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Extended Concavifications and Exact Games

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Abstract. In this paper we propose new version of cooperative games. In fact the notion of cooperative games and their concavifications are extended. As a consequence, in this new setting it turn out that $coreV \neq \emptyset$ if and only if $\mathbf{cav}(u)(C_\Omega) = u(C_\Omega)$.

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1. Introduction

Usually, a game V with a continum players is a bounded real valued function defined on $\sum$ the Borel subsets of $I = [0,1]$ such that $V(\emptyset) = 0$. Any member of $\sum$ is interpreted as coalition of player, $V(R)$ gives the maximum payoff achieved by efforts of all members in the coalition R. Of course with this interpretation usually it is assumed that V is non-negative and not identically zero. In [1] a cooperative game is viewed as a real valued function u defined on a finite set of points in the unit simplex, also a concavification of u used to characterize well-known classes of games.

2. Preliminaries

Let X be a normed space. The space of all continuous linear functionals defined on X is called the dual space of X and denoted by X^*. Let $\langle .,. \rangle : X \times X^* \to \mathbb{R}$ be the duality pairing in $X \times X^*$. The weakest topology on X that make continuous all elements of $x^* \in X^*$ is called the weak topology on X. Let $\phi : X \to X^{**}$ defined by $\phi(x) = g_x$ where $g_x(x^*) = \langle x^*, x \rangle$, $x^* \in X^*$ and $\|g_x\| = \|x\|$. The weakest topology on X^* that make continuous all $\phi(x)$ is called the weak* topology on X^*. The weak topology on X and the weak* topology on X^* are usually denoted by $\sigma(X, X^*)$ and $\sigma(X^*, X)$ respectively.

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Definition 1. *Let X be a normed space,*

(a) a net $\{x_n\}$ in X is called weak* convergent in X, if there exists an element $x \in X$ such that $\lim\limits_{n\to\infty} |x^*(x_n) - x^*(x)| = 0,\ \forall x^* \in X^*$;

(b) a subset A of X is called compact in weak* topology or weak* compact set if every net in A contains a subnet which is weak* convergent in A.

Definition 2. *A game V is called a balanced game if*

$$\sup \sum_{(R)} \alpha_R \mu(R) u(C_R) \leq u(C_\Omega),$$

where sup *is taken over all finite sums $\sum_{(R)} \alpha_R \mu(R) u(C_R)$, $\alpha_R \geq 0$ and $\sum_{(R)} \alpha_R \mu(R) = 1$.*

Definition 3. *Given such a function u, we consider the concavification of u, denoted by $\mathbf{cav}(u)$, which is a function defined on*

$$\Delta = \{g : g \geq 0, g \ \ is \ \ simple \ \ measurable \ \ function \ \ and \ \ \int_\Omega g d\mu = 1\},$$

as the infimum of all concave functions that are greater than or equal to u.

Since the infimum of a family of concave functions is concave, so $\mathbf{cav}(u)$ is concave and is greater than or equal to u as it is shown in Lemma 2.

Definition 4. *In the extended version of cooperative game, we consider a non-empty set Ω and a finite measure space $(\Omega, \sum, \mu)$, a game V is a bounded real valued function on $\sum$ such that $V(\emptyset) = 0$.*

For $R \in \sum$, we denote by χ_R the characteristic function of R. Let B be the Banach space spanned by the set $\{\chi_R : R \in \sum\}$ with the sup norm, where χ_R is the characteristic function of R. Then the space of all bounded additive functions on $\sum$ is denoted by BA would be isometrically isomorphic to the norm-dual of B. A payoff μ of V is an element of BA with $\mu(\Omega) = V(\Omega)$. The core of V consists of all payoffs μ such that $\mu(R) \geq V(R)$ for each $R \in \sum$. We can also identify the coalition R $C_R = \frac{\chi_R}{\mu(R)}$. Thus, the coalition will be identified with the uniform distribution over the members of R. A game V is converted a function u defined over the points C_R for $R \in \sum'$, where $\sum' = \{R \in \sum : \mu(R) \neq 0\}$. The value of u at C_R is the average of the worth of R, that is, $u(C_R) = \frac{V(R)}{\mu(R)}$.
We set

$$H = \{f : \Delta \to \mathbb{R} \mid \text{f is concave and } f \geq u \text{ on } \Delta'\}$$

where, $\Delta' = \{C_R : R \in \sum'\}$. For any $g \in \Delta$ we set

$$L_g = \{\textstyle\sum_{(R)} \alpha_R \mu(R) u(C_R) : g = \sum_{(R)} \alpha_R \chi_R \text{ and } \alpha_R > 0, \quad \sum_{(R)} \alpha_R \mu(R) = 1\}.$$

