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Characterization of $U_1(\mathbb{Z}[C_n \times K_4])$

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Abstract. Constructing the group of units $U(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$, for a finite group G, is a classical but open problem. In this study, it is shown that

$$U_1(\mathbb{Z}[C_n \times K_4]) = U_1(\mathbb{Z}C_n) \times (1 + K^x) \times (1 + K^y) \times (1 + K^{xy}).$$

This structure theorem is applied to give precise characterization of $U_1(\mathbb{Z}[C_n \times K_4])$ for cyclic groups C_5 and C_7.

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1. Introduction

Let us denote $\mathbb{Z}A$ the integral group ring of a finite abelian group A with the coefficients from the ring of integers $\mathbb{Z}$. Let $U(\mathbb{Z}A)$ be the group of units in $\mathbb{Z}A$. Higman [4] obtained the following result:

Theorem 1. *If A is a finite abelian group then*

$$U(\mathbb{Z}A) = \pm A \times F,$$

where F is a free abelian group.

Here torsion units are trivial, torsion free units are finite but the rank is not determined. The rank of torsion free part is determined by Ayoub and Ayoub [2].

Theorem 2. *If A is a finite abelian group then $U(\mathbb{Z}A) = \pm A \times F$ with the rank*

$$\rho = \frac{1}{2}(|A| + 1 + n_2 - 2l), \tag{1}$$

where n_2 is the number of elements of A of order 2 and l is the number of cyclic subgroups of A.

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The structures of the unit groups for $\mathbb{Z}C_5$, $\mathbb{Z}C_8$ were given by Karpilovsky [5] as follows:

$$U(\mathbb{Z}C_5) = \pm\, C_5 \times < -1 + a + a^4 > \text{ and}$$
$$U(\mathbb{Z}C_8) = \pm\, C_8 \times < 2 + (a + a^7) - (a^3 + a^5) - a^4 > .$$

Aleev and Panina [1] described the structure of $U(\mathbb{Z}C_7)$ and $U(\mathbb{Z}C_9)$:

$$U(\mathbb{Z}C_7) = \pm\, C_7 \times < -1 + a + a^6, -1 + 2a^2 - a^3 - a^4 + 2a^5 > \text{ and}$$
$$U(\mathbb{Z}C_9) = \pm\, C_9 \times < -1 - (a + a^8) - (a^2 + a^7) + 2(a^4 + a^5) > \times < -1 - (a + a^8) + (a^2 + a^7) > .$$

The unit group $U(\mathbb{Z}C_{12})$ was characterized by Bilgin [3] as,

$$U(\mathbb{Z}C_{12}) = \pm C_{12} \times < 3 + 2(a + a^{11}) + (a^2 + a^{10}) - (a^4 + a^8) - 2(a^5 + a^7) - 2a^6 > .$$

Low [6] gave a generalization of the structure of the the unit group for $C_n \times C_2$ using exact sequences.

Remark 1. *Since $U(\mathbb{Z}G) = \pm U_1(\mathbb{Z}G)$, we will use $U_1(\mathbb{Z}G)$ instead of $U(\mathbb{Z}G)$.*

In this study, we extend group epimorphisms linearly over $\mathbb{Z}$ to ring epimorphisms in the first place. After that, we determine their kernels to construct exact sequences at ring level. Then, by restricting these exact sequences to unit level, we characterize $U_1(\mathbb{Z}[C_n \times K_4])$ as an internal direct product of four subgroups. Finally we describe these subgroups explicitly and give two concrete examples for cyclic groups C_5 and C_7.

