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# On the structure of regular $\widetilde{\mathcal{H}}$-cryptogroups

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**Abstract.** We introduce the concepts of Green $\sim$-relations on $\widetilde{\mathcal{H}}$-abundant semigroups. By using the generalized strong semilattice of semigroups, we show that an $\widetilde{\mathcal{H}}$-cryptogroup is a regular $\widetilde{\mathcal{H}}$-cryptogroup if and only if it is an $\widetilde{\mathcal{H}}G$-strong semilattice of completely $\widetilde{\mathcal{J}}$-simple semigroups. This result not only extends a known result of Petrich from the class of completely regular semigroups to the class of semiabundant semigroups but also generalizes a well known result of Fountain on superabundant semigroups from the class of abundant semigroups to the class of semiabundant semigroups.

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**Key words**: The Green $\sim$-relations; Homomorphisms of $\widetilde{\mathcal{H}}$-abundant semigroups; $\widetilde{\mathcal{H}}$-cryptogroups.

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## 1. Introduction

It was proved by Clifford [1] that a regular semigroup is a union of groups if and only if it is a semilattice of completely simple semigroups. It is also known that if the set of all idempotents of a completely regular semigroup $S$ is the center of $S$, then $S$ can be expressed by a strong semilattice of groups (see [1]). Thus, we usually regard the completely regular semigroups as generalized groups. Moreover, by Petrich and Reilly, we call a completely regular semigroup $S$ a normal cryptogroup if the Green relation $\mathcal{H}$ on $S$ is a normal band congruence on $S$. In particular, a completely regular semigroup $S$ is a normal cryptogroup if and only if $S$ can be expressed by a strong semilattice of completely simple semigroups (see [12] and [13]). This result was further generalized by Fountain by proving that an abundant semigroup $S$ is a superabundant semigroup if and only if $S$ is a semilattice of completely $\mathcal{J}^*$ -simple semigroups [4]. The structure of superabundant semigroups whose set of idempotents forms a subsemigroup have been recently extensively investigated by Ren and Shum in [15] and [16].

The Green $*$-relations on a semigroup $S$ were first defined by Pastijn [11] which can be regarded as the Green relations in some oversemigroups of $S$. These relations were formulated by

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Fountain [4] as follows:

$$\begin{aligned}\mathcal{L}^* &= \{(a,b) \in S \times S : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\},\\ \mathcal{R}^* &= \{(a,b) \in S \times S : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb\},\\ \mathcal{H}^* &= \mathcal{L}^* \cap \mathcal{R}^*,\ \mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.\end{aligned}$$

Later on, El-Qallali further generalized the Green $*$-relations to Green $\sim$-relations [3] as follows:

$$\begin{aligned}\widetilde{\mathcal{L}} &= \{(a,b) \in S \times S : (\forall e \in E(S)) ae = a \Leftrightarrow be = b\},\\ \widetilde{\mathcal{R}} &= \{(a,b) \in S \times S : (\forall e \in E(S)) ea = a \Leftrightarrow eb = b\},\\ \widetilde{\mathcal{H}} &= \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{R}},\ \widetilde{\mathcal{D}} = \widetilde{\mathcal{L}} \vee \widetilde{\mathcal{R}}.\end{aligned}$$

We can easily see that $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ are equivalent relations on $S$, however, the $\widetilde{\mathcal{L}}$ relation is not necessary to be right compatible with the semigroup multiplication and the $\widetilde{\mathcal{R}}$ relation is not necessary to be left compatible with the semigroup multiplication. We now denote the $\widetilde{\mathcal{L}}$ -class containing the element $a$ of the semigroup $S$ by $\widetilde{L}_a$ and we observe that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$. Among the usual Green relations or the above relations, $\mathcal{L}$- or the generalized $\mathcal{L}$-relations are duals of the corresponding $\mathcal{R}$ -relations or generalized $\mathcal{R}$-relations. In what follows, we only discuss the properties which are related to the $\mathcal{L}$- relation and the generalized $\mathcal{L}$-relation, respectively. One can easily see that there is at most one idempotent of the semigroup $S$ in each $\widetilde{\mathcal{H}}$ -class. If $e \in \widetilde{\mathcal{H}}_a \cap E(S)$, for some $a \in S$, then we simply denote the idempotent $e$ by $x^0$, for any $x \in \widetilde{\mathcal{H}}_a$. Clearly, for any $x \in \widetilde{\mathcal{H}}_a$ with $a \in S$, we have $x = xx^0 = x^0x$.

If a semigroup $S$ is regular, then every $\mathcal{L}$-class of $S$ contains at least one idempotent, and so does every $\mathcal{R}$-class of $S$. If $S$ is a completely regular semigroup, then every $\mathcal{H}$-class of $S$ contains an idempotent. According to Fountain [4], a semigroup is *abundant* if every $\mathcal{L}^*$- and $\mathcal{R}^*$-class of $S$ contains some idempotents. In other words, the term “abundant” means that the semigroup has plenty of idempotents. Clearly, we have $\mathcal{L}^* = \mathcal{L}$ on the set of all regular elements of a semigroup. Thus, regular semigroups are obviously special abundant semigroups. Thus, Fountain called such semigroup *superabundant* [4] if its every $\mathcal{H}^*$-classes contains an idempotent. Obviously, completely regular semigroups are special superabundant semigroups. Following El-Qallali [3], we call a semigroup $S$ a *semiabundant* semigroup if every $\widetilde{\mathcal{L}}$-class and every $\widetilde{\mathcal{R}}$-class of $S$ contain at least one idempotent. A semigroup $S$ is called *$\widetilde{\mathcal{H}}$-abundant* if every $\widetilde{\mathcal{H}}$-class contains an idempotent of $S$. Clearly, the $\widetilde{\mathcal{H}}$-abundant semigroups are generalizations of superabundant semigroups in the class of semiabundant semigroups. One can easily see that $\widetilde{\mathcal{L}} = \mathcal{L}$ on the set of regular elements in any $\widetilde{\mathcal{H}}$-abundant semigroup.

Throughout this paper, we call a band $B$ a *regular band (right quasi normal band)* if $B$ satisfies the identity $axya = axaya(xya = xaya)$. According to Petrich and Reilly [12], a completely regular semigroup $S$ was called a *regular cryptogroup* if the Green relation $\mathcal{H}$ on $S$ is a regular band congruence on $S$. The structure of regular cryptogroup was investigated by Kong-Shum in [8] and [9]. In the class of abundant semigroups, Guo and Shum [5] called an abundant semigroup whose set of idempotents forms a regular band a *cyber* group. The semilattice structure of regular cyber groups have been recently investigated in [9].

Naturally, one would ask : can we establish an analogous result of superabundant semigroups [4] in the class of semiabundant semigroups or an analogous result of cryptogroups [12] in the class of $\widetilde{\mathcal{H}}$ -abundant semigroups? In this paper, we will establish a theorem for $\widetilde{\mathcal{H}}$-cryptogroups by using the Green $\sim$-relations and the $\mathcal{K}G$-strong semilattice of semigroups, as described in [10]. We will show that an $\widetilde{\mathcal{H}}$-cryptogroup is a regular $\widetilde{\mathcal{H}}$-cryptogroup if and only if it is an $\widetilde{\mathcal{H}}G$-strong semilattice of completely $\widetilde{\mathcal{J}}$-simple semigroups. Our results in this paper also generalize and enrich the corresponding results given in [1], [4], [7], [8] and [13].

## 2. $\mathcal{K}G$-strong semilattices

We now restate the concept of $G$-strong semilattice decomposition of semigroup $S$ given by Kong and Shum in [8] and [9].

Let $S = (Y; S_\alpha)$ be a semilattice of the semigroups $S_\alpha$, where each $S_\alpha$ is a subsemigroup of the semigroup $S$ and $Y$ is a semilattice. We define the $G$-strong semilattice of semigroups by generalizing the well known *strong semilattice of semigroups* ( see [9]).

**Definition 2.1** Let $S = (Y; S_\alpha)$ be a semigroup. Suppose that the following conditions $S$ are satisfied:

(i) $(\forall \alpha, \beta \in Y, \alpha \geqslant \beta)$, there exists a family of homomorphisms $\varphi_{d(\alpha,\beta)} : S_\alpha \longrightarrow S_\beta$, where $d(\alpha, \beta) \in D(\alpha, \beta)$ and $D(\alpha, \beta)$ is a non-empty index set.