We can define two functions $w : \Delta \to \mathbb{R}$ and $\mathbf{cav}u : \Delta \to \mathbb{R}$ by $w(g) = \sup L_g$ and $\mathbf{cav}u(g) = \inf H(g)$.

3. Main Results

Theorem 1. *For any game V, $core(V)$ is bounded and weak* compact.*

Proof. For each $\lambda \in core(V)$, $0 \leq \lambda(R) \leq \lambda(\Omega) = V(\Omega)$, $(\forall R \in \sum)$. Therefore, $core(V)$ is bounded. For each net $(\lambda_\alpha) \subseteq core(V)$, since bounded sets in B are relatively weak* compact, so (λ_α) has a subnet $(\lambda_{\alpha_\beta})_{\beta\in I}$ which converges in weak* topology to $\lambda_0 \in B$. But $\lambda_0(\Omega) = \lim \lambda_{\alpha_\beta}(\Omega) = V(\Omega)$ and $\lambda_{\alpha_\beta}(R) \geq V(R)$ $(\forall R \in \sum)$, it shows that $\lambda_0(R) \geq V(R)$. Both implies that $\lambda_0 \in core(V)$. These facts imply that $core(V)$ is weak* compact.

Lemma 1. *w is a concave map.*

Proof. For $\epsilon > 0$ there are two elements $\sum_{(R)} \alpha_R \mu(R) u(C_R)$ and $\sum_{(R')} \beta_{R'} \mu(R') u(C_{R'})$ such that

$$\begin{aligned} tw(g) + (1-t)w(h) - \epsilon &= t[w(g) - \epsilon] + (1-t)[w(h) - \epsilon] \\ &< t\sum_{(R)} \alpha_R \mu(R) u(C_R) \\ &+ (1-t)\sum_{(R')} \beta_{R'} \mu(R') u(C_{R'}) \\ &\leq w(tg + (1-t)h). \end{aligned}$$

Lemma 2. $\mathbf{cav}(u)(C_R) \geq u(C_R)$*.*

Proof. By concavity of $\mathbf{cav}(u)$ and $f \in H$ it follows that $f \geq u$. Hence $\mathbf{cav}(u) \in H$ and $\mathbf{cav}(u)(C_R) \geq u(C_R)$, for any $R \in \sum$.

Proposition 1. *$w(g) = \mathbf{cav}(u)(g)$ for any $g \in \triangle$.*

Proof. Suppose $\sum_{(R)} \alpha_R \mu(R) u(C_R) \in L_g$. Choosing $g = \sum_{(R)} \alpha_R \chi_R$ such that $\alpha_R \geq 0$ and $\sum_{(R)} \alpha_R \mu(R) = 1$. Then

$$\begin{aligned} \mathbf{cav}(u)(g) &= \mathbf{cav}(u)(\sum_{(R)} \alpha_R \chi_R) \\ &= \mathbf{cav}(u)(\sum_{(R)} \alpha_R \mu(R) \frac{\chi_R}{\mu(R)}) \\ &\geq \sum_{(R)} \alpha_R \mu(R) \mathbf{cav}(u)(\frac{\chi_R}{\mu(R)}) \\ &\geq \sum_{(R)} \alpha_R \mu(R) u(C_R). \end{aligned}$$

This shows that $w(g) \leq \mathbf{cav}(u)(g)$. For the converse, we note that w is concave from Lemma 1, $w(C_R) \geq u(C_R)$. Therefore, $\mathbf{cav}(u) \leq w$.