2. Main Structure Theorem

We can construct the following group epimorphisms by using abelian group

$$C_n \times K_4 = < a, x, y : a^n = x^2 = y^2 = 1, ax = xa, ay = ya, xy = yx >$$

as follows:

$$\begin{array}{cc} \pi_x : C_n \times K_4 \to C_n \times \langle y \rangle, & \pi_y : C_n \times K_4 \to C_n \times \langle x \rangle \\ a \mapsto a & a \mapsto a \\ x \mapsto 1 & x \mapsto x \\ y \mapsto y & y \mapsto 1 \end{array} .$$

If we denote the identity map by ι, then we get the following exact sequences:

$$\langle x \rangle \xrightarrow{\iota} C_n \times K_4 \xrightarrow{\pi_x} C_n \times \langle y \rangle,$$
$$\langle y \rangle \xrightarrow{\iota} C_n \times K_4 \xrightarrow{\pi_y} C_n \times \langle x \rangle .$$

By extending these epimorphisms linearly over $\mathbb{Z}$, the following ring epimorphisms are obtained:

$$\begin{aligned}\pi_x :& \mathbb{Z}[C_n \times K_4] \longrightarrow \mathbb{Z}[C_n \times \langle y \rangle] \\ & P_0 + P_1 x + P_2 y + P_3 xy \mapsto (P_0 + P_1) + (P_2 + P_3) y \\ \pi_y :& \mathbb{Z}[C_n \times K_4] \longrightarrow \mathbb{Z}[C_n \times \langle x \rangle] \\ & P_0 + P_1 x + P_2 y + P_3 xy \mapsto (P_0 + P_2) + (P_1 + P_3) x.\end{aligned}$$

Then, we can calculate the kernels of the epimorphisms.

$$\begin{aligned}N^x =& Ker\pi_x \\ =& \{P = P_0 + P_1 x + P_2 y + P_3 xy \in \mathbb{Z}C_n : \pi_x(P) = 0\} \\ =& \{P = P_0 + P_1 x + P_2 y + P_3 xy \in \mathbb{Z}C_n : (P_0 + P_1) + (P_2 + P_3) y = 0\} \\ =& \{P = P_0 + P_1 x + P_2 y + P_3 xy \in \mathbb{Z}C_n : P_0 = -P_1,\ P_2 = -P_3\} \\ =& \{(x-1)P_1 + y(x-1)P_3 : P_1, P_3 \in \mathbb{Z}C_n\} \\ =& \{(x-1)[P_1 + yP_3] : P_1, P_3 \in \mathbb{Z}C_n\} \\ =& (x-1)\mathbb{Z}[C_n \times \langle y \rangle].\end{aligned}$$

Similarly $N^y = (y-1)\mathbb{Z}[C_n \times \langle x \rangle]$. By restricting π_x and π_y to the kernels N^y and N^x we get the images $K^y = (y-1)\mathbb{Z}C_n$ and $K^x = (x-1)\mathbb{Z}C_n$ respectively. Of course, K^y is the kernel of π_y and K^x is also the kernel of π_x. On the other hand, the ring

$$K^{xy} = \{(x-1)(y-1)P_3 : P_3 \in \mathbb{Z}C_n\}$$

is the kernel of ring epimorphism :

$$\begin{aligned}\pi_y :& N^x \longrightarrow K^x \\ & (x-1)[P_1 + yP_3] \mapsto (x-1)[P_1 + P_3],\end{aligned}$$

while K^{xy} is the kernel of another ring epimorphism :

$$\begin{aligned}\pi_x :& N^y \longrightarrow K^y \\ & (y-1)[P_2 + xP_3] \mapsto (y-1)[P_2 + P_3].\end{aligned}$$

Hence, we have the following commutative diagram at ring level:

$$\begin{array}{ccccc} K^{xy} & \xrightarrow{\iota} & N^x & \xrightarrow{\pi_y} & K^x \\ {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota}\downarrow \\ N^y & \xrightarrow{\iota} & \mathbb{Z}[C_n \times K_4] & \xrightarrow{\pi_y} & \mathbb{Z}[C_n \times \langle x \rangle] \\ {\scriptstyle\pi_x}\downarrow & & {\scriptstyle\pi_x}\downarrow & & {\scriptstyle\pi_x}\downarrow \\ K^y & \xrightarrow{\iota} & \mathbb{Z}[C_n \times \langle y \rangle] & \xrightarrow{\pi_y} & \mathbb{Z}C_n. \end{array}$$

Theorem 3. $U_1(\mathbb{Z}[C_n \times K_4]) = U_1(\mathbb{Z}C_n) \times (1+K^x) \times (1+K^y) \times (1+K^{xy})$.