(ii) $(\forall \alpha \in Y)$, $D(\alpha, \alpha)$ is a singleton. Denote the element in $D(\alpha, \alpha)$ by $d(\alpha, \alpha)$. In this case, the homomorphism $\varphi_{d(\alpha,\alpha)} : S_\alpha \longrightarrow S_\alpha$ is the identity automorphism of the semigroup $S_\alpha$.

(iii) $(\forall \alpha, \beta, \gamma \in Y, \alpha \geqslant \beta \geqslant \gamma)$, if we write $\varphi_{\alpha,\beta} = \{\varphi_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$ then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} \subseteq \varphi_{\alpha,\gamma}$, where

$$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \{\varphi_{d(\alpha,\beta)}\varphi_{d(\beta,\gamma)} : \forall d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)\}.$$

(iv) for each $\alpha, \beta \in Y$, there is a mapping from $S_\alpha$ into the set $\varphi_{\beta,\alpha\beta}$ whose value at any given element $a \in S_\alpha$ is denoted by $\varphi^a_{d(\beta,\alpha\beta)}$ such that for all $b \in S_\beta$,

$$ab = (a\varphi^b_{d(\alpha,\alpha\beta)})(b\varphi^a_{d(\beta,\alpha\beta)}).$$

Then the above semilatttice of semigroups is called the generalized strong semilattice of semigroups $S_\alpha$ and in brevity , the "$G$-strong semilattice" of semigroups $S_\alpha$ and denoted it by $S = G[Y; S_\alpha, \varphi_{\alpha,\beta}]$.

The following definition is a more general version of $G$-strong semilattices.

**Definition 2.2** Let $\mathcal{K}$ be any equivalent relation on a $G$-strong semilattice of semigroups $S = G[Y; S_\alpha, \varphi_{\alpha,\beta}]$. Then, we call $S$ a "$\mathcal{K}G$-strong semilattice of semigroups $S_\alpha$" if for every $\alpha, \beta \in Y$, the mapping $a \longmapsto \varphi^a_{\alpha(\beta,\alpha\beta)}$ has the property that $\varphi^a_{d(\beta,\alpha\beta)} = \varphi^b_{d(\beta,\alpha\beta)}$ whenever the elements $a, b \in S_\alpha$ are in the same $\mathcal{K}$-class of $S$.

Thus, it is clear that the $G$-strong semilattice of semigroups $S$ can be determined by an equivalent relation $\mathcal{K}$. We therefore call the above generalized strong semilattice of semigroups $S_\alpha$ a "$\mathcal{K}G$ -strong semilattice of semigroups $S_\alpha$" and is denoted by $S = \mathcal{K}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$, where $\mathcal{K}$ is any one of the Green relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{H}$, respectively.

**Remark 2.3** It is clear that the $\mathcal{K}G$-strong semilattice is stronger than the $G$-strong semilattice but it is weaker than the usual strong semilattice. In fact, if $\rho$ and $\delta$ are equivalent relations on the semigroup $S = (Y; S_\alpha)$ with $\rho \subseteq \delta$, then one can observe that $\delta G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is "stronger" than $\rho G[Y; S_\alpha, \varphi_{\alpha,\beta}]$. As special cases, $1_S G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is the "weakest" $\mathcal{K}G$-strong semilattice of semigroups since $1_S$ is the "smallest" equivalent relation on $S$ and also $\eta G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is the strongest $\mathcal{K}G$-strong semilattice of semigroups since $\eta$ is the "greatest" equivalent relation on $S$, where $1_S$ is the identity relation on $S$ and $\eta$ is the semilattice congruence on $S$ which partitions the semigroup $S$ into disjoint subsemigroups $S_\alpha(\alpha \in Y)$ of $S$. Hence, we can easily see that $\eta G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is the usual strong semilattice of semigroups since in this case, every index set $D(\alpha, \beta)$ is a singleton for $\alpha \geqslant \beta$ on $Y$ and hence there exists one and only one structure homomorphism in the set of structure homomorphisms $\varphi_{\alpha,\beta}$.
We have already defined the Green $\sim$-relations $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{D}}$ on a semigroup $S$. In order to define the Green $\sim$-relation $\widetilde{\mathcal{J}}$ on $S$, we consider the left $\sim$-ideal $L$ of a semigroup $S$.

**Definition 2.4** A left (right) ideal $L$ $(R)$ of a semigroup $S$ is called a left $\sim$-ideal of $S$ if $\widetilde{L}_a \subseteq L(\widetilde{R}_a \subseteq R)$ holds, for all $a \in L(a \in R)$. We call a subset $I$ of a semigroup $S$ a $\sim$-ideal of $S$ if it is both a left $\sim$-ideal and a right $\sim$-ideal.
It is noteworthy that if $S$ is a regular semigroup, then every left (right, two-sided) ideal of $S$ is a left (right, two-sided) $\sim$-ideal. We also observe that for any idempotent $e$ in a semigroup $S$ , the left (right) ideal $Se(eS)$ is a left(right) $\sim$-ideal. For if $a \in Se$, then $a = ae$ , and hence for any element $b$ in $\widetilde{L}_a$, we have $b = be \in Se$.

By Definition 2.4, we see that the semigroup $S$ is always a $\sim$-ideal of itself, and we denote the smallest $\sim$-ideal containing the element $a$ of $S$ by $\widetilde{J}(a)$. Now, we define $\widetilde{\mathcal{J}} = \{(a, b) \in S \times S : \widetilde{J}(a) = \widetilde{J}(b)\}$.

**Definition 2.5** An $\widetilde{\mathcal{H}}$-abundant semigroup $S$ is called completely $\widetilde{\mathcal{J}}$-simple if $S$ does not contain any non-trivial proper $\sim$-ideal of $S$.
We now give some properties of the $\widetilde{\mathcal{H}}$-abundant semigroups. Some of the properties may have already been known or can be easily derived, however, for the sake of completeness, we provide here the proofs.

**Lemma 2.6** Let $S$ be an $\widetilde{\mathcal{H}}$-abundant semigroup. Then the following properties hold:

(i) The Green $\sim$-relation $\widetilde{\mathcal{H}}$ is a congruence on $S$ if and only if for any $a, b \in S$, $(ab)^0 = (a^0 b^0)^0$.

(ii) If $e, f$ are $\widetilde{\mathcal{D}}$-related idempotents of $S$, then $e\mathcal{D}f$.

(iii)
$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{L}} \circ \widetilde{\mathcal{R}} = \widetilde{\mathcal{R}} \circ \widetilde{\mathcal{L}}.$$

(iv) If $e, f$ are idempotents in $S$ such that $e\mathcal{J}f$, then $e\mathcal{D}f$.

*Proof.*

(i) *(Necessity).* For any $a, b \in S$, we have $a\widetilde{\mathcal{H}}a^0$ and $b\widetilde{\mathcal{H}}b^0$. Since $\widetilde{\mathcal{H}}$ is a congruence on $S$, $ab\widetilde{\mathcal{H}}a^0b^0$. But $ab\widetilde{\mathcal{H}}(ab)^0$, and so $(ab)^0 = (a^0b^0)^0$ since every $\widetilde{\mathcal{H}}$-class contains a unique idempotent.

*(Sufficiency).* We only need to show that $\widetilde{\mathcal{H}}$ is compatible with the semigroup multiplication of $S$ since $\widetilde{\mathcal{H}}$ is an equivalent relation on $S$. Let $(a, b) \in \widetilde{\mathcal{H}}$ and $c \in S$. Then $(ca)^0 = (c^0a^0)^0 = (c^0b^0)^0 = (cb)^0$ and hence, $\widetilde{\mathcal{H}}$ is left compatible to the semigroup multiplication. Dually, $\widetilde{\mathcal{H}}$ is right compatible with the semigroup multiplication and thus $\widetilde{\mathcal{H}}$ is a congruence on $S$.

(ii) Since $e\widetilde{\mathcal{D}}f$, there exist elements $a_1, \cdots, a_k$ of $S$ such that $e\widetilde{\mathcal{L}}a_1\widetilde{\mathcal{R}}a_2 \cdots a_k \ \widetilde{\mathcal{L}}\ f$. Since $S$ is an $\widetilde{\mathcal{H}}$-abundant semigroup, $e\mathcal{L}a_1^0\mathcal{R}a_2^0 \cdots a_k^0\mathcal{L}f$. Thus $e\mathcal{D}f$.