Definition 5. *$\lambda \in BA$ is called linear support of $f : \triangle \to \mathbb{R}$ at $g \in \triangle$ if*

$$f(g) = \int_\Omega g d\lambda \text{ and } f(g') \leq \int_\Omega g' d\lambda \quad (\forall g' \in \triangle).$$

Proposition 2. *$cav(u)(C_\Omega) = u(C_\Omega)$ if $coreV \neq \emptyset$.*

Proof. $coreV \neq \emptyset$ implies that there is $\lambda \in BA$ which satisfies $\lambda(\Omega) = V(\Omega)$ and $\lambda(R) \geq V(R) \quad (\forall R \in \sum')$. Set $f : \triangle \to \mathbb{R}$ by $f(g) = \int_\Omega g d\lambda$. It is clear that f would be a concave map. On the other hand, $f(C_R) \geq u(C_R)$. Therefore, $f \in H$, so $cav(u)(C_R) \leq f(C_R)$ for any $R \in \sum'$. But $f(C_\Omega) = u(C_\Omega)$ which implies that, $\mathbf{cav}(u)(C_\Omega) \leq u(C_\Omega)$. So from lemma 2 $\mathbf{cav}(u)(C_\Omega) = u(C_\Omega)$.

Corollary 1. *λ is a linear support for $\mathbf{cav}(u)$ at C_Ω if $\lambda \in core(V)$.*

Proposition 3. *V is balanced game if $core(V) \neq \emptyset$.*

Proof. Assuming $coreV \neq \emptyset$ by proposition 2 yields $\mathbf{cav}(u)(C_\Omega) = u(C_\Omega)$. Since

$$\begin{aligned}\mathbf{cav}(u)(C_\Omega) &= w(C_\Omega) \\ &= \sup\{\sum_{(R)} \alpha_R \mu(R) u(C_R) : \Sigma \alpha_R \mu(R) = 1, \alpha_R \geq 0, \sum_{(R)} \alpha_R \chi_R = C_\Omega\},\end{aligned}$$

so $\sum_{(R)} \alpha_R \mu(R) u(C_R) \leq cav(u)(C_\Omega) = u(C_\Omega)$. Hence, $\sup \sum_{(R)} \alpha_R \mu(R) u(C_R) \leq u(C_\Omega)$.

Lemma 3. *Suppose that S is the set of all simple functions on $(\Omega, \sum, \mu)$ and $f : \triangle \to \mathbb{R}$ is the concave map. Then for any $g \in S$, there is a linear map G such that*

$$G(g) = f(g) \;\; and \;\; f(h) \leq G(h), (\forall h \in S)$$

Proof. The function $-f$ is a convex function. Now applying Hahn Banach Theorem for $L =< \{g\} >$ and $-f$, there is a linear function $F : S \to \mathbb{R}$ such that

$$F(g) = -f(g) \;\; and \;\; F(h) \leq -f(h), (\forall h \in S).$$

Set $G = -F$, then $G(g) = f(g) \;\; and \;\; f(h) \leq G(h), (\forall h \in S)$.

Theorem 2. *$coreV \neq \emptyset$ if $cav(u)(C_\Omega) = u(C_\Omega)$(Here, we have not assumed that the elements of $core(V)$ are bounded).*

Proof. Since $cav(u)$ is a concave map, so from lemma 1, there is a linear map $G : S \to \mathbb{R}$ such that $\mathbf{cav}(u)(C_\Omega) = G(C_\Omega)$ and $\mathbf{cav}(u)(C_R) \leq G(C_R)$. Then $G(C_\Omega) = u(C_\Omega) = \frac{V(\Omega)}{\mu(\Omega)}$ and $u(C_R) \leq \mathbf{cav}(u)(C_R) \leq G(C_R)$. Define $\lambda : \sum \to \mathbb{R}$ by $\lambda(R) = G(\chi_R)$. It easy to see that λ is a finitely additive measure. Moreover,

$$\lambda(\Omega) \quad = \quad G(\chi_\Omega) = \mu(\Omega) G(\frac{\chi_\Omega}{\mu(\Omega)})$$

$$\begin{aligned} &= \mu(\Omega)G(C_\Omega) = \mu(\Omega)\frac{V(\Omega)}{\mu(\Omega)} \\ &= V(\Omega). \end{aligned}$$

Also

$$\begin{aligned} \lambda(R) &= G(\chi_R) = \mu(R)G(\frac{\chi_R}{\mu(R)}) \\ &= \mu(R)G(C_R) \geq \mu(R)u(C_R) \\ &= \mu(R)\frac{V(R)}{\mu(R)} = V(R). \end{aligned}$$

Therefore, $\lambda \in core(V)$ which it completes the proof.

