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Solving Bi-matrix Games with Pay-offs of Triangular Intuitionistic Fuzzy Numbers

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Abstract. This paper presents a solution methodology for bi-matrix games in which pay-off matrices are represented by triangular intuitionistic fuzzy numbers (TIFNs). In this methodology, a new ranking function is defined to defuzzify the TIFNs. A non-linear intuitionistic fuzzy (I-fuzzy) programming problem is constructed to conceptualize the term equilibrium solution for such type of bi-matrix games. It is shown that this non-linear I-fuzzy programming problem is a generalization of fuzzy non-linear programming problem. Finally, based on the ranking function the problem is transformed into a crisp non-linear programming problem which can be solved to obtain the equilibrium solution for each player. Numerical simulation is provided to show the validity and applicability of this methodology.

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Key Words and Phrases: Bi-matrix Game, Triangular intuitionistic fuzzy numbers, Mathematical programming, Ranking function.

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1. Introduction

Game theory is a formal way to analyze conflict of interest among rational agents. Bi-matrix game is a two players non-zero sum game which have been successfully applied in different areas such as competition, voting, artificial intelligence etc. In traditional bi-matrix games it assume that the pay-off values are known exactly by the players. But, in real situations, it often happens that the players are not able to evaluate exactly the pay-off values due to imprecision and unavailability of information. In such situations, the fuzzy set theory (Zadeh [36]) is used and has achieved a great success (Bector and Chandra [4], Vijay *et al.* [33, 34], Li [9, 10], Cevikel and Ahlatcioglu [5], Kocken *et al.* [7], Seikh *et al.* [29]). In recent past, various attempt have been made in fuzzy bi-matrix game theory namely (Nishizaki and

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Sakawa [21, 22], Sakawa and Nishizaki [24], Maeda [13], Vidyottama *et al.* [32], Nayak and Pal [18]). However, fuzzy set uses only a membership degree to describe the degree of belongingness. The non-membership degree is just automatically equal to the complement to 1. But, in some real situations, players/decision makers(DMs) could only know the pay-offs approximately with some imprecise degree. In other words, players/DMs may have some hesitation degree about the approximate pay-offs. Therefore, the fuzzy set has no means to incorporate the hesitation degree.

Intuitionistic fuzzy set (IFS), introduced by Atanassov [2, 3] has been found to be well suited than the fuzzy set to express and describe information under uncertainty. The IFS is characterized by two functions expressing the degree of membership and the degree of non-membership respectively, so that the sum of both values is less than or equal to 1. The hesitation degree is equal to 1 minus the degree of membership and the degree of non-membership. Therefore, the concept of an IFS can be seen in the literature (Nan et al. [15, 16], Seikh *et al.* [28, 30, 31], Aggarwal et al. [1]) as an alternative approach to define a fuzzy set in cases where available information is not sufficient. However, there exist less investigation on application of IFS in bi-matrix games.

Intuitionistic fuzziness in bi-matrix games can appear in so many ways, but two cases of fuzziness seem to be very natural. These being the one in which DMs have IF goals and the other in which the elements of the pay-off matrices are given by intuitionistic fuzzy numbers (Li [8], Seikh *et al.* [25, 27]). These two classes of fuzzy bi-matrix games are referred as bi-matrix games with I-fuzzy goals and bi-matrix games with I-fuzzy pay-off. Nayak and Pal [19, 20] studied bi-matrix games and multi-objective bi-matrix games in which goals are expressed by IFS. Li [12] implemented bi-linear programming models to solve bi-matrix games with pay-offs of IFS. Seikh *et al.* [26] used TIFNs in bi-matrix games though this method is limited to pure strategies only. Li and Yang [11] developed a difference index based bi-linear programming approach to solve bi-matrix games with pay-offs represented by trapezoidal intuitionistic fuzzy numbers.

In this paper, we have considered bi-matrix games in which the pay-offs are represented by TIFNs. A new ranking function is defined to find a order relation between two TIFNs. A non-linear I-fuzzy programming problem is formulated to find the equilibrium solution of this bi-matrix game. Utilizing the ranking function this non-linear programming problem is further transformed into a crisp equivalent non-linear programming problem which can be easily solved to find the equilibrium solution.

The paper is organized as follows: In Section 2, some definitions and preliminaries about TIFNs are recalled and a ranking function is defined. Section 3 describes concept of double I-fuzzy constraint conditions. The main problem about the bi-matrix games with I-fuzzy pay-offs is formulated in Section 4. The results are illustrated by considering a media marketing problem in Section 5. Section 6 concludes the paper.

2. Definitions and Preliminaries

2.1. Triangular Intuitionistic Fuzzy Number (TIFN)

Definition 1. *A TIFN $\tilde{a} = \langle(\underline{a}_\mu, a, \bar{a}_\mu); (\underline{a}_\nu, a, \bar{a}_\nu)\rangle$ is a convex IFS on the set $\Re$ of real numbers, whose membership and non-membership functions are defined as follows*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-\underline{a}_\mu}{a-\underline{a}_\mu} & \textit{for } \underline{a}_\mu \le x < a \\ \frac{\bar{a}_\mu - x}{\bar{a}_\mu - a} & \textit{for } a < x \le \bar{a}_\mu \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

and

$$\nu_{\tilde{a}}(x) = \begin{cases} \frac{a-x}{a-\underline{a}_\nu} & \textit{for } \underline{a}_\nu \le x < a \\ \frac{x-a}{\bar{a}_\nu - a} & \textit{for } a < x \le \bar{a}_\nu \\ 1 & \textit{otherwise} \end{cases} \tag{2}$$

respectively, where $\underline{a}_\nu \le \underline{a}_\mu \le a \le \bar{a}_\mu \le \bar{a}_\nu$, depicted as in Figure 1.

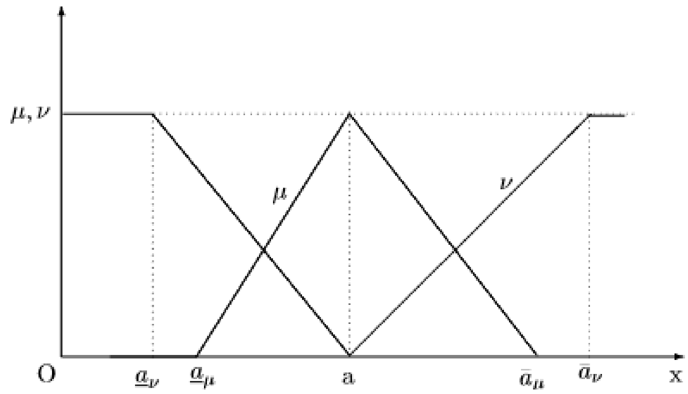

Figure 1: Membership and non-membership functions of TIFN

Note: Here $\mu_{\tilde{a}}(x)$ increases with constant rate for $x \in [\underline{a}_\mu, a]$ and decreases with constant rate for $x \in [a, \bar{a}_\mu]$ but $\nu_{\tilde{a}}(x)$ decreases with constant rate for $x \in [\underline{a}_\nu, a]$ and increases with constant rate for $x \in [a, \bar{a}_\nu]$. Let $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - \nu_{\tilde{a}}(x)$, which is called as the intuitionistic fuzzy index of an element x in the TIFN $\tilde{a}$.