(iii) If $a, b \in S$ and $a\widetilde{\mathcal{D}}b$, then by (ii), $a^0\mathcal{D}b^0$. Hence there exist elements $c, d$ in $S$ with $a^0\mathcal{L}c\mathcal{R}b^0$ and $a^0\mathcal{R}d\mathcal{L}b^0$, and consequently, $a\widetilde{\mathcal{L}}c\widetilde{\mathcal{R}}b$ and $a\widetilde{\mathcal{R}}d\widetilde{\mathcal{L}}b$. Thus the result is proved.

(iv) Since $SeS = SfS$, there exist elements $x, y, s, t$ in $S$ such that $f = set$ and $e = xfy$. Let $h = (fy)^0$ and $k = (se)^0$. Then $hfy = fy = ffy$ and so $h = h^2 = fh$ and $sek = se = see$, and thereby, $k = k^2 = ke$. Hence, $hf, ek$ are the idempotents satisfying the relations $hf\mathcal{R}h$ and $ek\mathcal{L}k$ . These imply that $ehf\mathcal{R}eh$ and $ekf\mathcal{L}kf$. Now by $eh = xfyh = xfy = e$ and $kf = kset = set = f$, we have $e\mathcal{R}ef\mathcal{L}f$. This shows that $e\mathcal{D}f$.

Similar to the definition of cyber group given by Guo and Shum [5], we formulate the following definition.

**Definition 2.7** An $\widetilde{\mathcal{H}}$-abundant semigroup $S$ is called an $\widetilde{\mathcal{H}}$*-cryptogroup* if the Green $\sim$-relation $\widetilde{\mathcal{H}}$ is a congruence on $S$. Also, we call an $\widetilde{\mathcal{H}}$-abundant semigroup $S$ a *regular* $\widetilde{\mathcal{H}}$*-cryptogroup* if $\widetilde{\mathcal{H}}$ is a congruence on $S$ such that $S/\widetilde{\mathcal{H}}$ is a regular band. Thus, $\widetilde{\mathcal{H}}$-cryptogroups are analogy of cryptogroups in the class of $\widetilde{\mathcal{H}}$-abundant semigroups. Also, we see in [5] that an $\widetilde{\mathcal{H}}$-cryptogroup is a generalized cyber groups.
The $\widetilde{\mathcal{H}}$-cryptogroup $S$ has the following properties:

**Lemma 2.8**

(i) For any element $a$ of the $\widetilde{\mathcal{H}}$-cryptogroup $S$ , $\widetilde{J}(a) = Sa^0S$.

(ii) For the $\widetilde{\mathcal{H}}$-cryptogroup $S$, $\widetilde{\mathcal{J}} = \widetilde{\mathcal{D}}$.

(iii) If the $\widetilde{\mathcal{H}}$-cryptogroup $S$ is completely $\widetilde{\mathcal{J}}$-simple, then the idempotents of $S$ are primitive.

(iv) If the $\widetilde{\mathcal{H}}$-cryptogroup $S$ is completely $\widetilde{\mathcal{J}}$-simple, then the regular elements of $S$ generate a regular subsemigroup of $S$.

*Proof.*

(i) Obviously, we have $a^0 \in \widetilde{J}(a)$ and so $Sa^0S \subseteq \widetilde{J}(a)$. We need to show that the ideal $Sa^0S$ is in fact a $\sim$-ideal and since $a = aa^0a^0 \in Sa^0S$, $\widetilde{J}(a) \subseteq Sa^0S$. Let $b = xa^0y \in Sa^0S (x, y \in S)$ and $k = (a^0y)^0$. Then $a^0a^0y = a^0y = ka^0y$ so that $a^0(a^0y)^0 = k^2 = k$. Also since $\widetilde{\mathcal{H}}$ is a congruence, $xa^0y\widetilde{\mathcal{H}}xk$. Now let $h = (xk)^0 = (xa^0y)^0$. Then $xkh = xk = xkk$ so that $h = h^2 = hk = ha^0k \in Sa^0S$. Hence if $c \in \widetilde{L}_b, d \in \widetilde{R}_b$, then $c = ch, d = hd \in Sa^0S$ and hence, $Sa^0S$ is a $\sim$-ideal, as required.

(ii) Suppose that $(a, b) \in S$ with $a\widetilde{\mathcal{J}}b$. Then by (i), we have $Sa^0S = Sb^0S$. Now, by Lemma 2.6 (iv), $a^0\mathcal{D}b^0$ and so $a\widetilde{\mathcal{H}}a^0\mathcal{D}b^0\widetilde{\mathcal{H}}b$. This implies that $a\widetilde{\mathcal{D}}b$ and hence $\widetilde{\mathcal{J}} \subseteq \widetilde{\mathcal{D}}$. Conversely, let $a, b \in S$ with $a\widetilde{\mathcal{D}}b$. Then by Lemma 2.6 (iii), there exists an element $c \in S$ such that $a\widetilde{\mathcal{L}}c\widetilde{\mathcal{R}}b$. This leads to $a^0\mathcal{L}c^0\mathcal{R}b^0$ and so $Sa^0S = Sc^0S = Sb^0S$. Now, by (i), $(a, b) \in \widetilde{\mathcal{J}}$ and hence $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{J}}$. Therefore, $\widetilde{\mathcal{J}} = \widetilde{\mathcal{D}}$.

(iii) Let $e, f$ be idempotents in $S$ with $e \leqslant f$. Since $S$ is completely $\widetilde{\mathcal{J}}$-simple, $f \in SeS$. Now by the first part of Exercise 3 in [14][§8.4], there exists an idempotent $g$ of $S$ such that $f\mathcal{D}g$ and $g \leqslant e$. Let $a \in S$ be such that $f\mathcal{L}a\mathcal{R}g$. Then $f\mathcal{L}a^0\mathcal{R}g$ and since $g \leqslant f$, we have

$$a^0 = ga^0(gf)a^0 = g(fa^0) = gf = g.$$

Now by noting that $g \leqslant f$ and $g\mathcal{L}f$, we have $f = fg = g$. However, since $g \leqslant e$ , we obtain $e = f$ and hence all idempotents of $S$ are primitive.

(iv) Let $a, b$ be regular elements of $S$. Since $S$ consists of a single $\widetilde{\mathcal{D}}$-class, by (ii) and by Lemma 2.6 (iii), there exists an element $c \in S$ such that $a\widetilde{\mathcal{L}}c\widetilde{\mathcal{R}}b$. Hence $a\widetilde{\mathcal{L}}c^0\widetilde{\mathcal{R}}b$. This leads to $c^0b = b$ and $a\mathcal{L}c^0$ since $a$ is regular. Now we have $ab\mathcal{L}b$ and so the regularity of $ab$ follows from the regularity of $b$.

We now establish the following theorem for $\widetilde{\mathcal{H}}$ -cryptogroups.

**Theorem 2.9** Let $S$ be an $\widetilde{\mathcal{H}}$ -cryptogroup. Then $S$ is a semilattice $Y$ of completely $\widetilde{\mathcal{J}}$-simple semigroups $S_\alpha (\alpha \in Y)$ such that for every $\alpha \in Y$ and $a \in S_\alpha$, we have $\widetilde{L}_a(S) = \widetilde{L}_a(S_\alpha)$ and $\widetilde{R}_a(S) = \widetilde{L}_a(S_\alpha)$.