Proof. By restricting π_x and π_y to the the unit group $U_1(\mathbb{Z}[C_n \times K_4])$, we get the following commutative diagram for groups.

$$\begin{array}{ccccc}
1+K^{xy} & \xrightarrow{\iota} & 1+N^x & \xrightarrow{\pi_y} & 1+K^x \\
{\scriptstyle\iota}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota}\downarrow \\
1+N^y & \xrightarrow{\iota} & U_1(\mathbb{Z}[C_n \times K_4]) & \xrightarrow{\pi_y} & U_1(\mathbb{Z}[C_n \times \langle x \rangle]) \\
{\scriptstyle\pi_x}\downarrow & & {\scriptstyle\pi_x}\downarrow & & {\scriptstyle\pi_x}\downarrow \\
1+K^y & \xrightarrow{\iota} & U_1(\mathbb{Z}[C_n \times \langle y \rangle]) & \xrightarrow{\pi_y} & U_1(\mathbb{Z}C_n).
\end{array}$$

In the diagram, each row and column are exact sequences. If we define τ as the identity function in the reverse directions of π_x and π_y, we can say that each exact sequence splits. Thus, from column-wise split-short exact sequences we can write,

$$\begin{aligned}
1+N^y =&(1+K^{xy}) \times (1+K^y), \\
U_1(\mathbb{Z}[C_n \times K_4]) =&(1+N^x) \times U_1(\mathbb{Z}[C_n \times \langle y \rangle]), \\
U_1(\mathbb{Z}[C_n \times \langle x \rangle]) =&(1+K^x) \times U_1(\mathbb{Z}C_n).
\end{aligned}$$

Equivalently, from row-wise split-short exact sequences we get

$$\begin{aligned}
1+N^x =&(1+K^{xy}) \times (1+K^x), \\
U_1(\mathbb{Z}[C_n \times K_4]) =&(1+N^y) \times U_1(\mathbb{Z}[C_n \times \langle x \rangle]), \\
U_1(\mathbb{Z}[C_n \times \langle y \rangle]) =&(1+K^y) \times U_1(\mathbb{Z}C_n).
\end{aligned}$$

Finally, the unit group $U_1(\mathbb{Z}[C_n \times K_4])$ can be described as an internal direct product of four subgroups :

$$\begin{aligned}
U_1(\mathbb{Z}[C_n \times K_4]) =&(1+N^x) \times U_1(\mathbb{Z}[C_n \times \langle y \rangle]) \\
&= U_1(\mathbb{Z}C_n) \times (1+K^x) \times (1+K^y) \times (1+K^{xy}).
\end{aligned}$$

□

Lemma 1. *In $U_1(\mathbb{Z}[C_n \times K_4])$, the subgroups $(1+K^x), (1+K^y), (1+K^{xy})$ satisfy the following conditions.*

(i) $1+K^x = \{1+(x-1)P : 1-2P \in U_1(\mathbb{Z}C_n)\}$.

(ii) $1+K^y = \{1+(y-1)P : 1-2P \in U_1(\mathbb{Z}C_n)\}$.

(iii) $1+K^{xy} = \{1+(x-1)(y-1)P : 1+4P \in U_1(\mathbb{Z}C_n)\}$.

Proof. $u \in 1+K^x \Leftrightarrow u = 1+(x-1)P$ and there exists $v = 1+(x-1)Q$ for some $P, Q \in \mathbb{Z}C_n$ such that $u.v = 1$ then

$$\begin{aligned} uv = 1 &\Leftrightarrow [1+(x-1)P][1+(x-1)Q] = 1 \\ &\Leftrightarrow 1+(x-1)[P+Q-2PQ] = 1 \\ &\Leftrightarrow P+Q-2PQ = 0 \\ &\Leftrightarrow 1-2P-2Q+4PQ = 1 \\ &\Leftrightarrow (1-2P)(1-2Q) = 1 \\ &\Leftrightarrow 1-2P \in U_1(\mathbb{Z}C_n). \end{aligned}$$