Obviously, if $\underline{a}_\nu = \underline{a}_\mu = \underline{a}$ and $\bar{a}_\mu = \bar{a}_\nu = \bar{a}$, then $\mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) = 1$, $\forall x \in \Re$. In this case TIFN $\tilde{a} = \langle(\underline{a}_\mu, a, \bar{a}_\mu); (\underline{a}_\nu, a, \bar{a}_\nu)\rangle$ is reduced to $\tilde{a} = \langle(\underline{a}, a, \bar{a})\rangle$ which is just a triangular fuzzy number (TFN). Thus, it is easy to see that the definition of a TIFN is a generalization of that of the TFN, introduced by Dubois and Prade [6]. The set of all TIFNs is denoted by $\tilde{\mathscr{F}}(\Re)$.

Definition 2 (Arithmetic Operations). *Let $\tilde{a}$ and $\tilde{b}$ be two TIFNs, denoted by $\tilde{a} = \langle(\underline{a}_\mu, a, \bar{a}_\mu); (\underline{a}_\nu, a, \bar{a}_\nu)\rangle$ and $\tilde{b} = \langle(\underline{b}_\mu, b, \bar{b}_\mu); (\underline{b}_\nu, b, \bar{b}_\nu)\rangle$ then the addition and scalar multiplication are defined as follows:*

Addition:

$$\tilde{a}+\tilde{b}=\left\langle(\underline{a}_\mu+\underline{b}_\mu,a+b,\bar{a}_\mu+\bar{b}_\mu);(\underline{a}_\nu+\underline{b}_\nu,a+b,\bar{a}_\nu+\bar{b}_\nu)\right\rangle.$$

Scalar Multiplication:

$$k\tilde{a}=\begin{cases}\langle(k\underline{a}_\mu,ka,k\bar{a}_\mu);(k\underline{a}_\nu,ka,k\bar{a}_\nu)\rangle & \text{if } k>0\\ \langle(k\bar{a}_\mu,ka,k\underline{a}_\mu);(k\bar{a}_\nu,ka,k\underline{a}_\nu)\rangle & \text{if } k<0\end{cases},$$

where k is a real number.

Definition 3 (Cut sets of TIFNs). *For any $\alpha\in[0,1]$, a α-cut set of TIFN $\tilde{a}=\langle(\underline{a}_\mu,a,\bar{a}_\mu);(\underline{a}_\nu,a,\bar{a}_\nu)\rangle$ can be expressed as a crisp subset of $\Re$, denoted by $\tilde{a}_\alpha=\{x|\mu_{\tilde{a}}(x)\geq\alpha,x\in\Re\}$. According to the definition of the TIFN, it can be easily seen that $\tilde{a}_\alpha$ is a closed interval, denoted by $\tilde{a}_\alpha=[L_\alpha(\tilde{a}),R_\alpha(\tilde{a})]$. It directly follows from (1) that*

$$[L_\alpha(\tilde{a}),R_\alpha(\tilde{a})]=\left[\underline{a}_\mu+\alpha(a-\underline{a}_\mu),\bar{a}_\mu-\alpha(\bar{a}_\mu-a)\right].$$

Similarly, for any $\beta\in[0,1]$, a β-cut set of an TIFN $\tilde{a}=\langle(\underline{a}_\mu,a,\bar{a}_\mu);(\underline{a}_\nu,a,\bar{a}_\nu)\rangle$ can be expressed as a crisp subset of $\Re$, denoted by $\tilde{a}_\beta=\{x|\nu_{\tilde{a}}(x)\leq\beta,x\in\Re\}$. Obviously, $\tilde{a}_\beta$ is a closed interval, denoted by $\tilde{a}_\beta=[L_\beta(\tilde{a}),R_\beta(\tilde{a})]$. It directly follows from (2) that

$$[L_\beta(\tilde{a}),R_\beta(\tilde{a})]=\left[a-\beta(a-\underline{a}_\nu),a+\beta(\bar{a}_\nu-a)\right].$$

In the next context, we discuss about the value index and ambiguity index of a TIFN, inspired by Li [8].

Definition 4. *Let $\tilde{a}_\alpha=[L_\alpha(\tilde{a}),R_\alpha(\tilde{a})]$ and $\tilde{a}_\beta=[L_\beta(\tilde{a}),R_\beta(\tilde{a})]$ be any α-cut set and a β-cut set of a TIFN $\tilde{a}=\langle(\underline{a}_\mu,a,\bar{a}_\mu);(\underline{a}_\nu,a,\bar{a}_\nu)\rangle$, respectively. Then the values of the membership and non-membership functions are defined as follows.*

$$G_\mu(\tilde{a})=\int_0^1\frac{L_\alpha(\tilde{a})+R_\alpha(\tilde{a})}{2}f(\alpha)d\alpha$$

and

$$G_\nu(\tilde{a})=\int_0^1\frac{L_\beta(\tilde{a})+R_\beta(\tilde{a})}{2}g(\beta)d\beta$$

respectively.

Here $f(\alpha)$ is a non-negative and non-decreasing function on the interval $[0,1]$, satisfying the conditions, $f(0)=0$ and $f(1)=1$. Also $g(\beta)$ is a non-negative and non-increasing function on the interval $[0,1]$, satisfying $g(0)=1$ and $g(1)=0$.

The functions $f(\alpha)$ and $g(\beta)$ may be considered as weighting functions and are chosen as

$$f(\alpha)=2\alpha,\alpha\in[0,1] \tag{3}$$

and

$$g(\beta) = 2(1-\beta), \beta \in [0,1] \tag{4}$$

Then the values of membership an non-membership functions of a TIFN $\tilde{a} = \langle(\underline{a}_\mu, a, \bar{a}_\mu); (\underline{a}_\nu, a, \bar{a}_\nu)\rangle$ are calculated as follows:

$$G_\mu(\tilde{a}) = \int_0^1 \left[\underline{a}_\mu + \alpha(a - \underline{a}_\mu) + \bar{a}_\mu - \alpha(\bar{a}_\mu - a)\right]\alpha d\alpha = \frac{(\underline{a}_\mu + 4a + \bar{a}_\mu)}{6} \tag{5}$$

$$G_\nu(\tilde{a}) = \int_0^1 \left[a - \beta(a - \underline{a}_\nu) + a + \beta(\bar{a}_\nu - a)\right](1-\beta)d\beta = \frac{(\underline{a}_\nu + 4a + \bar{a}_\nu)}{6}. \tag{6}$$

Obviously, $G_\mu(\tilde{a})$ and $G_\nu(\tilde{a})$ synthetically reflect information on membership degrees and non-membership degrees at all levels, respectively. Also it directly follows that $G_\mu(\tilde{a}) \leq G_\nu(\tilde{a})$.

Similarly, the ambiguities of the membership and non-membership functions for any TIFN $\tilde{a}$ are defined by

$$H_\mu(\tilde{a}) = \int_0^1 [R_\alpha(\tilde{a}) - L_\alpha(\tilde{a})] f(\alpha) d\alpha$$

and

$$H_\nu(\tilde{a}) = \int_0^1 [R_\beta(\tilde{a}) - L_\beta(\tilde{a})] g(\beta) d\beta,$$

respectively. Obviously, $R_\alpha(\tilde{a}) - L_\alpha(\tilde{a})$ and $R_\beta(\tilde{a}) - L_\beta(\tilde{a})$ represent the lengths of the intervals $\tilde{a}_\alpha$ and $\tilde{a}_\beta$. Therefore, $H_\mu(\tilde{a})$ and $H_\nu(\tilde{a})$ measure the uncertainty in $\tilde{a}$.