*Proof.* If $a \in S$, then $a\widetilde{\mathcal{H}}a^2$ and so, $\widetilde{J}(a) = \widetilde{J}(a^2)$. Now for $a, b \in S$, we have $(ab)^2 \in SbaS$, and hence, it follows that

$$\widetilde{J}(ab) = \widetilde{J}((ab)^2) \subseteq \widetilde{J}(ba).$$

Now, by symmetry, we obtain $\widetilde{J}(ab) = \widetilde{J}(ba)$. Since, by Lemma 2.8 (i), we have $\widetilde{J}(a) = Sa^0S$ and $\widetilde{J}(b) = Sb^0S$ so that if $c \in \widetilde{J}(a) \cap \widetilde{J}(b)$, then $c = xa^0y = zb^0t$ for some $x, y, z, t \in S$. Now $c^2 = zb^0txa^0y \in Sb^0txa^0S \subseteq \widetilde{J}(b^0txa^0)$ and hence, $\widetilde{J}(b^0txa^0) = \widetilde{J}(a^0b^0tx)$ by using previous arguments. Thus, $c^2 \in \widetilde{J}(a^0b^0)$ and since $c\widetilde{\mathcal{H}}c^2$, we have $c \in \widetilde{J}(a^0b^0)$. Since $a\widetilde{\mathcal{H}}a^0$,

$b\widetilde{\mathcal{H}}b^0$ and $\widetilde{\mathcal{H}}$ is a congruence on $S$, we have $ab\widetilde{\mathcal{H}}a^0b^0$ . Consequently, $c \in \widetilde{J}(ab)$, and thereby $\widetilde{J}(a) \cap \widetilde{J}(b) \subseteq \widetilde{J}(ab)$. The converse containment is clear so that $\widetilde{J}(a) \cap \widetilde{J}(b) = \widetilde{J}(ab)$. We can easily see that the set $Y$ of all $\sim$-ideals $\widetilde{J}(a)(a \in S)$ forms a semilattice under set intersection and that the mapping $a \mapsto \widetilde{J}(a)$ is a homomorphism from $S$ onto $Y$. The inverse image of $\widetilde{J}(a)$ is just the $\widetilde{\mathcal{J}}$-class $\widetilde{J}_a$ which is a subsemigroup of $S$. Hence $S$ is a semilattice $Y$ of the semigroups $\widetilde{J}_a$. Now let $a, b$ be elements of $\widetilde{\mathcal{J}}$-class $\widetilde{J}$ and suppose that $(a, b) \in \widetilde{\mathcal{L}}(\widetilde{J})$. Then, $a^0, b^0 \in \widetilde{J}$ so that $(a^0, b^0) \in \widetilde{\mathcal{L}}(\widetilde{J})$, that is, $a^0b^0 = a^0, b^0a^0 = b^0$ and $(a^0, b^0) \in \widetilde{\mathcal{L}}(S)$. It follows that $(a, b) \in \widetilde{\mathcal{L}}(S)$ and consequently, by $\widetilde{L}_a(S) \subseteq \widetilde{J}$, we have $\widetilde{L}_a(S) = \widetilde{L}_a(\widetilde{J})$. By using a similar argument, we can show that $\widetilde{R}_a(S) = \widetilde{R}_a(\widetilde{J})$. From the above discussion, we can deduce that $\widetilde{H}_a(\widetilde{J}) = \widetilde{H}_a(S)$ and so $\widetilde{J}$ is indeed an $\widetilde{\mathcal{H}}$-abundant semigroup. Furthermore, if $a, b \in \widetilde{J}$, then by Lemma 2.8 (i), $(a, b) \in \widetilde{\mathcal{D}}(S)$ and hence, by Lemma 2.6 (iii), there exists an element $c$ in $\widetilde{L}_a(S) \cap \widetilde{R}_b(S) = \widetilde{L}_a(\widetilde{J}) \cap \widetilde{R}_b(\widetilde{J})$. Thus $a, b$ are $\widetilde{\mathcal{D}}$-related in $\widetilde{J}$ and so $\widetilde{J}$ is $\widetilde{\mathcal{J}}$-simple.

For the $\widetilde{\mathcal{H}}$-cryptogroups, we have the following theorem.
**Theorem 2.10** Let $S$ be an $\widetilde{\mathcal{H}}$ -cryptogroup which is expressed by the semilattice of semigroups $S = (Y; S_\alpha)$. Then the following statements hold:

(i) For $\alpha$, and $\beta$ in the semilattice $Y$ with $\alpha \geqslant \beta$, if $a \in S_\alpha$ then there exists $b \in S_\beta$ with $a \geqslant b$;

(ii) For $a, b, c \in S$ with $b\widetilde{\mathcal{H}}c$, if $a \geqslant b,\ a \geqslant c$ then $b = c$;

(iii) For $a \in E(S)$ and $b \in S$, if $a \geqslant b$ then $b \in E(S)$ .

*Proof.* (i) Let $c \in S_\beta$. Then, by Lemma 2.6 (i), we see that $a(aca)^0$, $(aca)^0a$ and $(aca)^0$ are all in the same $\widetilde{\mathcal{H}}$-class of the semigroup $S$ and hence, $a(aca)^0 = (aca)^0a(aca)^0 = (aca)^0a$. Write $b = a(aca)^0$. Then $b \in S_\beta$ and $a \geqslant b$. (ii) By the definition of "$\geqslant$", there exist $e, f, g, h \in E(S)$ such that $b = ea = af$, $c = ga = ah$. From $eb = b$ and $b\widetilde{\mathcal{H}}b^0$, we have $eb^0 = b^0$. Similarly, $c^0h = c^0$. Thus $ec = ec^0c = eb^0c = b^0c = c$. By using similar arguments, we have $bh = b$ and so, $b = bh = eah = ec = c$, as required. (iii) We have $b = ea = af$ for some $e, f \in E(S)$, and whence

$$b^2 = (ea)(af) = ea^2f = b.$$

The following fact can be easily observed:

**Fact 2.11** Let $\varphi$ be a homomorphism which maps an $\widetilde{\mathcal{H}}$-cryptogroup $S$ into another $\widetilde{\mathcal{H}}$ -cryptogroup $T$. Then $(a\varphi)^0 = a^0\varphi$.

## 3. Properties of regular $\widetilde{\mathcal{H}}$-cryptogroups

**Lemma 3.1** Let $S$ be a regular $\widetilde{\mathcal{H}}$ -cryptogroup(that is, $\widetilde{\mathcal{H}}$ is a congruence on the $\widetilde{\mathcal{H}}$-abundant semigroup $S$ such that $S/\widetilde{\mathcal{H}}$ is a regular band). For every $a \in S$, we define a relation $\rho_a$ on $S$ by $(b_1, b_2) \in \rho_a$ if and only if $(ab_1a)^0 = (ab_2a)^0$, $(b_1, b_2 \in S)$. Then the following properties hold on $S$:

(i) $\rho_a$ is a band congruence on $S$;

(ii) $(\forall a, a_1 \in S_\alpha)$, $\rho_a = \rho_{a_1}$, that is, $\rho_a$ depends only on the component $S_\alpha$ containing the element $a$ rather than on the element itself, hence we can write $\rho_\alpha = \rho_a$, for all $a \in S_\alpha$.

(iii) $(\forall \alpha, \beta \in Y$ with $\alpha \geqslant \beta)$, $\rho_\alpha \subseteq \rho_\beta$ and $\rho_\beta|_{S_\alpha} = \omega_{S_\alpha}$, where $\omega_{S_\alpha}$ is the universal relation on $S_\alpha$.

*Proof.* (i) It is easy to see that $\rho_a$ is an equivalent relation on $S$, for all $a \in S$. We now prove that $\rho_a$ is left compatible with the semigroup multiplication. For this purpose, we let $(x, y) \in \rho_a$ and $c \in S$. Then, by the definition of $\rho_a$, we have $(axa)^0 = (aya)^0$. Since $S$ is a regular $\widetilde{\mathcal{H}}$-cryptogroup, by Lemma 2.6 (i) and the regularity of the band $S/\widetilde{\mathcal{H}}$, we obtain that