Similarly we can see that $1+K^y = \{1+(y-1)P : 1-2P \in U_1(\mathbb{Z}C_n)\}$ and
$1+K^{xy} = \{1+(x-1)(y-1)P : 1+4P \in U_1(\mathbb{Z}C_n)\}$. □

Now consider the surjective ring homomorphism $\rho_m : \mathbb{Z}C_n \longrightarrow \mathbb{Z}_m C_n$, where ρ_m reduces the coefficients modulo m. If we denote the kernel of ρ_m by M_m, we have $M_m = (m\mathbb{Z})C_n$ and the following exact sequence is obtained at ring level:

$$M_m = (m\mathbb{Z})C_n \xrightarrow{\iota} \mathbb{Z}C_n \xrightarrow{\rho_m} \mathbb{Z}_m C_n.$$

By restricting to the unit group $U_1(\mathbb{Z}C_n)$ we get exact sequence of groups

$$1+M_m \xrightarrow{\iota} U_1(\mathbb{Z}C_n) \xrightarrow{\rho_m} U(\mathbb{Z}_m C_n).$$

We can define two group isomorphisms,

$$\begin{array}{cc} \sigma_x : C_n \times K_4 \to \pm C_n \times \langle y \rangle, & \sigma_y : C_n \times K_4 \to \pm C_n \times \langle x \rangle \\ a \mapsto a & a \mapsto a \\ x \mapsto -1 & x \mapsto x \\ y \mapsto y & y \mapsto -1 \end{array}$$

By extending these isomorphisms linearly over $\mathbb{Z}$, we get the following ring epimorphisms:

$$\begin{aligned} \sigma_x :& \mathbb{Z}[C_n \times K_4] \longrightarrow \mathbb{Z}[C_n \times \langle y \rangle] \\ & P_0 + P_1 x + P_2 y + P_3 xy \mapsto (P_0 - P_1) + (P_2 - P_3)y, \\ \sigma_y :& \mathbb{Z}[C_n \times K_4] \longrightarrow \mathbb{Z}[C_n \times \langle x \rangle] \\ & P_0 + P_1 x + P_2 y + P_3 xy \mapsto (P_0 - P_2) + (P_1 - P_3)x. \end{aligned}$$

This leads to the diagrams at ring level. For K^x we write

$$\begin{array}{ccccc} K^x & \xrightarrow{\iota} & Z[C_n \times < x >] & \xrightarrow{\pi_x} & ZC_n \\ {\scriptstyle \sigma_x} \downarrow & & {\scriptstyle \sigma_x} \downarrow & & {\scriptstyle \rho_2} \downarrow \\ M_2 & \xrightarrow{\iota} & ZC_n & \xrightarrow{\rho_2} & Z_2 C_n, \end{array}$$

for K^y we have

$$\begin{array}{ccccc} K^y & \xrightarrow{\iota} & \mathbb{Z}[C_n\times <y>] & \xrightarrow{\pi_y} & \mathbb{Z}C_n \\ \sigma_y \downarrow & & \sigma_y \downarrow & & \rho_2 \downarrow \\ M_2 & \xrightarrow{\iota} & \mathbb{Z}C_n & \xrightarrow{\rho_2} & \mathbb{Z}_2C_n, \end{array}$$

and for K^{xy} we get

$$\begin{array}{ccccc} K^{xy} & \xrightarrow{\iota} & \mathbb{Z}[C_n\times K_4] & \xrightarrow{\pi_x\pi_y} & \mathbb{Z}C_n \\ \sigma_x\sigma_y \downarrow & & \sigma_x\sigma_y \downarrow & & \rho_4 \downarrow \\ M_4 & \xrightarrow{\iota} & \mathbb{Z}C_n & \xrightarrow{\rho_4} & \mathbb{Z}_4C_n. \end{array}$$

Corollary 1. *The following maps are group isomorphisms:*

(i) $\sigma_x : 1+K^x \longrightarrow 1+M_2$,

(ii) $\sigma_y : 1+K^y \longrightarrow 1+M_2$,

(iii) $\sigma_x\sigma_y : 1+K^{xy} \longrightarrow 1+M_4$.