Using (3) and (4), the ambiguities of membership and non-membership functions of a TIFN $\tilde{a}$ are calculated as

$$H_\mu(\tilde{a}) = \int_0^1 \left[\bar{a}_\mu - \alpha(\bar{a}_\mu - a) - \underline{a}_\mu - \alpha(a - \underline{a}_\mu)\right]2\alpha d\alpha = \frac{\bar{a}_\mu - \underline{a}_\mu}{3} \tag{7}$$

and

$$G_\nu(\tilde{a}) = \int_0^1 \left[\beta(\bar{a}_\nu - a) - \beta(a - \underline{a}_\nu)\right]2(1-\beta)d\beta = \frac{\bar{a}_\nu - \underline{a}_\nu}{3}. \tag{8}$$

It can be easily shown that $E_\mu(\tilde{a}) \leq E_\nu(\tilde{a})$.

Proposition 1. *Let $\tilde{a}$ and $\tilde{b}$ be two any TIFNs and k be any nonnegative real number. Then the following equalities are valid.*

(i) $G_\mu(k\tilde{a} + \tilde{b}) = kG_\mu(\tilde{a}) + G_\mu(\tilde{b})$

(ii) $G_\nu(k\tilde{a}+\tilde{b}) = kG_\nu(\tilde{a}) + G_\nu(\tilde{b})$

(iii) $H_\mu(k\tilde{a}+\tilde{b}) = kH_\mu(\tilde{a}) + H_\mu(\tilde{b})$

(iv) $H_\nu(k\tilde{a}+\tilde{b}) = kH_\nu(\tilde{a}) + H_\nu(\tilde{b})$.

Definition 5 (Value index and Ambiguity index)**.** *The value index and ambiguity index of any TIFN $\tilde{a}$ are defined as follows:*

$$\text{value index } V(\tilde{a},\lambda) = \lambda G_\nu(\tilde{a}) + (1-\lambda)G_\mu(\tilde{a})$$

and

$$\text{ambiguity index } A(\tilde{a},\lambda) = \lambda H_\mu(\tilde{a}) + (1-\lambda)H_\nu(\tilde{a}),$$

respectively, where $\lambda \in [0,1]$ is the weight represents the players/DMs preference information. $\lambda \in [0,\frac{1}{2})$ indicates players/DMs pessimistic attitude towards uncertainty while $\lambda \in (\frac{1}{2},1]$ indicates players/DMs optimistic attitude towards uncertainty. Also $\lambda = \frac{1}{2}$ shows that players/DMs are indifferent. Thus, the value index and ambiguity index may reflect players/DMs attitude to the TIFNs.

If we choose $\lambda = \frac{1}{2}$, then

$$V(\tilde{a},\frac{1}{2}) = \frac{G_\nu(\tilde{a}) + G_\mu(\tilde{a})}{2} \text{ and } A(\tilde{a},\frac{1}{2}) = \frac{H_\mu(\tilde{a}) + H_\nu(\tilde{a})}{2}.$$

Proposition 2. *Let $\tilde{a}$ and $\tilde{b}$ be two any TIFNs. Then for any real number k, the following equalities are valid:*

(i) $V(k\tilde{a}+\tilde{b},\frac{1}{2}) = kV(\tilde{a},\frac{1}{2}) + V(\tilde{b},\frac{1}{2})$

(ii) $A(k\tilde{a}+\tilde{b},\frac{1}{2}) = kA(\tilde{a},\frac{1}{2}) + A(\tilde{b},\frac{1}{2})$.

It can be easily seen that more is the value and lesser is the ambiguity of the TIFN, larger the TIFN. In the following, a ranking function is defined based on difference of value index and ambiguity index to find an order relation between two TIFNs.

Definition 6. *A ranking function (or defuzzification function) is a function $R : \tilde{\mathscr{F}}(\mathfrak{R}) \to \mathfrak{R}$, where $\tilde{\mathscr{F}}(\mathfrak{R})$ is a set of all TIFNs defined on $\mathfrak{R}$, which maps each TIFN into the real line, where a natural order exists. Let $\tilde{a}$ be a TIFN, then $R(\tilde{a}) = V(\tilde{a},\frac{1}{2}) - A(\tilde{a},\frac{1}{2})$.*

Suppose that $\tilde{a}$ and $\tilde{b}$ be two TIFNs and $\lambda \in [0,1]$ be any real number. Then a new order relation between $\tilde{a}$ and $\tilde{b}$ is defined as follows:

(i) $\tilde{a} \tilde{\geq} \tilde{b}$ *iff* $R(\tilde{a}) \geq R(\tilde{b})$

(ii) $\tilde{a} \tilde{\leq} \tilde{b}$ *iff* $R(\tilde{a}) \leq R(\tilde{b})$

(iii) $\tilde{a} \tilde{=} \tilde{b}$ *iff* $R(\tilde{a}) = R(\tilde{b})$.

The symbol "$\widetilde{\geq}$" is an intuitionistic fuzzy version of the order relation "$\geq$" on the set of real numbers and has the linguistic interpretation as "essentially greater than or equal to"Ṡimilarly, the symbols "$\widetilde{\leq}$" and "$\widetilde{=}$" are the intuitionistic fuzzy versions of the order relations "$\leq$" and "$=$" on the set of real numbers and have the linguistic interpretations "essentially less than or equal to" and "essentially equal to", respectively.

It can be easily verified that the above proposed ranking method satisfies some of the axioms namely the reasonable properties proposed by Wang and Kerre [35].

Proposition 3. *Let $\tilde{a}$ and $\tilde{b}$ be any two TIFNs, then for any real number k, the following equality is valid*

$$R(k\tilde{a}+\tilde{b}) = kR(\tilde{a}) + R(\tilde{b}).$$

This shows that the proposed ranking function is linear.

In the next section the concept of double I-fuzzy inequalities, *i.e.*, the I-fuzzy constraints involving I-fuzzy numbers is interpreted.

3. Interpretation of Double I-fuzzy Constraints

Let us recall the concept of double fuzzy constraints (Vidyottama *et al.* [32]), *i.e.*, constraints which are expressed as fuzzy inequalities involving fuzzy numbers. For this, let $\mathcal{N}(\mathfrak{R})$ be the set of all fuzzy numbers. Also let $\tilde{S}$, $\tilde{w}$, respectively, be $m \times n$ matrix and $m \times 1$ vector having entries from $\mathcal{N}(\mathfrak{R})$ and the double fuzzy constraints under consideration be given by $X^T\tilde{S}Y \preceq_{\tilde{p}} \tilde{w}$ and $X^T\tilde{S}Y \succeq_{\tilde{p'}} \tilde{w}$, with adequacies $\tilde{p}$ and $\tilde{p'}$, respectively. Then the double fuzzy constraints $X^T\tilde{S}Y \preceq_{\tilde{p}} \tilde{w}$ and $X^T\tilde{S}Y \succeq_{\tilde{p'}} \tilde{w}$ can be expressed as

$$X^T\tilde{S}Y \preceq_F \tilde{w} + \tilde{p}(1-\rho),\ \rho \in [0,1]$$

and

$$X^T\tilde{S}Y \succeq_F \tilde{w} - \tilde{p'}(1-\rho),\ \rho \in [0,1],$$

where $\tilde{p}$ and $\tilde{p'}$ measure the adequacy between the fuzzy numbers $X^T\tilde{S}Y$ and $\tilde{w}$. Here $\preceq_F$ and $\succeq_F$ are the relations between fuzzy numbers.

