F}(n).\end{aligned}$$

So D is a derivation which is not inner.

Since $(lip_\alpha\mathbb{T})''=Lip_\alpha\mathbb{T}$, then for $F,G\in Lip_\alpha\mathbb{T}$ there are $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ in $lip_\alpha\mathbb{T}$ such that $F=w^*-\lim_\alpha\hat{f}_\alpha$ and $G=w^*-\lim_\beta\hat{g}_\beta$. Then we have

$$\begin{aligned}DF_\alpha(g_\beta)=&\sum_{n=-\infty}^{+\infty}n\hat{g}_\beta(n)\hat{f}_\alpha(-n)=\sum_{n=-\infty}^{+\infty}(-n)\hat{g}_\beta(-n)\hat{f}_\alpha(n)\\=&-\sum_{n=-\infty}^{+\infty}n\hat{g}_\beta(-n)\cdot\hat{f}_\alpha(n)=-(Dg_\beta)(f_\alpha).\end{aligned}$$

On the other hand

$$\begin{aligned}\lim_\beta\lim_\alpha Df_\alpha(g_\beta)=\lim_\beta DF(g_\beta)=\lim_\beta\hat{g}_\beta DF=G(DF),\\\lim_\beta\lim_\alpha Dg_\beta(f_\alpha)=\lim_\alpha\lim_\beta Dg_\beta(f_\alpha)=\lim_\alpha DG(f_\alpha)=F(DG).\end{aligned}$$

So $F(DG)=-G(DF)$ where D is a non-inner derivation.

Theorem 5. *Let A be a unital Banach algebra and A'' is commutative and (-1)-weakly amenable. Then $Z^1(A'', E) = (0)$, for each Banach A''-module E.*

Proof. Let E be a Banach left A''-module and define $x \cdot F =: F \cdot x$ for each $F \in A''$ and $x \in E$. Then E is a Banach right A''-module and commutativity of A'' implies that E is a Banach A''-bimodule (of course E is an A-bimodule and E' is an A''-bimodule).

Let e be the unit element in A, and let D be a non-zero derivation in $Z^1(A'', E)$. Then for some $F_0 \in A''$, we have $DF_0 \neq 0$, so there exists $\lambda \in E'$ such that $\lambda(DF_0) = 1$. We define

$$\begin{aligned} R : E \longrightarrow & A' \\ R(x)(a) = & \lambda(\hat{a} \cdot x), \quad (a \in A, x \in X). \end{aligned}$$

R is a bounded linear map. Now $R \circ D : A'' \longrightarrow A'$ is a bounded derivation since

$$\begin{aligned} R \circ D(F \cdot G)(a) = & R(DF \cdot G + F \cdot DG)(a) = \lambda(\hat{a} \cdot (DF \cdot G) + \hat{a} \cdot (F \cdot DG)) \\ = & G \cdot \lambda(\hat{a} \cdot DF) + F \cdot \lambda(\hat{a} \cdot DG). \end{aligned}$$

On the other hand for $G = w^* - \lim_\alpha \hat{b}_\alpha$ and $x \in E$, the net $(\hat{b}_\alpha \cdot x)_\alpha$ is a bounded net in E'', so $\widehat{b_\alpha \cdot x} \xrightarrow{w^*} G \cdot x$, especially $\lambda(G \cdot x) = \lim \lambda(\hat{b}_\alpha \cdot x)$ and we have

$$\begin{aligned} (R(DF) \cdot G)(a) = & \lim_\alpha (R(DF) \cdot a)(b_\alpha) \\ = & \lim_\alpha R(DF)(a \cdot b_\alpha) = \lim_\alpha \lambda(\widehat{a \cdot b}_\alpha \cdot DF) \\ = & \lambda \cdot G(\hat{a} \cdot DF) = G \cdot \lambda(\hat{a} \cdot DF). \end{aligned}$$

Similarly $(F \cdot R(DG))(a) = F \cdot \lambda(a \cdot DG)$.