$$(acxa)^0 = (ac(axa))^0 = ((ac)^0(axa)^0)^0 = ((ac)^0(aya)^0)^0 = (acya)^0.$$

Hence, $(cx, cy) \in \rho_a$. Dually, we can prove that $\rho_a$ is right compatible with the semigroup multiplication. Thus $\rho_a$ is a congruence on $S$. Obviously, $\widetilde{\mathcal{H}} \subseteq \rho_a$ and so $\rho_a$ is a band congruence on $S$. (ii) Let $(x, y) \in \rho_a$. Then, by the definition of $\rho_a$, we have $(axa)^0 = (aya)^0$ and so $a_1^0(axa)^0a_1^0 = a_1^0(aya)^0a_1^0$. This leads to $(a_1^0(axa)^0a_1^0)^0 = (a_1^0(aya)^0a_1^0)^0$. Since $S/\widetilde{\mathcal{H}} = (Y; S_\alpha/\widetilde{\mathcal{H}})$ is a regular band and by Lemma 2.6 (i), we obtain $(a_1aa_1xa_1aa_1)^0 = (a_1aa_1ya_1aa_1)^0$. However, since $a, a_1$ are elements of the completely $\widetilde{\mathcal{J}}$-simple semigroup $S_\alpha$, $(a_1aa_1)^0 = a_1^0$. Thereby, by Lemma 2.6 (i) again, we have $(a_1xa_1)^0 = (a_1ya_1)^0$, that is, $(x, y) \in \rho_{a_1}$. This shows that $\rho_a \subseteq \rho_{a_1}$. Similarly, we also have $\rho_{a_1} \subseteq \rho_a$. Thus, $\rho_a = \rho_{a_1}$. Since this relation holds for all $a \in S_\alpha$, we usually write $\rho_a = \rho_\alpha$. (iii) Let $a \in S_\alpha, b \in S_\beta$ and $\alpha \geqslant \beta$. We need to prove that $\rho_\alpha \subseteq \rho_\beta$. For this purpose, we let $(x, y) \in \rho_\alpha = \rho_a$, by (ii). Then, by the definition of $\rho_a$, we have $(axa)^0 = (aya)^0$ and hence $b(axa)^0b = b(aya)^0b$. By Lemma 2.6 (i) and the regularity of the band, we have $(babxbab)^0 = (babybab)^0$. Since $\alpha \geqslant \beta$ in $Y$ and $a \in S_\alpha, b \in S_\beta$, we have $(bab)^0 = b^0$. By using Lemma 2.6 (i) again, we can show that $(bxb)^0 = (byb)^0$, that is, $(x, y) \in \rho_b = \rho_\beta$. Thus, $\rho_\alpha \subseteq \rho_\beta$ as required. Furthermore, it is trivial that $\rho_\beta|_{S_\alpha} = \omega_{S_\alpha}$, which is the universal relation on the semigroup $S_\alpha$.

We now use the band congruence $\rho_\alpha$ defined in Lemma 3.1 to describe the structural homomorphisms for the $\widetilde{\mathcal{H}}$ -cryptogroup $S = (Y; S_\alpha)$, where each $S_\alpha$ is a completely $\widetilde{\mathcal{J}}$-simple semigroup.

We first consider the congruence $\rho_{\alpha,\beta} = \rho_\alpha|_{S_\beta}$ for $\alpha, \beta \in Y$, which is a band congruence on the semigroup $S_\beta$. Now, we denote all the $\rho_{\alpha,\beta}$-classes of $S_\beta$ by $\{S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$, where $D(\alpha, \beta)$ is a non-empty index set. In particular, the set $D(\alpha, \alpha)$ is a singleton and we can therefore write $d(\alpha, \alpha) = D(\alpha, \alpha)$. We have the following lemma.

**Lemma 3.2** Let $S = (Y; S_\alpha)$ be a regular $\widetilde{\mathcal{H}}$-cryptogroup. Then, for all $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$, the following statements hold for all $d(\alpha, \beta) \in D(\alpha, \beta)$.

(i) For all $a \in S_\alpha$, there exists a unique $a_{d(\alpha,\beta)} \in S_{d(\alpha,\beta)}$ satisfying $a \geqslant a_{d(\alpha,\beta)}$ ;

(ii) For all $a \in S_\alpha$ and $x \in S_{d(\alpha,\beta)}$, if $a^0 \geqslant e$ for some idempotent $e \in S_{d(\alpha,\beta)}$ then $eax = ax, xae = xa$, $ea = ae$ and $(ea)^0 = e$;

(iii) Let $a \in S_\alpha$. Define $\varphi_{d(\alpha,\beta)} : S_\alpha \longrightarrow S_{d(\alpha,\beta)}$ by $a\varphi_{d(\alpha,\beta)} = a_{d(\alpha,\beta)}$, where $a_{d(\alpha,\beta)} \in S_{d(\alpha,\beta)}$ and $a \geqslant a_{d(\alpha,\beta)}$. Then $\varphi_{d(\alpha,\beta)}$ is a homomorphism and $a_{d(\alpha,\beta)} = a(aba)^0 = (aba)^0a$ for any $b \in S_{d(\alpha,\beta)}$.

*Proof.* (i) We first show that for any $a \in S_\alpha$ and $b \in S_{d(\alpha,\beta)}$, we have $ab \in S_{d(\alpha,\beta)}$, that is, $(ab, b) \in \rho_{\alpha,\beta}$. In fact, since $S = (Y, S_\alpha)$ is an $\widetilde{\mathcal{H}}$ -cryptogroup, each $S_\alpha$ is a completely $\widetilde{\mathcal{J}}$-simple semigroup. Hence, we have $(xax)^0 = x^0$, for all $x \in S_\alpha$. This leads to $(xabx)^0 = (xaxbx)^0 = (xbx)^0$ by the regularity of the band $S/\widetilde{\mathcal{H}}$ and Lemma 2.6 (i). Thereby, $(ab, b) \in \rho_{\alpha,\beta}$. Similarly, we also have $ba \in S_{d(\alpha,\beta)}$. Invoking the above results, we have $aba \in S_{d(\alpha,\beta)}$ for any $b \in S_{d(\alpha,\beta)}$. Since $\widetilde{\mathcal{H}}$ is a band congruence on $S$, by Lemma 2.6 (i) again, we see that $a(aba)^0, (aba)^0$ and $(aba)^0a$ are in the same $\widetilde{\mathcal{H}}$-class of $S$ so that $a(aba)^0 = (aba)^0a(aba)^0 = (aba)^0a$. Let $a(aba)^0 = a_{d(\alpha,\beta)}$. Then by the natural partial order imposed on $S$, we have $a \geqslant a_{d(\alpha,\beta)}$. In order to show the uniqueness of $a_{d(\alpha,\beta)}$, we assume that there is another $a^*_{d(\alpha,\beta)} \in S_{d(\alpha,\beta)}$ satisfying $a \geqslant a^*_{d(\alpha,\beta)}$. Then, by the definition of "$\leqslant$", we can write $a^*_{d(\alpha,\beta)} = ea = af$ for some $e, f \in E(S)$ and so $a^*_{d(\alpha,\beta)}a^0 = a^*_{d(\alpha,\beta)} = a^0a^*_{d(\alpha,\beta)}$. By the fact $a^*_{d(\alpha,\beta)}\widetilde{\mathcal{H}}a^0$, we have $(a^*_{d(\alpha,\beta)})^0a^0 = (a^*_{d(\alpha,\beta)})^0$ and $a^0(a^*_{d(\alpha,\beta)})^0 = (a^*_{d(\alpha,\beta)})^0$. Consequently, by the definition of "$\leqslant$", we have $a^0 \geqslant (a^*_{d(\alpha,\beta)})^0$. By Lemma 2.6 (i) again, we deduce that

$$(a^*_{d(\alpha,\beta)})^0 = (a^0(a^*_{d(\alpha,\beta)})^0a^0)^0 = (aa^*_{d(\alpha,\beta)}a)^0 = (aba)^0.$$

Hence, $(a^*_{d(\alpha,\beta)}, a_{d(\alpha,\beta)}) \in \widetilde{\mathcal{H}}$, and consequently, by Theorem 2.10 (ii), $a^*_{d(\alpha,\beta)} = a_{d(\alpha,\beta)}$. This shows the uniqueness of $a_{d(\alpha,\beta)}$. (ii) It is easy to see that, by the definition of "$\leqslant$", $a^0 \geqslant (a^0(ax)^0a^0)^0$. Also, since $a \in S_\alpha$ and $x \in S_{d(\alpha,\beta)}$, we have $ax \in S_{d(\alpha,\beta)}$ by (i). Moreover, since $S_{d(\alpha,\beta)}$ is a $\rho_{\alpha,\beta}$ -congruence class, $(ax)^0 \in S_{d(\alpha,\beta)}$. Thus, by (i) again, we have $(a^0(ax)^0a^0)^0 \in S_{d(\alpha,\beta)}$ and $e = (a^0(ax)^0a^0)^0$. Thereby, we have $eax = (a^0(ax)^0a^0)^0a^0(ax)^0a^0ax = ax$. Similarly, we have $xae = xa$. Since $x$ is arbitrarily chosen element in $S_{d(\alpha,\beta)}$, we can particularly choose $x = e$. In this way, we obtain $ea = ae$ and consequently, by Lemma 2.6 (i), we have $(ea)^0 = (ea^0)^0 = e$. (iii) By using the result in (i), we can define $\varphi_{d(\alpha,\beta)} : S_\alpha \longrightarrow S_{d(\alpha,\beta)}$ by $a\varphi_{d(\alpha,\beta)} = a_{d(\alpha,\beta)} = a(aca)^0 = (aca)^0a$, for any $a \in S_\alpha$ and $c \in S_{d(\alpha,\beta)}$. Then, for any $a, b \in S_\alpha$, we have, by (ii),

$$\begin{aligned}(a\varphi_{d(\alpha,\beta)})(b\varphi_{d(\alpha,\beta)}) &= a_{d(\alpha,\beta)}b_{d(\alpha,\beta)}\\ &= (aca)^0ab(bcb)^0\\ &= (aca)^0(ab(bcb)^0)\\ &= ab(bcb)^0.\end{aligned}$$

Similarly, we can show that $(a\varphi_{d(\alpha,\beta)})(b\varphi_{d(\alpha,\beta)}) = (aca)^0ab$. Hence, $ab \geqslant (a\varphi_{d(\alpha,\beta)})(b\varphi_{d(\alpha,\beta)})$. Thus $(ab)\varphi_{d(\alpha,\beta)} = (a\varphi_{d(\alpha,\beta)})(b\varphi_{d(\alpha,\beta)})$, by the definition of $\varphi_{d(\alpha,\beta)}$. This shows that $\varphi_{d(\alpha,\beta)}$ is indeed a homomorphism.