We now extend the interpretation of double fuzzy constraints to the I-fuzzy sense. Let $\tilde{S}, \tilde{B}$ and $\tilde{C}$ respectively, be $m \times n$ matrix, $m \times 1$ and $n \times 1$ vectors having entries from $\mathscr{F}(\mathfrak{R})$, and the double I-fuzzy constraints under consideration be given by $\tilde{S}^T Y \widetilde{\preceq}_{\tilde{p},\tilde{q}} \tilde{C}$ and $\tilde{S}X \widetilde{\succeq}_{\tilde{r},\tilde{s}} \tilde{B}$, with the adequacies/tolerances $\tilde{p}, \tilde{q}$ and $\tilde{r}, \tilde{s}$, respectively, which are also I-fuzzy vectors. Based on the resolution method discussed above we extend the interpretation of I-fuzzy inequalities to the case where the parameters and the adequacies are also I-fuzzy number. Therefore, the double I-fuzzy constraint conditions are to be understood as

$$\tilde{S}^T Y \widetilde{\preceq}_{\tilde{p},\tilde{q}} \tilde{C} \Rightarrow \begin{cases} \tilde{S}_i Y \widetilde{\preceq} \tilde{C}_i + \tilde{p}_i(1-\xi), & 0 \leq \xi \leq 1 \\ \tilde{S}_i Y \widetilde{\preceq} (\tilde{C}_i + \tilde{p}_i) - \tilde{q}_i(1-\eta), & 0 \leq \eta \leq 1 \end{cases} \tag{9}$$

and

$$\tilde{S}X \widetilde{\succeq}_{\tilde{r},\tilde{s}} \tilde{B} \Rightarrow \begin{cases} \tilde{S}_j X \widetilde{\succeq} \tilde{B}_j - \tilde{r}_j(1-\gamma), & 0 \leq \gamma \leq 1 \\ \tilde{S}_j X \widetilde{\succeq} (\tilde{B}_j - \tilde{r}_j) + \tilde{s}_j(1-\delta), & 0 \leq \delta \leq 1 \end{cases} \tag{10}$$

respectively, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Here $\tilde{\succeq}$ and $\tilde{\preceq}$ are relation between I-fuzzy numbers which preserves the ranking when I-fuzzy numbers are multiplied by positive scalars. Also, $\tilde{p}_j, \tilde{q}_j (j = 1, 2, \ldots, n)$ represent the j^{th} component of I-fuzzy vectors $\tilde{p}$ and $\tilde{q}$, respectively. Similarly, $\tilde{r}_i, \tilde{s}_i (i = 1, 2, \ldots, m)$ represent the i^{th} component of I-fuzzy vectors $\tilde{r}$ and $\tilde{s}$, respectively.

4. Mathematical Model of a Bi-matrix Game

A bi-matrix game can be considered as a natural extension of the matrix game. Let I, II denote two players and let $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ be the sets of all pure strategies available for players I, II respectively. By a_{ij} and b_{ij}, we denote the pay-offs that the player I and II receive when player I plays the pure strategy i and player II plays the pure strategy j. Then we have the following pay-off matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix}; B = \begin{pmatrix} b_{11} & b_{12} & \cdots b_{1n} \\ b_{21} & b_{22} & \cdots b_{2n} \\ \cdots & \cdots & \\ b_{m1} & b_{m1} & \cdots b_{mn} \end{pmatrix}.$$

where we assume that each of the two players chooses a strategy, a pay-off for each of them is represented as a crisp number. We denote the game by $\Gamma = \langle \{I, II\}, A, B \rangle$.

4.1. Nash Equilibrium Solution

Nash [17] defined the concept of Nash equilibrium solutions (NES) in bi-matrix games for single pair of payoff matrices and presented methodology for obtaining them.

Definition 7 (Pure strategy). *Let I and II denote two players and let $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ be the sets of all pure strategies available for players I and II respectively. A pair of strategies (row r, column s) is said to constitute a NES to a bi-matrix game Γ if the following pair of inequalities is satisfied for all $i = 1, 2, \ldots, m$ and for all $j = 1, 2, \ldots, n$:*

$$a_{is} \leq a_{rs}; b_{rj} \leq b_{rs}$$

Since the strategy sets are finite, these expressions may exist and in such case, bi-matrix game admits a NES for pure strategy. The pair (a_{rs}, b_{rs}) is known as a *Nash equilibrium outcome* of the bi-matrix game in pure strategies. A bi-matrix game can admit more than one NES, with the equilibrium outcomes being different in each case.

4.2. Mixed Strategy

We denote the sets of all mixed strategies, called strategy spaces, available for players I, II by

$$S_I = \Big\{ \mathbf{x} = (x_1, x_2, \ldots, x_m) \in \Re^m_+ : x_i \geq 0; i = 1, 2, \ldots, m \text{ and } \sum_{i=1}^{m} x_i = 1 \Big\}$$

$$S_{II} = \Big\{\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Re^n_+ : y_j \geq 0; j = 1, 2, \ldots, n \text{ and } \sum_{j=1}^{n} y_j = 1\Big\},$$

where $\Re^m_+$ denotes the m—dimensional non negative Euclidean space. Thus by a crisp two person zero-sum bi-matrix game BG we mean the triplet $BG = (S_I \times S_{II}, A, B)$. Since the player is uncertain about what strategy he/she will choose, he/she will choose a probability distribution over the aet of alternatives available to him/her or a mixed strategy in terms of game theory.

Definition 8 (Expected payoff). *If the mixed strategies* $\mathbf{x}$ *and* $\mathbf{y}$ *are proposed by the player I and player II respectively, then the expected pay-offs of the players I and II are respectively*

$$\mathbf{x}^T A\mathbf{y} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j \text{ and } \mathbf{x}^T B\mathbf{y} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i b_{ij} y_j.$$

Definition 9 (Equilibrium Solution). *A pair* $(\mathbf{x}^*, \mathbf{y}^*) \in S_I \times S_{II}$ *is said to be an equilibrium solution of the bi-matrix game BG if*

$$\mathbf{x}^T A\mathbf{y}^* \leq \mathbf{x}^{*T} A\mathbf{y}^*, \forall \mathbf{x} \in S_I \text{ and } \mathbf{x}^{*T} B\mathbf{y} \leq \mathbf{x}^{*T} B\mathbf{y}^*, \forall \mathbf{y} \in S_{II}.$$

$\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are also called the optimal strategies for the player I and II respectively. The pair of numbers* $V = \langle \mathbf{x}^{*T} A\mathbf{y}^*, \mathbf{x}^{*T} B\mathbf{y}^* \rangle$ *is said to be the Nash equilibrium outcome of BG and the triplet* $(\mathbf{x}^*, \mathbf{y}^*, V)$ *is called the solution the bi-matrix game.*

The following theorem due to Nash, guarantees the existence of an equilibrium solution of the bi-matrix game BG.

Theorem 1 (Owen [23] (Nash Existence Theorem)). *Every bi-matrix game BG has at least one equilibrium solution.*

A Nash equilibrium solution of the bi-matrix game BG can be obtained by solving an appropriate quadratic programming problem as discussed below.