Proof. Consider the following diagram

$$\begin{array}{ccccc} 1+K^x & \xrightarrow{\iota} & U_1(\mathbb{Z}[C_n\times <x>]) & \xrightarrow{\pi_x} & U_1(\mathbb{Z}C_n) \\ \sigma_x \downarrow & & \sigma_x \downarrow & & \rho_2 \downarrow \\ 1+M_2 & \xrightarrow{\iota} & U_1(\mathbb{Z}C_n) & \xrightarrow{\rho_2} & U_1(\mathbb{Z}_2C_n). \end{array}$$

Here if we restrict the ring homomorphism σ_x to the unit group $1+K^x$, we get the group homomorphism $\sigma_x(1+(x-1)P)=1-2P$. By Lemma 1 σ_x is surjective. For $u\in 1+K^x$,

$$\begin{aligned} u\in Ker\sigma_x &\Leftrightarrow u=1+(x-1)P \text{ and } \sigma_x(u)=1 \\ &\Leftrightarrow 1-2P=1 \text{ and } P\in\mathbb{Z}C_n \\ &\Leftrightarrow P=0 \\ &\Leftrightarrow u=1. \end{aligned}$$

Hence σ_x is injective. Similarly applying the same method to σ_y and $\sigma_x\sigma_y$ one can see that $1+K^y\cong 1+M_2$ and $1+K^{xy}\cong 1+M_4$ respectively. □

Remark 2.

$$\begin{aligned} f :& \mathbb{Z}[C_n\times K_4]\longrightarrow \mathbb{Z}[C_n\times K_4] \\ & P_0+P_1x+P_2y+P_3xy \mapsto P_0+P_2x+P_1y+P_3xy \end{aligned}$$

is a ring isomorphism.

3. Applications

Theorem 4.

$$U_1(\mathbb{Z}[C_5 \times K_4]) = C_5 \times K_4 \times <v> \times <1+(x-1)P> \times <1+(y-1)P> \\ \times <1+(x-1)(y-1)Q>,$$

where $v = -1 + a + a^4$, $P = 4 - 3(a+a^4) + (a^2+a^3)$ *and* $Q = 32 - 26(a+a^4) + 10(a^2+a^3)$.

Proof. Karpilovsky [5] showed if $C_5 =< a : a^5 = 1 >$ then $U_1(\mathbb{Z}C_5) = C_5 \times < v >$, where $v = -1 + a + a^4$. In order to describe $1 + K^x$ consider the following commutative diagram :

$$\begin{array}{ccccc} 1+K^x & \xrightarrow{\iota} & U_1(\mathbb{Z}[C_5 \times <x>]) & \xrightarrow{\pi_x} & U_1(\mathbb{Z}C_5) \\ {}^{\sigma_x}\downarrow & & {}^{\sigma_x}\downarrow & & {}^{\rho_2}\downarrow \\ 1+M_2 & \xrightarrow{\iota} & U_1(\mathbb{Z}C_5) & \xrightarrow{\rho_2} & U_1(\mathbb{Z}_2C_5) \\ \iota\uparrow & & \iota\uparrow & & \iota\uparrow \\ (1+M_2)\cap F & \xrightarrow{\iota} & F & \xrightarrow{\rho_2} & \rho_2(F). \end{array}$$