We now proceed to show that the homomorphisms given in Lemma 3.2 (iii) are the structural homomorphisms for the $G$-strong semilattice $G[Y;\, S_\alpha\ ,\, \varphi_{\alpha,\beta}]$ induced by the semigroup $S = (Y; S_\alpha)$ under the band congruence $\rho_\alpha$ on the semigroup $S_\alpha$.

**Lemma 3.3** Let $S = (Y; S_\alpha)$ be an $\widetilde{\mathcal{H}}$ -cryptogroup and $\varphi_{\alpha,\beta} = \{\varphi_{d(\alpha,\beta)} \mid d(\alpha,\beta) \in D(\alpha,\beta)\}$ for $\alpha \geqslant \beta$ on $Y$ , where $D(\alpha,\beta)$ is a non-empty index set. Then

(i) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} \subseteq \varphi_{\alpha,\gamma}$ for $\alpha \geqslant \beta \geqslant \gamma$ on $Y$.

(ii) For $a \in S_\alpha$ and $\beta \in Y$,

$$a\varphi_{\alpha,\alpha\beta} = \{a\varphi_{d(\alpha,\alpha\beta)} | \forall d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq S_{d(\beta,\alpha\beta)},$$

for some $\rho_{\beta,\alpha\beta}$-class $S_{d(\beta,\alpha\beta)}$.

*Proof.* (i) Clearly, $\varphi_{d(\alpha,\alpha)}$ is an identity automorphism of $S_\alpha$. We now prove that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} \subseteq \varphi_{\alpha,\gamma}$ for $\alpha \geqslant \beta \geqslant \gamma$ on $Y$. Pick $\varphi_{d(\alpha,\beta)} : S_\alpha \longrightarrow S_{d(\alpha,\beta)} \subseteq S_\beta$ and $\varphi_{d(\beta,\gamma)} : S_\beta \longrightarrow S_{d(\beta,\gamma)} \subseteq S_\gamma$. We show that $\varphi_{d(\alpha,\beta)}\varphi_{d(\beta,\gamma)} = \varphi_{d(\alpha,\gamma)}$ for some $\varphi_{d(\alpha,\gamma)} : S_\alpha \longrightarrow S_{d(\alpha,\gamma)} \subseteq S_\gamma$. For this purpose, we let $a \in S_\alpha$, $b_1, b_2 \in S_{d(\alpha,\beta)}$ and $c \in S_{d(\beta,\gamma)}$. Then, because $S/\widetilde{\mathcal{H}}$ is a band, by Lemma 3.2, we have $b_1\varphi_{d(\beta,\gamma)} = b_1(b_1cb_1)^0$, $b_2\varphi_{d(\beta,\gamma)} = b_2(b_2cb_2)^0$. Since $b_1, b_2 \in S_{d(\alpha,\beta)}$, by the definition of $\rho_{\alpha,\beta}$, $(b_1, b_2) \in \rho_{\alpha,\beta}$. This leads to $(ab_1a)^0 = (ab_2a)^0$. Now, by the regularity of the band $S/\widetilde{\mathcal{H}}$, we can easily deduce that

$$\begin{aligned}(a(b_1\varphi_{d(\beta,\gamma)})a)^0 &= (ab_1(b_1cb_1)^0a)^0 = ((ab_1a)^0c(ab_1a)^0)^0 \\ &= ((ab_2a)^0c(ab_2a)^0)^0 = (a(b_2(b_2cb_2)^0)a)^0 \\ &= (a(b_2\varphi_{d(\beta,\gamma)})a)^0.\end{aligned}$$

Thus, by the definition of $\rho_{\alpha,\gamma}$, we have $(b_1\varphi_{d(\beta,\gamma)}, b_2\varphi_{d(\beta,\gamma)}) \in \rho_{\alpha,\gamma}$. In other words, there exists a $\rho_{\alpha,\gamma}$-class $S_{d(\alpha,\gamma)}$ satisfying $S_{d(\alpha,\beta)}\varphi_{d(\beta,\gamma)} \subseteq S_{d(\alpha,\gamma)}$. Also, $\varphi_{d(\alpha,\beta)}\varphi_{d(\beta,\gamma)}$ clearly maps $S_\alpha$ into $S_{d(\alpha,\gamma)}$ by the transitivity of "$\leqslant$", and hence $\varphi_{d(\alpha,\beta)}\varphi_{d(\beta,\gamma)} = \varphi_{d(\alpha,\gamma)}$. This proves that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} \subseteq \varphi_{\alpha,\gamma}$. (ii) It suffices to show that for any $\varphi_{d(\alpha,\alpha\beta)}$ and $\varphi_{d'(\alpha,\alpha\beta)} \in \varphi_{\alpha,\alpha\beta}$ , we have $(a\varphi_{d(\alpha,\alpha\beta)}, a\varphi_{d'(\alpha,\alpha\beta)}) \in \rho_{\beta,\alpha\beta}$. For this purpose, we let $x \in S_{d(\alpha,\alpha\beta)}$ and $x' \in S_{d'(\alpha,\alpha\beta)}$. Then, by Lemma 3.2 (iii), we have $a\varphi_{d(\alpha,\alpha\beta)} = a(axa)^0$ and $a\varphi_{d'(\alpha,\alpha\beta)} = a(ax'^0$. Let $b \in S_\beta$. Since $S_{\alpha\beta}$ is a completely $\widetilde{\mathcal{J}}$-simple semigroup, and $bab$, $a\varphi_{d(\alpha,\alpha\beta)}$, $a\varphi_{d'(\alpha,\alpha\beta)}$ are elements in $S_{\alpha\beta}$, we obtain that $(bab,\ (bab)(a\varphi_{d(\alpha,\alpha\beta)})(bab)) \in \widetilde{\mathcal{H}}$ and $(bab,\ (bab)(a\varphi_{d'(\alpha,\alpha\beta)})(bab)) \in \widetilde{\mathcal{H}}$. Since every $\widetilde{\mathcal{H}}$-class of $S_{\alpha\beta}$ contains a unique idempotent, $((bab)(a\varphi_{d(\alpha,\alpha\beta)})(bab))^0 = ((bab)(a\varphi_{d'(\alpha,\alpha\beta)})(bab))^0$. In other words, we have $((bab)(a(axa)^0)(bab))^0 = ((bab)(a(ax'^0)(bab))^0$. Thus, by the regularity of the band $S/\widetilde{\mathcal{H}}$, we can further simplify the above equality to $(b(a(axa)^0)b)^0 = (b(a(ax'^0)b)^0$, that is, $(b(a\varphi_{d(\alpha,\alpha\beta)})b)^0 = (b(a\varphi_{d'(\alpha,\alpha\beta)})b)^0$. By the definition of $\rho_{\beta,\alpha\beta}$, we see that $(a\varphi_{d(\alpha,\alpha\beta)}, a\varphi_{d'(\alpha,\alpha\beta)}) \in \rho_{\beta,\alpha\beta}$.

Finally we show that $S = (Y; S_\alpha)$ equipped with the above structural homomorphisms acting on the $\rho_{\alpha,\beta}$-equivalence class of $S$ forms a $G$-strong semilattice of semigroups $S_\alpha$. We need the following lemma.