Theorem 2 (Mangasarian and Stone [14] (Equivalence Theorem)). *For a given bi-matrix game* $BG = (S_I \times S_{II}, A, B)$ *a necessary and sufficient condition that* $(\mathbf{x}^*, \mathbf{y}^*)$ *be an equilibrium solution of BG is that it is a solution of the following quadratic programming problem (QPP).*

$$\begin{aligned}
&\max\ \mathbf{x}^T (A+B)\mathbf{y} - p - q\\
&S.T.\ A\mathbf{y} \leq pe,\\
&\qquad B^T \mathbf{x} \leq qe,\\
&\qquad \mathbf{x} \in S_I,\\
&\qquad \mathbf{y} \in S_{II};\\
&\qquad p, q \in \Re.
\end{aligned}$$

Further, if $(\mathbf{x}^*, \mathbf{y}^*, p^*, q^*)$ *is a solution of the above problem then*

$$p^* = \mathbf{x}^{*T} A\mathbf{y}^*,\ q^* = \mathbf{x}^{*T} B\mathbf{y}^* \text{ and } \mathbf{x}^{*T}(A+B)\mathbf{y}^* - p^* - q^* = 0$$

4.3. Bi-matrix Games with Pay-offs of TIFNs

Let S_I, S_{II} be the strategy spaces for player I and player II, respectively as defined in above section. Also let $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{m\times n}$ be the pay-off matrices for players I and II, respectively, where $\tilde{a}_{ij} = \langle(\underline{a}_{ij\mu}, a_{ij}, \bar{a}_{ij\mu}); (\underline{a}_{ij\nu}, a_{ij}, \bar{a}_{ij\nu})\rangle$ $(i = 1,2,\dots m;\ j = 1,2,\dots n)$ and $\tilde{b}_{ij} = \langle(\underline{b}_{ij\mu}, b_{ij}, \bar{b}_{ij\mu}); (\underline{b}_{ij\nu}, b_{ij}, \bar{b}_{ij\nu})\rangle$ $(i = 1,2,\dots m;\ j = 1,2,\dots n)$ are the TIFNs as defined in 2.1. Then a two person bi-matrix game with pay-offs of TIFNs is defined by $(S_I, S_{II}, \tilde{A}, \tilde{B})$. In the following, we shall often call a two-person bi-matrix game with pay-offs of TIFNs as intuitionistic fuzzy bi-matrix game, denoted by $\widetilde{IFBG} = (S_I, S_{II}, \tilde{A}, \tilde{B})$.

Definition 10 (Expected Pay-offs). *Let player I chooses any mixed strategy $\mathbf{x} \in S_I$ and player II chooses any mixed strategy $\mathbf{y} \in S_{II}$, then the expected pay-offs for player I and player II are*

$$\begin{aligned}
\tilde{E}_1(\tilde{A}) = \mathbf{x}^T\tilde{A}\mathbf{y} = &\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{a}_{ij}x_iy_j\\
=&\Big\langle\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}\underline{a}_{ij\mu}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{a}_{ij\mu}x_iy_j\Big);\\
&\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}\underline{a}_{ij\nu}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{a}_{ij\nu}x_iy_j\Big)\Big\rangle\\
\tilde{E}_2(\tilde{B}) = \mathbf{x}^T\tilde{B}\mathbf{y} = &\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{b}_{ij}x_iy_j\\
=&\Big\langle\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}\underline{b}_{ij\mu}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}b_{ij}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{b}_{ij\mu}x_iy_j\Big);\\
&\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}\underline{b}_{ij\nu}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}b_{ij}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{b}_{ij\nu}x_iy_j\Big)\Big\rangle
\end{aligned}$$

respectively, which are TIFNs.

In the next section, we have described the meaning of equilibrium solution of bi-matrix $\widetilde{IFBG}$.

Definition 11. *Let $\tilde{u}$ and $\tilde{v}$ be two TIFNs. Then $(\tilde{u}, \tilde{v})$ is called a reasonable solution of the bi-matrix game $\widetilde{IFBG}$ if there exists $\mathbf{x}^* \in S_I, \mathbf{y}^* \in S_{II}$ such that*

(i) $\mathbf{x}^T\tilde{A}\mathbf{y}^* \tilde{\preceq}_{\tilde{p},\tilde{q}}\ \tilde{u}$, *for all* $\mathbf{x} \in S_I$

(ii) $\mathbf{x}^{*T}\tilde{B}\mathbf{y} \tilde{\preceq}_{\tilde{r},\tilde{s}}\ \tilde{v}$, *for all* $\mathbf{y} \in S_{II}$

(iii) $\mathbf{x}^{*T}\tilde{A}\mathbf{y}^* \tilde{\succeq}_{\tilde{p_0},\tilde{q_0}}\ \tilde{u}$

(iv) $\mathbf{x}^{*T}\tilde{B}\mathbf{y}^* \tilde{\succeq}_{\tilde{r_0},\tilde{s_0}}\ \tilde{v}$.

If $(\tilde{u}, \tilde{v})$ be a reasonable solution of $\widetilde{IFBG}$ then $\tilde{u}, \tilde{v}$ are called the reasonable values for player I and II, respectively.

Definition 12. *Let $\tilde{U}$ and $\tilde{V}$ be the set of all reasonable values $\tilde{u}$ and $\tilde{v}$ for player I and II, respectively. Also let there exist $\tilde{u}^* \in \tilde{U}$, $\tilde{v}^* \in \tilde{V}$ such that $R(\tilde{u}^*) \geq R(\tilde{u})$, $\forall \tilde{u} \in \tilde{U}$ and $R(\tilde{v}^*) \geq R(\tilde{v})$, $\forall \tilde{v} \in \tilde{V}$, where R is the ranking function defined in Section 2. Then the pair $(\mathbf{x}^*, \mathbf{y}^*)$ is called an equilibrium point of the game $\widetilde{IFBG}$ and $\mathbf{x}^*$, $\mathbf{y}^*$ are called Nash equilibrium strategies of player I and II, respectively. Also $\tilde{u}^* = \mathbf{x}^T \tilde{A} \mathbf{y}$ and $\tilde{v}^* = \mathbf{x}^T \tilde{B} \mathbf{y}$ are called the Nash equilibrium values of the game $\widetilde{IFBG}$ for player I and II, respectively, and $(\mathbf{x}^*, \mathbf{y}^*, \tilde{u}^*, \tilde{v}^*)$ is called the Nash equilibrium solution of the bi-matrix game $\widetilde{IFBG}$.*

By using the above definitions we can now construct the following I-fuzzy non-linear programming problem as

$$\begin{aligned} & \max\ \{\tilde{u} + \tilde{v}\} \\ & \text{subject to } \mathbf{x}^T \tilde{A} \mathbf{y} \tilde{\preceq}_{\tilde{p},\tilde{q}}\ \tilde{u},\ \text{for all } \mathbf{x} \in S_I, \\ & \qquad \mathbf{x}^T \tilde{B} \mathbf{y} \tilde{\preceq}_{\tilde{r},\tilde{s}}\ \tilde{v},\ \text{for all } \mathbf{y} \in S_{II}, \\ & \qquad \mathbf{x}^T \tilde{A} \mathbf{y} \tilde{\succeq}_{\tilde{p_0},\tilde{q_0}}\ \tilde{u}, \\ & \qquad \mathbf{x}^T \tilde{B} \mathbf{y} \tilde{\succeq}_{\tilde{r_0},\tilde{s_0}}\ \tilde{v}, \\ & \qquad \mathbf{x} \in S_I, \mathbf{y} \in S_{II}, \\ & \qquad \tilde{u}, \tilde{v} \in \mathscr{F}(\Re) \end{aligned} \tag{11}$$