Therefore $R \circ D$ is a derivation in $Z^1(A'', A')$. Now, since A'' is (-1)-weakly amenable and commutative then $R \circ D = 0$. But $R \circ D(F_0)(e) = R(DF_0)(e) = \lambda(e \cdot DF_0) = 1$, which is a contradiction. So $D = 0$ and we have $Z^1(A'', E) = 0$ □

Now we recall some Theorems which are used in the following corollaries.

Theorem 6. *For a commutative Banach algebra A, if A is weakly amenable, then $Z^1(A, E) = (0)$ for each Banach A-module E.*

Theorem 7. *Let A be a commutative Banach algebra. Then A is weakly amenable if and only if $A^\#$ is weakly amenable.*

See [2] and [4] for proofs of Theorems 6 and 7, respectively.

Corollary 2. *Let A be an Arens regular commutative Banach algebra. Then $A^{\#''}$ is (-1)-weakly amenable if and only if A'' is weakly amenable.*

Proof. Let A'' be weakly amenable then by Theorem 7, $A^\#$ is weakly amenable. Since A is Arens regular then by Lemma 1, $A^{\#'}$ is a Banach $A^{\#''}$-bimodul, so $H^1(A^{\#''}, A^{\#'}) = \{0\}$.

For the converse, let $A^{\#''}$ is (-1)-weakly amenable. Using Theorem 5, $A^{\#''}$ is weakly amenable, so by Theorem 7, A'' is weakly amenable. □

Corollary 3. *Let A be a Banach algebra and A'' be commutative and (-1)-weakly amenable, for which $A'' \cdot A = A''$. Then A'' is (-1)-weakly amenable if and only if $A^{\#''}$ is (-1)-weakly amenable.*

Proof. If A'' is commutative and (-1)-weakly amenable, and also $A'' \cdot A = A''$ then it is proved that $Z1(A'', E) = 0$ for each Banach A''-module E. Now use Theorems 6 and 7. □

References

[1] F. E. Alexander. *Some algebraic properties of $F(X)$ and $K(X)$*, Proceedings of the Edinburgh Mathematical Society, 19, 353–361. 1975.

[2] W. G. Bade, P. C. Curtis, and H. G. Dales. *Amenability and weak amenability for Beurling and Lipschits algebra*, Proceedings of the London Mathematical Society, 3(55), 359–377. 1987.

[3] M. C. F. Berglund. *Ideal C^*-algebras*, Duke Mathematical Journal, 40, 241–257. 1973.

[4] H. G. Dales. *Banach algebra and Automatic continuity*, Oxford University Press, 2000.

[5] H. G. Dales, A. Rodriguez-Palacios, and M. V. Velasco. *The second transpose of a derivation*, Journal of the London Mathematical Society, 2(64), 707–721. 2001.

[6] J. Duncan and S. A. Hosseiniun, *The second dual of banach algeba*. Proceedings of the Royal Society of Edinburgh, Section A, 84, 309–325. 1979.

[7] M. Eshaghi, S. A. R. Hosseinioun, and A. Valadkhani. *On (-1)-weak amenability of Banach algebras*, Mathematical Reports, 15(3), 271–279. 2013.

[8] B. E. Johnson. *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, 127, 96. 1972.

[9] B. E. Johnson. *Weak amenability of group algebras*, Bulletin of the London Mathematical Society, 23, 281–284. 1991.

[10] A. Medghalchi and T. Yazdanpanah. *Problems concerning n-weak amenability of Banach algebras*, Czechoslovak Mathematical Journal, 55(4), 863–876. 2005.

[11] S. Watanabe. *A Banach algebra which is an ideal in the second dual space*, Science Reports of Niigata University, Series A, 11, 95–101. 1974.

[12] S. Watanabe. *A Banach algebra which is an ideal in the second dual space II*, Science Reports of Niigata University, Series A, 13, 43–48. 1976.

[13] P. K. Wong. *On the Arens product and certain Banach algebras*, Transactions of the American Mathematical Society, 37, 111–113. 1973.