**Lemma 3.4** Let $S = (Y; S_\alpha)$ be a regular $\widetilde{\mathcal{H}}$-cryptogroup. For any $a \in S_\alpha, b \in S_\beta$, suppose that $a\varphi_{\alpha,\alpha\beta} \subseteq S_{d(\beta,\alpha\beta)}, b\varphi_{\beta,\alpha\beta} \subseteq S_{d(\alpha,\alpha\beta)}$, where $\varphi_{\alpha,\alpha\beta}$ and $\varphi_{\beta,\alpha\beta}$ are the structural homomorphisms defined in Lemma 3.3. Then we have

$$ab = (a\varphi_{d(\alpha,\alpha\beta)})(b\varphi_{d(\beta,\alpha\beta)}).$$

*Proof.* Let $c_1 \in S_{d(\alpha,\alpha\beta)}$, $c_2 \in S_{d(\beta,\alpha\beta)}$. Then $(ac_1a)^0 \in S_{d(\alpha,\alpha\beta)}$ because $S_{d(\alpha,\alpha\beta)}$ is a $\rho_{\alpha,\alpha\beta}$-equivalence class of $S_{\alpha\beta}$. Now, by Lemma 3.2, $a\varphi_{d(\alpha,\alpha\beta)} = (ac_1a)^0a$ and $b\varphi_{d(\beta,\alpha\beta)} = b(bc_2b)^0$ for $\varphi_{d(\alpha,\alpha\beta)} \in \varphi_{\alpha,\alpha\beta}$ and $\varphi_{d(\beta,\alpha\beta)} \in \varphi_{\beta,\alpha\beta}$. Since we assume that $a\varphi_{\alpha,\alpha\beta} \subseteq S_{d(\beta,\alpha\beta)}$, we have $a\varphi_{d(\alpha,\alpha\beta)} = (ac_1a)^0a \in S_{d(\beta,\alpha\beta)}$. Similarly, we have $b\varphi_{d(\beta,\alpha\beta)} \in S_{d(\alpha,\alpha\beta)} \cap S_{d(\beta,\alpha\beta)}$. Thus, by Lemma 3.2 (ii), we have

$$(a\varphi_{d(\alpha,\alpha\beta)})(b\varphi_{d(\beta,\alpha\beta)}) = (ac_1a)^0(ab(bc_2b)^0) = ab(bc_2b)^0$$

and also

$$(a\varphi_{d(\alpha,\alpha\beta)})(b\varphi_{d(\beta,\alpha\beta)}) = ((ac_1a)^0ab)(bc_2b)^0 = (ac_1a)^0ab.$$

However, by the definition of the natural partial order "$\leqslant$", we have $ab \geqslant (a\varphi_{d(\alpha,\alpha\beta)})(b\varphi_{d(\beta,\alpha\beta)})$. On the other hands, since every semigroup $S_{\alpha\beta}$ is primitive, we obtain

$$ab = (a\varphi_{d(\alpha,\alpha\beta)})(b\varphi_{d(\beta,\alpha\beta)}).$$

## 4. Structure of regular $\widetilde{\mathcal{H}}$-cryptogroups

In this section, we use the $\mathcal{K}G$-strong semilattice to characterize regular $\widetilde{\mathcal{H}}$-cryptogroups. Also, we consider the question when will the Green $\sim$-relation $\widetilde{\mathcal{H}}$ to be a right quasi-normal band congruence? By using the $\mathcal{K}G$-strong semilattice, we are able to give a description for the normal $\widetilde{\mathcal{H}}$ -cryptogroups. We note here that the orthodox regular $\widetilde{\mathcal{H}}$ -cryptogroups with $\mathcal{K}G$-strong semilattices have been studies in [10]. A construction theorem of orthodox regular $\widetilde{\mathcal{H}}$ -cryptogroups was also given in [8].

**Theorem 4.1** An $\widetilde{\mathcal{H}}$-cryptogroup $S$ is a regular $\widetilde{\mathcal{H}}$-cryptogroup if and only if $S$ is an $\widetilde{\mathcal{H}}G$-strong semilattice of completely $\widetilde{\mathcal{J}}$-simple semigroups, that is, $S = \widetilde{\mathcal{H}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$.

*Proof.* By the definition of the $\mathcal{K}G$-strong semilattice and the results obtained in §3, we have already proved the necessity part of Theorem 4.1 since it is obvious that $\widetilde{\mathcal{H}}|_{S_\beta} \subseteq \rho_{\alpha,\beta}$ for $\alpha \geqslant \beta$ on $Y$. We now prove the sufficiency part of the theorem. To prove that $S/\widetilde{\mathcal{H}}$ is a regular band, we use a result in [14]. What we need is to prove that the usual Green relations $\mathcal{L}$ and $\mathcal{R}$ are congruences on $S/\widetilde{\mathcal{H}}$. In fact, we only need to verify that $\mathcal{L}$ is a left congruence on $S/\widetilde{\mathcal{H}}$ since $\mathcal{R}$ is a right congruence on $S/\widetilde{\mathcal{H}}$ can be proved in a similar fashion. Since $S = (Y; S_\alpha)$ is an $\widetilde{\mathcal{H}}$-cryptogroup, we can let $e\widetilde{\mathcal{H}}, f\widetilde{\mathcal{H}}$ and $g\widetilde{\mathcal{H}} \in S/\widetilde{\mathcal{H}}$, where $e, f \in S_\alpha \cap E(S)$, $g \in S_\beta \cap E(S)$ with $(e, f) \in \mathcal{L}$. Then, we have $ef = e$ and $fe = f$. By the definition of $\widetilde{\mathcal{H}}G$-strong semilattice $\widetilde{\mathcal{H}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$, we can find the homomorphisms $\varphi^{ef}_{d(\beta,\alpha\beta)}$ and $\varphi^{f}_{d(\beta,\alpha\beta)} \in \varphi_{\beta,\alpha\beta}$, $\varphi^{g}_{d(\alpha,\alpha\beta)} \in \varphi_{\alpha,\alpha\beta}$ such that

$$\begin{aligned}(gegf)\widetilde{\mathcal{H}} &= \{[g(ef)](gf)\}\widetilde{\mathcal{H}} \\ &= \{[(g\varphi^{ef}_{d(\beta,\alpha\beta)})((ef)\varphi^{g}_{d(\alpha,\alpha\beta)})][(g\varphi^{f}_{d(\beta,\alpha\beta)})(f\varphi^{g}_{d(\alpha,\alpha\beta)})]\}\widetilde{\mathcal{H}} \\ &= [(g\varphi^{ef}_{d(\beta,\alpha\beta)})(f\varphi^{g}_{d(\alpha,\alpha\beta)})]\widetilde{\mathcal{H}}\end{aligned}$$

and

$$\begin{aligned}(ge)\widetilde{\mathcal{H}} &= [g(ef)]\widetilde{\mathcal{H}} \\ &= [(g\varphi^{ef}_{d(\beta,\alpha\beta)})((ef)\varphi^{g}_{d(\alpha,\alpha\beta)})]\widetilde{\mathcal{H}} \\ &= [(g\varphi^{ef}_{d(\beta,\alpha\beta)})(f\varphi^{g}_{d(\alpha,\alpha\beta)})]\widetilde{\mathcal{H}}.\end{aligned}$$

Thereby, $(gegf)\widetilde{\mathcal{H}} = (ge)\widetilde{\mathcal{H}}$. Analogously, we can also prove that $(gfge)\widetilde{\mathcal{H}} = (gf)\widetilde{\mathcal{H}}$ . This proves that $\mathcal{L}$ is left compatible with the multiplication of $S/\widetilde{\mathcal{H}}$. Since $\mathcal{L}$ is always right congruence, $\mathcal{L}$ is a congruence on $S/\widetilde{\mathcal{H}}$, as required. Dually, $\mathcal{R}$ is also a congruence on $S/\widetilde{\mathcal{H}}$. Thus by [14] (see II. 3.6 Proposition ), $S/\widetilde{\mathcal{H}}$ forms a regular band and hence $S$ is indeed a regular $\widetilde{\mathcal{H}}$-cryptogroup. Our proof is completed.