Since S_I and S_{II} are convex polytopes, it is sufficient to consider only the extreme points (*i.e.* pure strategies) of S_I and S_{II}. This observation leads to the following I-fuzzy non-linear programming problem as

$$\begin{aligned} & \max\ \{\tilde{u} + \tilde{v}\} \\ & \text{subject to } \tilde{A}_i \mathbf{y} \tilde{\preceq}_{\tilde{p},\tilde{q}}\ \tilde{u}\ (i = 1, 2, \ldots, m), \\ & \qquad \mathbf{x}^T \tilde{B}_j \tilde{\preceq}_{\tilde{r},\tilde{s}}\ \tilde{v}\ (j = 1, 2, \ldots, n), \\ & \qquad \mathbf{x}^T \tilde{A} \mathbf{y} \tilde{\succeq}_{\tilde{p_0},\tilde{q_0}}\ \tilde{u}, \\ & \qquad \mathbf{x}^T \tilde{B} \mathbf{y} \tilde{\succeq}_{\tilde{r_0},\tilde{s_0}}\ \tilde{v}, \\ & \qquad \mathbf{x} \in S_I, \mathbf{y} \in S_{II}, \\ & \qquad \tilde{u}, \tilde{v} \in \mathscr{F}(\Re) \end{aligned} \tag{12}$$

Here $\tilde{A}_i$ $(i = 1, 2, \ldots, m)$ denotes the i^{th} row of the pay-off matrix $\tilde{A}$ and $\tilde{B}_j$ $(j = 1, 2, \ldots, n)$ represents the j^{th} column of $\tilde{B}$. Now by using the resolution procedure for the double I-fuzzy constraints described in Section 3, we obtain the following I-fuzzy non-linear programming problem as

$$\begin{aligned} & \max\ \{\tilde{u} + \tilde{v}\} \\ & \text{subject to } \tilde{A}_i \mathbf{y} \tilde{\preceq}\ \tilde{u} + (1 - \xi)\tilde{p}_i\ (i = 1, 2, \ldots, m), \\ & \qquad \tilde{A}_i \mathbf{y} \tilde{\preceq}\ (\tilde{u} + \tilde{p}_i) - (1 - \eta)\tilde{q}_i\ (i = 1, 2, \ldots, m), \end{aligned}$$

$$
\begin{aligned}
&\mathbf{x}^T\tilde{B}_j \tilde{\preceq}\ \tilde{v} + (1-\gamma)\tilde{r}_j\ (j = 1,2,\ldots,n),\\
&\mathbf{x}^T\tilde{B}_j \tilde{\preceq}\ (\tilde{v} + \tilde{r}_j) - (1-\delta)\tilde{s}_j\ (j = 1,2,\ldots,n),\\
&\mathbf{x}^T\tilde{A}\mathbf{y} \tilde{\succeq}\ \tilde{u} - (1-\xi)\tilde{p}_0,\\
&\mathbf{x}^T\tilde{A}\mathbf{y} \tilde{\succeq}\ (\tilde{u} - \tilde{p}_0) + (1-\eta)\tilde{q}_0,\\
&\mathbf{x}^T\tilde{B}\mathbf{y} \tilde{\succeq}\ \tilde{v} - (1-\gamma)\tilde{r}_0,\\
&\mathbf{x}^T\tilde{B}\mathbf{y} \tilde{\succeq}\ (\tilde{v} - \tilde{r}_0) + (1-\delta)\tilde{s}_0,\\
&\mathbf{x} \in S_I, \mathbf{y} \in S_{II},\\
&0 \le \xi \le 1, 0 \le \eta \le 1\\
&0 \le \gamma \le 1, 0 \le \delta \le 1
\end{aligned}
\tag{13}
$$

Here $\tilde{\preceq}$ and $\tilde{\succeq}$ are the relations between TIFNs, which preserve the ranking when I-fuzzy numbers are multiplied by positive scalar. Now by utilizing the ranking function R(defined in Section 2, Definition 6), the above problem can be transformed into crisp equivalent non-linear programming problem as follows:

$$
\begin{aligned}
&\max\ \{R(\tilde{u}) + R(\tilde{v})\}\\
\text{subject to } &\sum_{j=1}^{n} R(\tilde{a}_{ij})y_j \le R(\tilde{u}) + (1-\xi)R(\tilde{p}_i)\ (i = 1,2,\ldots,m),\\
&\sum_{j=1}^{n} R(\tilde{a}_{ij})y_j \le R(\tilde{u}) + R(\tilde{p}_i) - (1-\eta)R(\tilde{q}_i)\ (i = 1,2,\ldots,m),\\
&\sum_{i=1}^{m} R(\tilde{b}_{ij})x_i \le R(\tilde{v}) + (1-\gamma)R(\tilde{r}_j)\ (j = 1,2,\ldots,n),\\
&\sum_{i=1}^{m} R(\tilde{b}_{ij})x_i \le R(\tilde{v}) + R(\tilde{r}_j) - (1-\delta)R(\tilde{s}_j)\ (j = 1,2,\ldots,n),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{a}_{ij})x_i y_j \ge R(\tilde{u}) - (1-\xi)R(\tilde{p}_0),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{a}_{ij})x_i y_j \ge R(\tilde{u}) - R(\tilde{p}_0) + (1-\eta)R(\tilde{q}_0),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{b}_{ij})x_i y_j \ge R(\tilde{v}) - (1-\gamma)R(\tilde{r}_0),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{b}_{ij})x_i y_j \ge R(\tilde{v}) - R(\tilde{r}_0) + (1-\delta)R(\tilde{s}_0),\\
&\sum_{i=1}^{m} x_i = 1, \sum_{j=1}^{n} y_j = 1,\\
&0 \le \xi \le 1, 0 \le \eta \le 1,
\end{aligned}
\tag{14}
$$

$$0 \leq \gamma \leq 1, 0 \leq \delta \leq 1.$$

Thus from above discussion we observed that for solving a I-fuzzy bi-matrix game $\widetilde{IFBG}$ we have to solve the crisp non-linear programming problem (14). Therefore, if $(\mathbf{x}^*, \mathbf{y}^*, R(\tilde{u}^*), R(\tilde{v}^*))$ be an optimal solution of the crisp non-linear programming problem (14), then $(\mathbf{x}^*, \mathbf{y}^*)$ is the Nash equilibrium point of the game $\widetilde{IFBG}$. Also, the players can take the values $\tilde{u}$ and $\tilde{v}$ for which $R(\tilde{u})$ and $R(\tilde{v})$ are close to $R(\tilde{u}^*)$ and $R(\tilde{v}^*)$, respectively.

The above discussion can be summarized in the following theorem.