Since $F =< v >$, $\rho_2(F) =< 1 + a + a^4 >= \{1 + a + a^4, 1 + a^2 + a^3, 1\}$. So, we get

$$(1+M_2)\cap F = Ker\rho_2 = \{u \in F : \rho_2(u) = 1\} =< v^3 > .$$

On the other hand, $u \in 1 + K^x \Rightarrow u = 1 + (x-1)P$, $(P \in \mathbb{Z}C_5)$. Since σ_x is an isomorphism and $\sigma_x(u) = 1 - 2P$, we conclude that $1 - 2P = v^3$. This leads us to,

$$P = 4 - 3(a + a^4) + (a^2 + a^3).$$

Consequently we get the second generator as

$$u = 1 + (x-1)[4 - 3(a + a^4) + (a^2 + a^3)].$$

By Remark 2 we write third generator as

$$1 + K^y =< 1 + (y-1)[4 - 3(a + a^4) + (a^2 + a^3)] > .$$

In order to construct $1 + K^{xy}$, consider the following commutative diagram:

$$\begin{array}{ccccc} 1+K^{xy} & \xrightarrow{\iota} & 1+N^y & \xrightarrow{\pi_x} & 1+K^y \\ {}^{\sigma_x\sigma_y}\downarrow & & {}^{\sigma_x\sigma_y}\downarrow & & {}^{\sigma_y}\downarrow \\ 1+M_4 & \xrightarrow{\iota} & U_1(\mathbb{Z}C_5) & \xrightarrow{\rho_4} & U_1(\mathbb{Z}_4C_5) \\ \iota\uparrow & & \iota\uparrow & & \iota\uparrow \\ (1+M_4)\cap F & \xrightarrow{\iota} & F & \xrightarrow{\rho_4} & \rho_4(F). \end{array}$$

As $F =< v >$ and $\rho_4(F) =< -1 + a + a^4 >$ we write,

$$(1+M_4)\cap F = Ker\rho_4 = \{u \in F : \rho_4(u) = 1\} =< v^6 > .$$

For $u \in 1+K^{xy} \Rightarrow u = 1+(x-1)(y-1)Q$, $(Q \in \mathbb{Z}C_5)$. Since $\sigma_x\sigma_y$ is an isomorphism and $(\sigma_x\sigma_y)(u) = 1+4Q$, we conclude that $1+4Q = v^6$. That is,

$$Q = 32-26(a+a^4)+10(a^2+a^3),$$

then the last generator is

$$u = 1+(x-1)(y-1)[32-26(a+a^4)+10(a^2+a^3)].$$

□

Theorem 5.

$$\begin{aligned} U_1(\mathbb{Z}[C_7\times K_4]) =& C_7\times K_4 \times <v_1, v_2> \times <1+(x-1)P_1> \times <1+(x-1)P_2> \\ & \times <1+(y-1)P_1> \times <1+(y-1)P_2> \\ & \times <1+(x-1)(y-1)Q_1> \times <1+(x-1)(y-1)Q_2>, \end{aligned}$$

where

$$\begin{aligned} v_1 =& -1+(a+a^6), \\ v_2 =& -1+(a^2+a^6), \\ P_1 =& -4+(a+a^6)+4(a^2+a^5)-3(a^3+a^4), \\ P_2 =& 4-(a+a^6)-3(a^2+a^5)+2(a^3+a^4), \\ Q_1 =& 72-16(a+a^6)-65(a^2+a^5)+45(a^3+a^4), \\ Q_2 =& 40-9(a+a^6)-36(a^2+a^5)+25(a^3+a^4). \end{aligned}$$

Proof. Let $C_7 =< a : a^7 = 1 >$. Aleev and Panina [1] showed that,

$$U_1(\mathbb{Z}C_7) = C_7 \times <-1+a+a^6, -1+2a^2-a^3-a^4+2a^5> .$$

Since $(-1+a+a^6)^2(-1+a^2+a^5) = -1+2a^2-a^3-a^4+2a^5$, we can write

$$<-1+a+a^6, -1+2a^2-a^3-a^4+2a^5> = <-1+a+a^6, -1+a^2+a^5> .$$

Take $v_1 = -1+(a+a^6)$ and $v_2 = -1+(a^2+a^6)$.