Recall that a right quasi-normal band is a band satisfying the identity $yxa = yaxa$ [6]. Also, a left quasi-normal band is a band satisfying the identity $axy = axay$. Thus, we can easily observe that both the right quasi-normal bands and the left quasi-normal bands are special cases of the regular bands. Also, a normal band (that is, a band satisfies the identity $axya = ayxa$) is a special right quasi-normal band and a left quasi-normal band. Based on the above observation, we are able to establish the following theorem for right quasi-normal $\widetilde{\mathcal{H}}$-cryptogroups.

**Theorem 4.2** An $\widetilde{\mathcal{H}}$-abundant semigroup $S$ is a right quasi-normal $\widetilde{\mathcal{H}}$-cryptogroup if and only if $S$ is an $\widetilde{\mathcal{L}}G$-strong semilattice of completely $\widetilde{\mathcal{J}}$-simple semigroups, that is, $S = \widetilde{\mathcal{L}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$.

*Proof.* $(Necessity)$ Let $S$ be a right quasi-normal $\widetilde{\mathcal{H}}$ -cryptogroup. Then $S/\widetilde{\mathcal{H}}$ is a right quasi-normal band. To show that $S$ is an $\widetilde{\mathcal{L}}G$-strong semilattice, by invoking Lemma 3.3 and its proof, we only need to show that for any $\delta \geqslant \gamma$ on $Y$, $\widetilde{\mathcal{L}}|_{S_\gamma} \subseteq \rho_{\delta,\gamma}$. In fact, for $a \in S_\delta, x, y \in S_\gamma$ with $(x, y) \in \widetilde{\mathcal{L}}$, we have $(axa)\widetilde{\mathcal{H}} = ((axy)a)\widetilde{\mathcal{H}} = (ayxya)\widetilde{\mathcal{H}} = (aya)\widetilde{\mathcal{H}}$ by the right quasi-normality of the band $S/\widetilde{\mathcal{H}}$. Thus, by the definition of $\rho_{\delta,\gamma}$, we have $\widetilde{\mathcal{L}}|_{S_\gamma} \subseteq \rho_{\delta,\gamma}$ as required. This shows that $S = \widetilde{\mathcal{L}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$. $(Sufficiency)$ Let $a \in S_\alpha, x \in S_\beta$, and $y \in S_\gamma$. Then, since $S = \widetilde{\mathcal{L}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$ is an $\widetilde{\mathcal{H}}G$-strong semilattice of $S_\alpha$ and by Theorem 4.1, $\widetilde{\mathcal{H}}$ is a congruence on $S$. Moreover, we have $xa = (x\varphi^a_{d(\beta,\alpha\beta)})(a\varphi^x_{d(\alpha,\alpha\beta)})$ and thereby, $axa = (a\varphi^x_{d(\alpha,\alpha\beta)})(x\varphi^a_{d(\beta,\alpha\beta)})(a\varphi^x_{d(\alpha,\alpha\beta)})$. By the fact $((xa)^0, (axa)^0) \in \mathcal{L}$, we can easily see that $(xa, axa) \in \widetilde{\mathcal{L}}|_{S_{\alpha\beta}}$, and so, by our hypothesis, $S = \widetilde{\mathcal{L}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$. This implies that there exist some homomorphisms $\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)} \in \varphi_{\alpha\beta,\alpha\beta\gamma}$ and $\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)} \in \varphi_{\gamma,\alpha\beta\gamma}$ satisfying the conditions $y(xa) = (y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})((xa)\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)})$ and $y(axa) = (y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})((axa)\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)})$. Hence, it follows that

$$\begin{aligned}(y(xa))\widetilde{\mathcal{H}} &= [(y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})((xa)\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)})]\widetilde{\mathcal{H}} \\ &= \{(y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})\{[(x\varphi^{a}_{d(\beta,\alpha\beta)})(a\varphi^{x}_{d(\alpha,\alpha\beta)})]\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)}\}\}\widetilde{\mathcal{H}} \\ &= [(y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})((a\varphi^{x}_{d(\alpha,\alpha\beta)})\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)})]\widetilde{\mathcal{H}}\end{aligned}$$

and

$$\begin{aligned}(y(axa))\widetilde{\mathcal{H}} &= \{(y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})\{[(a\varphi^{x}_{d(\alpha,\alpha\beta)})(x\varphi^{a}_{d(\beta,\alpha\beta)})(a\varphi^{x}_{d(\alpha,\alpha\beta)})]\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)}\}\}\widetilde{\mathcal{H}} \\ &= [(y\varphi^{\widetilde{\mathcal{L}}}_{d(\gamma,\alpha\beta\gamma)})((a\varphi^{x}_{d(\alpha,\alpha\beta)})\varphi^{\widetilde{\mathcal{L}}}_{d(\alpha\beta,\alpha\beta\gamma)})]\widetilde{\mathcal{H}}.\end{aligned}$$

This leads to $(yxa)\widetilde{\mathcal{H}} = (yaxa)\widetilde{\mathcal{H}}$ and so $S/\widetilde{\mathcal{H}}$ is a right quasi-normal band. Thus, $S$ is indeed a right quasi-normal $\widetilde{\mathcal{H}}$-cryptogroup.

Since we have already mentioned that a band $B$ is a normal band if for all elements $e, f, g$ in $B$, the identity $efge = egfe$ holds in $B$( see [6]). In closing this paper, we characterize the normal $\widetilde{\mathcal{H}}$ -cryptogroups. In fact, this result gives a modified version of the theorem of Petrich and Reilly in [11] on normal cryptogroups, in particular, the theorem on normal cryptogroups in [11] and also the theorem of Fountain on superabundant semigroups in [4] is now refined and amplified in the class of quasiabundant semigroups.

**Theorem 4.3** An $\widetilde{\mathcal{H}}$-abundant semigroup $S$ is a normal $\widetilde{\mathcal{H}}$-cryptogroup if and only if $S$ is a $\widetilde{\mathcal{D}}G$-strong semilattice of completely $\widetilde{\mathcal{J}}$-simple semigroups, that is, $S = \widetilde{\mathcal{D}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$.

*Proof.* $(Necessity)$ The proof is similar to the necessity part given in Theorem 4.2, that is, we only need to prove that $\widetilde{\mathcal{D}}|_{S_\beta} \subseteq \rho_{\alpha,\beta}$ for all $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$. Since every semigroup $S_\alpha$ can be regarded as a $\widetilde{\mathcal{D}}$-class of $S$, we can just let $a \in S_\alpha$, $x, y \in S_\beta$. Recall that $S = (Y; S_\alpha)$ is a normal $\widetilde{\mathcal{H}}$-cryptogroup, $S/\widetilde{\mathcal{H}}$ is a normal band. Now, by the normality of the band $S/\widetilde{\mathcal{H}}$, we have

$$(axa)\widetilde{\mathcal{H}} = (a(xyx)a)\widetilde{\mathcal{H}} = (ayxya)\widetilde{\mathcal{H}} = (aya)\widetilde{\mathcal{H}}.$$

Thus, by Lemma 3.1, we see that $(x, y) \in \rho_{\alpha,\beta}$ and whence $\widetilde{\mathcal{D}}|_{S_\beta} \subseteq \rho_{\alpha,\beta}$. This proves that $S = \widetilde{\mathcal{D}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$. $(Sufficency)$ Let $S = \widetilde{\mathcal{D}}G[Y; S_\alpha, \varphi_{\alpha,\beta}]$, where each $S_\alpha$ is a completely $\widetilde{\mathcal{J}}$ -simple semigroup, for all $\alpha \in Y$. Then by definition, $S$ is an $\widetilde{\mathcal{L}}G$-strong semilattice of semigroups $S_\alpha$ and also $S$ is an $\widetilde{\mathcal{R}}G$-strong semilattice of semigroups $S_\alpha$ . By applying Theorem 4.2 and its dual, we immediately deduce that $\widetilde{\mathcal{H}}$ is a congruence on $S$ and for all $a, x, y \in S$, we have

$$[(axy)a]\widetilde{\mathcal{H}} = [ay(xya)]\widetilde{\mathcal{H}} = (ayxyxa)\widetilde{\mathcal{H}} = (ayxa)\widetilde{\mathcal{H}}.$$

This shows that $S/\widetilde{\mathcal{H}}$ is a normal band. Moreover, since each $S_\alpha$ is a $\widetilde{\mathcal{D}}$-class of $S$, for every $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$, the set $D(\alpha, \beta)$ is just a singleton. This means that $S$ is a strong semilattice of completely $\widetilde{\mathcal{J}}$-simple semigroups $S_\alpha$. Our proof is completed.

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