Theorem 3. *The I-fuzzy bi-matrix game $\widetilde{IFBG}$ described by $\widetilde{IFBG} = (S_I, S_{II}, \tilde{A}, \tilde{B})$ is equivalent to the crisp non-linear programming problem* (14)*, which can be easily solved by ordinary simplex method.*

Remark 1. *It may be noted that when $\tilde{p}_i = \tilde{q}_i$, $\tilde{r}_j = \tilde{s}_j$, $\tilde{p}_0 = \tilde{q}_0$, $\tilde{r}_0 = \tilde{s}_0$, $\eta = 1 - \xi$ and $\delta = 1 - \gamma$, the I-fuzzy bi-matrix game $\widetilde{IFBG}$ reduces to the fuzzy bi-matrix game BGFP studied by Vidyottama et al. [32]. Further, the I-fuzzy non-linear programming problem* (13) *reduced to fuzzy non-linear programming problem of Vidyottama et al. [32] as*

$$\begin{aligned}
&\max\ \{\tilde{u} + \tilde{v}\} \\
\textit{subject to}\ &\tilde{A}_i\mathbf{y} \preceq_F\ \tilde{u} + (1-\xi)\tilde{p}_i\ (i = 1, 2, \ldots, m), \\
&\mathbf{x}^T\tilde{B}_j \preceq_F\ \tilde{v} + (1-\gamma)\tilde{q}_j\ (j = 1, 2, \ldots, n), \\
&\mathbf{x}^T\tilde{A}\mathbf{y} \succeq_F\ \tilde{u} - (1-\xi)\tilde{p}_0, \\
&\mathbf{x}^T\tilde{B}\mathbf{y} \succeq_F\ \tilde{v} - (1-\gamma)\tilde{r}_0, \\
&\mathbf{x} \in S_I, \mathbf{y} \in S_{II}, \\
&0 \leq \xi \leq 1, 0 \leq \gamma \leq 1 \\
&\tilde{u}, \tilde{v} \in \mathcal{N}(\mathfrak{R})
\end{aligned} \tag{15}$$

where $\mathcal{N}(\mathfrak{R})$ is the set of all fuzzy numbers. The relations $\preceq_F$ and $\succeq_F$ are the relations between fuzzy numbers. Therefore, I-fuzzy non-linear programming problem (14) *is a generalization of fuzzy non-linear programming problem* (15).

Remark 2. *In general, it is very much difficult to obtain the exact membership and non-membership functions for $\tilde{u}^*$ and $\tilde{v}^*$ as there are several number of parameters involved in their representation. For example, if $\tilde{u} = \langle(\underline{u}_\mu, u, \bar{u}_\mu); (\underline{u}_\nu, u, \bar{u}_\nu)\rangle$ be a TIFN then to determine $\tilde{u}$ completely we need all of these variables. Thus, from computational point of view we take $R(\tilde{u})$ and $R(\tilde{v})$ as real variables u and v and modify the non-linear programming problem* (14) *as follows*

$$\begin{aligned}
&\max\ \{u + v\} \\
\textit{subject to}\ &\sum_{j=1}^{n} R(\tilde{a}_{ij})y_j \leq u + (1-\xi)R(\tilde{p}_i)\ (i = 1, 2, \ldots, m), \\
&\sum_{j=1}^{n} R(\tilde{a}_{ij})y_j \leq u + R(\tilde{p}_i) - (1-\eta)R(\tilde{q}_i)\ (i = 1, 2, \ldots, m),
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{m} R(\tilde{b}_{ij})x_i \leq v + (1-\gamma)R(\tilde{r}_j)\ (j=1,2,\ldots,n),\\
&\sum_{i=1}^{m} R(\tilde{b}_{ij})x_i \leq v + R(\tilde{r}_j) - (1-\delta)R(\tilde{s}_j)\ (j=1,2,\ldots,n),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{a}_{ij})x_i y_j \geq u - (1-\xi)R(\tilde{p}_0),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{a}_{ij})x_i y_j \geq u - R(\tilde{p}_0) + (1-\eta)R(\tilde{q}_0),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{b}_{ij})x_i y_j \geq v - (1-\gamma)R(\tilde{r}_0),\\
&\sum_{i=1}^{m}\sum_{j=1}^{n} R(\tilde{b}_{ij})x_i y_j \geq v - R(\tilde{r}_0) + (1-\delta)R(\tilde{s}_0),\\
&\sum_{i=1}^{m} x_i = 1, \sum_{j=1}^{n} y_j = 1,\\
&0 \leq \xi \leq 1, 0 \leq \eta \leq 1,\\
&0 \leq \gamma \leq 1, 0 \leq \delta \leq 1.
\end{aligned} \tag{16}$$

In this situation, we shall get only the numerical values u^* and v^* instead of $\tilde{u}^*$ and $\tilde{v}^*$, respectively for player I and II. Therefore, we are not able to get exact membership and non-membership values for $\tilde{u}^*$ and $\tilde{v}^*$ which are very much desirable and be satisfied with u^* and v^* that are close to the actual I-fuzzy values for the player I and II, respective.

5. An Application to Media Industry

In this section, decision making problem in media industry is considered to show the validity and applicability of the proposed methodology in real life problem.

Let us consider two major TV station companies T_1 and T_2 aiming to enhance TRPs by increasing their number of viewers. Assume that management of both the companies are rational *i.e.*, they will choose optimal strategies to maximize their own TRPs without co-operation. Let the manager of TV stations T_1 and T_2 make decision to show what kind of TV program to broadcast every day at the peak watching hours (6 P.M.-10 P.M.). They choose two options called strategies-TV Serials (strategy ϵ_1) and Reality Show (strategy ϵ_2). The above problem may be regarded as a bi-matrix game. Namely, the TV station companies T_1 and T_2 are regarded as Players I and II, respectively. They may use strategies ϵ_1 and ϵ_2. Due to a lack of information or imprecision of the available information, the managers of two companies usually are not able to forecast the number of viewers exactly. They estimate the same with a certain confidence degree, but it is possible that they are not so sure about it. Thus, there may survive a hesitation degree. In order to deal with uncertainty, TIFNs are used to express the number of viewers for a particular TV station for a specified time period. Let there are about 20 million of viewers. The marketing research department of both the companies supplied the following pay-off matrices.

$$\tilde{A} = \begin{array}{r} \\ \text{TV Serials} \\ \text{Reality Show} \end{array} \left[\begin{array}{cc} \text{TV Serials} & \text{Reality Show} \\ \langle (7,8,9);(6.5,8,9.5) \rangle & \langle (10.5,12,13);(10,12,14) \rangle \\ \langle (9.5,10,10.5);(9,10,11) \rangle & \langle (4.5,6,7);(4,6,7.5) \rangle \end{array} \right]$$

$$\tilde{B} = \begin{array}{r} \\ \text{TV Serials} \\ \text{Reality Show} \end{array} \left[\begin{array}{cc} \text{TV Serials} & \text{Reality Show} \\ \langle (7,8,9);(6.5,8,9.5) \rangle & \langle (4.5,6,7);(4,6,7.5) \rangle \\ \langle (4.5,6,7);(4,6,7.5) \rangle & \langle (9.5,10,10.5);(9,10,11) \rangle \end{array} \right]$$

The entries of these matrices are TIFNs in millions of viewers who would watch TV station T_1 or TV station T_2 respectively, at the specified time. Other viewers may watch other minor TV stations. Here $\langle (7,8,9);(6.5,8,9.5) \rangle$ in the matrix $\tilde{A}$ is an TIFN, which indicates that expected number of viewers in favour of TV station T_1 is "about 8 million" when both the companies T_1 and T_2 use the strategy ϵ_1 (TV Serials) simultaneously. Other elements (i.e., TIFNs) in the matrices $\tilde{A}$ and $\tilde{B}$ are explained similarly.