Definition 6. [4] *A game V is called an exact game if for each coalition R there is $\lambda \in core(V)$ such that $\lambda(R) = V(R)$.*

Theorem 3. *Suppose V is an exact game. Then u is continuous at C_Ω if and only if each $\lambda \in core(V)$ is countably additive.*

Proof. It is well known that $\lambda \in BA$ is countably additive if and only if it is continuous at Ω. Assume $\lambda \in core(V)$, u is continuous at C_Ω and $(R_n)_n$ is a monotone sequence in Ω such that $\bigcup R_n = \Omega$. We must show that $\lambda(R_n) \to \lambda(\Omega)$. From the assumption $u(C_{R_n}) \to u(C_\Omega)$. But $u(C_{R_n}) = \frac{V(R_n)}{\mu(R_n)} \leq \frac{\lambda(R_n)}{\mu(R_n)} \leq \frac{\lambda(\Omega)}{\mu(R_n)} = \frac{V(\Omega)}{\mu(R_n)}$. Tending $n \to \infty$ and since $u(C_{R_n})$ and $\frac{V(\Omega)}{\mu(R_n)} \to u(C_\Omega)$, so $\frac{\lambda(R_n)}{\mu(R_n)} \to u(C_\Omega) = \frac{V(\Omega)}{\mu(\Omega)}$. But μ is a measure so $\mu(R_n) \to \mu(\Omega)$. This shows that $\lambda(R_n) \to V(\Omega) = \lambda(\Omega)$. For the converse, we assume that $\lambda \in coreV$ is countably additive, $(R_n) \subseteq \sum', \bigcup R_n = \Omega$ and a is a limit point for $(u(C_{R_n}))_n$. Without loss of generality one can assume that $u(C_{R_n}) \to a$ (otherwise we can pass to a subsequence). From exactness of V for each R_n there is $\lambda_n \in coreV$ such that $\lambda_n(R_n) = V(R_n)$. From the compactness of $core(V)$, one can assume $\lambda_n \to \lambda$, where $\lambda \in coreV$. Assume $\epsilon > 0$, there is $k \in \mathbb{N}$ satisfying in $\lambda(R_n) > \lambda(\Omega) - \epsilon\mu(\Omega)$ for any $n \geq k$. There is $m' \in \mathbb{N}$ such that $|\lambda_m(R_n) - \lambda(R_n)| < \epsilon$ and so $\lambda(R_n) < \lambda_m(R_n) + \epsilon$ for each $m \geq m'$. Consider $n \geq k$ and $l' \geq \max\{n, m'\}$, now for each $l \geq l'$,

$$\begin{aligned} u(C_\Omega) &= \frac{V(\Omega)}{\mu(\Omega)} = \frac{\lambda(\Omega)}{\mu(\Omega)} < \frac{\lambda(R_n) + \epsilon\mu(\Omega)}{\mu(\Omega)} \\ &= \frac{\lambda(R_n)}{\mu(\Omega)} + \epsilon < \frac{\lambda_l(R_n)}{\mu(\Omega)} + 2\epsilon \\ &\leq \frac{\lambda_l(R_l)}{\mu(\Omega)} + 2\epsilon = \frac{V(R_l)}{\mu(\Omega)} + 2\epsilon \\ &\leq \frac{V(R_l)}{\mu(R_l)} + 2\epsilon = u(C_l) + 2\epsilon \\ &= a + 2\epsilon. \end{aligned}$$

Therefore, $u(C_\Omega) \leq a$. On the other hand, $u(C_{R_n}) = \frac{V(R_n)}{\mu(R_n)} \leq \frac{\lambda(R_n)}{\mu(R_n)} \leq \frac{\lambda(\Omega)}{\mu(R_n)}$. Let $n \to \infty$, then $a \leq \frac{\lambda(\Omega)}{\mu(\Omega)} = \frac{V(\Omega)}{\mu(\Omega)} = u(C_\Omega)$. Hence, $a = u(C_\Omega)$.