To describe $1+K^x$, consider the following commutative diagram :

$$\begin{array}{ccccc} 1+K^x & \stackrel{\iota}{\longrightarrow} & U_1(\mathbb{Z}[C_7\times <x>]) & \stackrel{\pi_x}{\longrightarrow} & U_1(\mathbb{Z}C_7) \\ {\scriptstyle\sigma_x}\downarrow & & {\scriptstyle\sigma_x}\downarrow & & {\scriptstyle\rho_2}\downarrow \\ 1+M_2 & \stackrel{\iota}{\longrightarrow} & U_1(\mathbb{Z}C_7) & \stackrel{\rho_2}{\longrightarrow} & U_1(\mathbb{Z}_2C_7) \\ \iota\uparrow & & \iota\uparrow & & \iota\uparrow \\ (1+M_2)\cap F & \stackrel{\iota}{\longrightarrow} & F & \stackrel{\rho_2}{\longrightarrow} & \rho_2(F) \end{array}$$

$F=<v_1,v_2>$ implies $\rho_2(F)=<\rho_2(v_1),\rho_2(v_2)>=<1+a+a^6,1+a^2+a^5>$. Since $(1+a+a^6)(1+a^2+a^5)^3=1$ and $(1+a+a^6)^3(1+a^2+a^5)^2=1$. The kernel of ρ_2 is

$$(1+M_2)\cap F=<v_1v_2^3,v_1^3v_2^2>.$$

On the other hand, $u\in 1+K^x \Rightarrow u=1+(x-1)P$, $(P\in\mathbb{Z}C_7)$. Since σ_x is an isomorphism and $\sigma_x(u)=1-2P$, we conclude that

$$\left.\begin{array}{l}1-2P_1=v_1v_2^3\\ 1-2P_2=v_1^3v_2^2\end{array}\right\}\Rightarrow \left.\begin{array}{l}P_1=-4+(a+a^6)+4(a^2+a^5)-3(a^3+a^4)\\ P_2=4-(a+a^6)-3(a^2+a^5)+2(a^3+a^4)\end{array}\right\}.$$

Hence $1+K^x=<1+(x-1)P_1,1+(x-1)P_2>$. By Remark 2 we have $1+K^y=<1+(y-1)P_1,1+(y-1)P_2>$. In order to construct $1+K^{xy}$ consider the following commutative diagram :

$$\begin{array}{ccccc}1+K^{xy} & \xrightarrow{\iota} & 1+N^y & \xrightarrow{\pi_x} & 1+K^y\\ \sigma_x\sigma_y\downarrow & & \sigma_x\sigma_y\downarrow & & \sigma_y\downarrow\\ 1+M_4 & \xrightarrow{\iota} & U_1(\mathbb{Z}C_7) & \xrightarrow{\rho_4} & U_1(\mathbb{Z}_4C_7)\\ \iota\uparrow & & \iota\uparrow & & \iota\uparrow\\ (1+M_4)\cap F & \xrightarrow{\iota} & F & \xrightarrow{\rho_4} & \rho_4(F)\end{array}$$

Since $F=<v_1,v_2>$, $\rho_4(F)=<-1+a+a^6,-1+a^2+a^5>$. Then, the kernel of ρ_4 is

$$\left.\begin{array}{l}(-1+a+a^6)^2(-1+a^2+a^5)^6=1\\ (-1+a+a^6)^6(-1+a^2+a^5)^4=1\end{array}\right\}\Rightarrow (1+M_4)=<v_1^2v_2^6,v_1^6v_2^4>.$$

For $u\in 1+K^{xy}\Rightarrow u=1+(x-1)(y-1)Q$, $(Q\in\mathbb{Z}C_7)$. Since $\sigma_x\sigma_y$ is an isomorphism and $(\sigma_x\sigma_y)(u)=1+4Q$, we conclude that

$$\left.\begin{array}{l}1+4Q_1=v_1^2v_2^6\\ 1+4Q_2=v_1^6v_2^4\end{array}\right\}\Rightarrow \begin{array}{l}Q_1=72-16(a+a^6)-65(a^2+a^5)+45(a^3+a^4),\\ Q_2=40-9(a+a^6)-36(a^2+a^5)+25(a^3+a^4).\end{array}$$

Thus

$$1+K^{xy}=<1+(x-1)(y-1)Q_1,1+(x-1)(y-1)Q_2>.$$

□

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