We assume that the player I and II have the tolerances

$$\begin{aligned}
\tilde{p}_1 =& \tilde{p}_2 = \tilde{p}_0 = \langle (0.08, 0.10, 0.11); (0.06, 0.10, 0.12) \rangle, \\
\tilde{q}_1 =& \tilde{q}_2 = \tilde{q}_0 = \langle (0.13, 0.15, 0.17); (0.12, 0.15, 0.18) \rangle, \\
\tilde{r}_1 =& \tilde{r}_2 = \tilde{r}_0 = \langle (0.10, 0.12, 0.13); (0.9, 0.12, 0.14) \rangle, \\
\tilde{s}_1 =& \tilde{s}_2 = \tilde{s}_0 = \langle (0.12, 0.15, 0.16); (0.10, 0.15, 0.17) \rangle,
\end{aligned}$$

respectively. The crisp equivalent of the TIFNs $\tilde{a}_{ij} = \langle (\underline{a}_{ij\mu}, a_{ij}, \bar{a}_{ij\mu}); (\underline{a}_{ij\nu}, a_{ij}, \bar{a}_{ij\nu}) \rangle$ $(i = 1,2; j = 1,2)$ can be obtained by using the ranking function defined in Section 2 as follows

$$\begin{aligned}
R(\tilde{a}_{ij}) =& V(\tilde{a}_{ij}, \frac{1}{2}) - A(\tilde{a}_{ij}, \frac{1}{2}) \\
=& \frac{G_\nu(\tilde{a}_{ij}) + G_\mu(\tilde{a}_{ij})}{2} - \frac{H_\mu(\tilde{a}_{ij}) + H_\nu(\tilde{a}_{ij})}{2} \\
=& \frac{(\underline{a}_{ij\nu} + 4a_{ij} + \bar{a}_{ij\nu}) + (\underline{a}_{ij\mu} + 4a_{ij} + \bar{a}_{ij\mu})}{12} - \frac{(\bar{a}_{ij\mu} - \underline{a}_{ij\mu}) + (\bar{a}_{ij\nu} - \underline{a}_{ij\nu})}{6} \\
=& \frac{8(\underline{a}_{ij\mu} + a_{ij} + \underline{a}_{ij\nu}) - (\bar{a}_{ij\mu} + \bar{a}_{ij\nu})}{12}
\end{aligned}$$

Therefore,

$$R(\tilde{a}_{11}) = 12.79, \ R(\tilde{a}_{12}) = 19.42, \ R(\tilde{a}_{21}) = 17.21, \ R(\tilde{a}_{22}) = 8.46.$$

Similarly,

$$R(\tilde{b}_{11}) = 12.79, \ R(\tilde{b}_{12}) = 8.46, \ R(\tilde{b}_{21}) = 8.46, \ R(\tilde{b}_{22}) = 17.21.$$

Also,

$$R(\tilde{p}_0) = 0.14, \ R(\tilde{q}_0) = 0.24, \ R(\tilde{r}_0) = 0.18, \ R(\tilde{s}_0) = 0.22.$$

Thus according to equation (16) the non-linear programming models can be written as:

$$
\begin{aligned}
&\max \{u+v\}\\
\text{subject to } & 12.79y_1+19.42y_2 \le u+0.14(1-\xi)\\
& 17.21y_1+8.46y_2 \le u+0.14(1-\xi)\\
& 12.79y_1+19.42y_2 \le u+0.14-0.24(1-\eta)\\
& 17.21y_1+8.46y_2 \le u+0.14-0.24(1-\eta)\\
& 12.79x_1+8.46x_2 \le v+0.18(1-\gamma)\\
& 8.46x_1+17.21x_2 \le v+0.18(1-\gamma)\\
& 12.79x_1+8.46x_2 \le v+0.18-0.22(1-\delta)\\
& 8.46x_1+17.21x_2 \le v+0.18-0.22(1-\delta)\\
& 12.79x_1y_1+19.42x_1y_2+17.21x_2y_1+8.46x_2y_2 \ge u-0.14(1-\xi)\\
& 12.79x_1y_1+19.42x_1y_2+17.21x_2y_1+8.46x_2y_2 \ge u-0.14+0.24(1-\eta)\\
& 12.79x_1y_1+8.46x_1y_2+8.46x_2y_1+17.21x_2y_2 \ge v-0.18(1-\gamma)\\
& 12.79x_1y_1+8.46x_1y_2+8.46x_2y_1+17.21x_2y_2 \ge v-0.18+0.22(1-\delta)\\
& x_1+x_2=1\\
& y_1+y_2=1\\
& 0\le\xi\le 1, 0\le\eta\le 1\\
& 0\le\gamma\le 1, 0\le\delta\le 1\\
& x_1,x_2,y_1,y_2\ge 0.
\end{aligned}
\tag{17}
$$

Solving (17) with the help of LINGO software we obtained the optimal solution as

Table 1: Solution of the non-linear programming problem.

x_1^*	x_2^*	u^*	y_1^*	y_2^*	v^*
0.746	0.254	15.03	0.641	0.359	11.51

It can be easily seen that the Nash equilibrium values of the bi-matrix game $\widetilde{IFBG}$ for player I and II are, respectively

$$
\begin{aligned}
\tilde{u}^* =& \tilde{E}(\tilde{A}) = \langle(8.12, 9.22, 10.13); (7.62, 9.22, 10.77)\rangle\\
\tilde{v}^* =& \tilde{E}(\tilde{B}) = \langle(6.15, 7.32, 8.27); (5.65, 7.32, 8.77)\rangle
\end{aligned}
$$

which are TIFNs and indicates that the expected number of viewers for TV station T_1 and T_2 are respectively "about 9.22" millions and "about 7.32" millions respectively, when TV station T_1 chooses mixed strategies $(0.746, 0.254)^T$ and TV station T_2 chooses mixed strategies $(0.641, 0.359)^T$. In other words, the expected number of viewers for TV station T_1 is "about 9.22" millions when it broadcast "TV serials" with probability 75% and "reality show" with probability 25%. Similarly, the expected number of viewers for TV station T_2 is "about 7.32" millions when it broadcast "TV serials" with probability 64% and "reality show" with probability 36%.

6. Conclusion

The solution concept for the bi-matrix games with pay-offs of TIFNs is discussed here. It is shown that the equilibrium solution for each player is obtained by solving a crisp non-linear programming problem which is derived from a I-fuzzy non-linear programming problem by employing a suitable ranking function. Further, it should be noted that the proposed I-fuzzy non-linear programming problem is a generalization of fuzzy non-linear programming problem studied by Vidyottama *et al.* [32].

The major limitation of this proposed methodology is that it has not been possible to establish a Nash existence theorem (Theorem 1), so as to guarantee that the all I-fuzzy bi-matrix games will have an "equilibrium solution".

However, it is expected that a more effective methodology will be investigated in near future. Further, in this methodology the bi-matrix games with I-fuzzy pay-offs are considered only but the study on bi-matrix games with I-fuzzy goals as well as I-fuzzy pay-offs is our future work.

Although, the proposed method is illustrated with a media industry problem, it can be applied in decision making theory such as economics, operations research, management, war science etc.

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