Set $BA_R = \{\lambda \in BA, \lambda(R) = V(R)\}$. Then if $\lambda \in BA_R$ we can define $f_\lambda(g) = \int_\Omega g d\lambda$. So we define $core_R V = \{\lambda \in core(V); \lambda(R) = V(R)\}$.

Lemma 4. *$BA_R \neq \emptyset$ if $R \in \sum$ and $R \neq \emptyset$.*

Proof. It is easy to see that, there is $\lambda_0 \in BA$, such that $\lambda_0(S) \neq 0$. Set $\lambda = \frac{V(S)}{\lambda_0(S)}\lambda_0$. Then $\lambda \in BA_R$.

Lemma 5. *(a) Let $\lambda \in BA_R$ and $f_\lambda : \triangle \to \mathbb{R}$ by $f_\lambda(g) = \int_\Omega g d\lambda$. Then $f_\lambda(C_R) = u(C_R)$.*

(b) Let $\lambda \in core_R(V)$, then $f_\lambda(C_R) = u(C_R)$ and $f_\lambda(C_S) \geq u(C_S)$, $(\forall S \in \sum)$.

Proof.

(a) $f_\lambda(C_R) = \int_\Omega C_R d\lambda = \frac{\lambda(R)}{\mu(R)} = \frac{V(R)}{\mu(R)} = u(C_R)$.

(b) It is similar to (a) $f_\lambda(C_R) = u(C_R)$. For an arbitrary element $S \in \sum$, $f_\lambda(C_S) = \int_\Omega C_S d\lambda = \frac{\lambda(S)}{\mu(S)} \geq \frac{V(S)}{\mu(S)} = u(C_S)$.

Theorem 4. *Suppose that V is an exact game. Then $u = \inf\{f_\lambda : \lambda \in core_R V, R \in \sum\}$.*

Proof. For each $\lambda \in core_R V$, then $f_\lambda(C_R) \geq u(C_R)$. Therefore, $\inf\{f_\lambda : \lambda \in core_R(V), R \in \sum\} \geq u$. Since V is an exact game so for each $R \in \sum$, there is a $\lambda \in core_R(V)$. It follows by Lemma 3 $f_\lambda(C_R) = u(C_R)$. Hence, $u = \inf\{f_\lambda : \lambda \in core_R(V), R \in \sum\}$.

Theorem 5. *Let $u = \inf\{f_\lambda : S \in \sum, \lambda \in BA_S, \lambda(\Omega) = V(\Omega)\}$. Then the equation $\sum \alpha_R C_R = \beta C_T + (1-\beta)C_\Omega$ implies $\sum \alpha_R u(C_R) \leq \beta u(C_T) + (1-\beta)u(C_\Omega)$, where $\alpha_R > 0, \sum \alpha_R = 1, \beta \in [0,1]$ and T is a coalition.*

Proof. Consider $R \in \sum$. Then $u(C_R) = f_\lambda(C_R)$ where, $\lambda \in BA_R$ is suitable element with $\lambda(\Omega) = V(\Omega)$. It is easy to see that $f_\lambda(C_\Omega) = u(C_\Omega)$. Suppose that L denotes the segment connecting $(C_R, u(C_R))$ to $(C_\Omega, u(C_\Omega))$. Then L lies on the graph of f_λ. Since **cav**(u) is concave, L is below the graph of **cav**(u). As **cav**$(u) \leq f_\lambda$, L is above the graph of **cav**(u). Thus, L is on the graph **cav**(u). Now by concavity of **cav**(u),

$$\begin{aligned}\sum_R \alpha_R cav(u)(C_R) &\leq \mathbf{cav}(u)(\sum_R \alpha_R C_R) \\ &= \mathbf{cav}(u)(\beta C_T + (1-\beta)C_\Omega) \\ &= \beta u(C_T) + (1-\beta)u(C_\Omega).\end{aligned}$$

That is $\sum_R \alpha_R u(C_R) \leq \beta u(C_T) + (1-\beta)u(C_\Omega)$.

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