EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 8, No. 2, 2015, 239-254
ISSN 1307-5543 – www.ejpam.com

Gaussian Radial Basis Functions for the Solution of an Inverse Problem of Mixed Parabolic-Hyperbolic Type

Farzaneh Parzlivand, Alimardan Shahrezaee *

Department of Mathematics, Alzahra University, Vanak, Tehran, Iran

Abstract. In this paper, we consider an inverse problem of mixed parabolic-hyperbolic type. This inverse problem related to finding the unknown right-hand side of the equation of mixed parabolic-hyperbolic type in a ectangular domain. We proposed a n umerical approach to solve this problem. This method is a combination of collocation method and Gaussian radial basis functions (GA-RBFs). The operational matrix of derivative for GA-RBFs is introduced. The operational matrix of derivative is utilized to reduce the problem to a set of algebraic equations. Using this method, a rapid convergent solution is produced which tends to the exact solution of the problem. The accuracy of the method is tested in term of RMS error. Some examples is included to demonstrate the validity and applicability of the technique.

2010 Mathematics Subject Classifications: 35M10, 65N21, 65N35

Key Words and Phrases: Mixed parabolic-hyperbolic problem; inverse problem; RBFs; collocation method

1. Introduction

Let us consider the equation

$$Lu = \begin{cases} u_t - u_{xx} + B^2 u = f_1(x); & > 0, \\ u_{tt} - u_{xx} + B^2 u = f_2(x); & < 0, \end{cases} \tag{1}$$

of mixed parabolic-hyperbolic type with the unknown right-hand side in the rectangular domain $D = \{(x,t)|0 < x < 1, -\alpha < t < \beta\}$, where $B \geq 0$, $\alpha > 0$ and $\beta > 0$ are given real numbers and let us pose the following inverse problem.

Inverse problem. For any nonnegative integer m let $C^m(D)$ denote the vector space consisting of all functions μ which, together with all their partial derivatives $D^\alpha \mu$ of orders $|\alpha| \leq m$, are

*Corresponding author.

Email addresses: `fparzlivand@gmail.com` (F. Parzlivand), `ashahrezaee@alzahra.ac.ir` (A. Shahrezaee)

continuous on D. In the domain D, it is required to find the functions $u(x,t)$, $f_1(x)$ and $f_2(x)$ satisfying the following conditions:

$$u(x,t) \in C^1(\bar{D}), \quad u_t(x,t) \in C^2(D_-) \cap C^{2,1}_{x,t}(D_+), \tag{2}$$

$$f_1(x) \in C(D_+), \quad f_2(x) \in C(D_-), \tag{3}$$

$$Lu(x,t) = \begin{cases} u_t - u_{xx} + B^2 u = f_1(x); & > 0, \\ u_{tt} - u_{xx} + B^2 u = f_2(x); & < 0, \end{cases} \quad (x,t) \in D_+ \cup D_-, \tag{4}$$

$$u(0,t) = g(t); \quad -\alpha \le tleq\beta, \tag{5}$$

$$u(1,t) = h(t); \quad -\alpha \le tleq\beta, \tag{6}$$

$$u(x,-\alpha) = \psi(x); \quad 0 \le x \le 1, \tag{7}$$

$$u_t(x,-\alpha) = q(x); \quad 0 \le x \le 1, \tag{8}$$

$$u(x,\beta) = \varphi(x); \quad 0 \le x \le 1, \tag{9}$$

where $g(t), h(t), q(t), \psi(x)$ and $\varphi(x)$ are given sufficiently smooth functions, $D_- = D \cap \{t < 0\}$ and $D_+ = D \cap \{t > 0\}$. The existence and uniqueness of the solution of this problem are discussed in [30].

The first fundamental research on the theory of mixed type equations are works of F. Tricomi, and S. Gellerstedt, which were published in the 1920's. Due to the research of F.I. Frankl, I.N. Vekua, M.A. LavrentâĂŹev and A.V.Bitsadze, K.I. Babenko, P. Germain and R. Bader, M. Protter, K. Morawetz, M.S. Salakhitdinov, T.D. Djuraev, A.M. Nakhushev, V.N. Vragov and many other authors, this theory became one of the main directions of the modern theory of partial differential equations [1].

The necessity of the consideration of the parabolic-hyperbolic type equation was specified in 1959 by I. M. Gelfand [7]. He considered the problem on the motion of a gas in a channel surrounded by a porous medium; the motion is described by the wave equation in the channel and by the diffusion equation outside the channel.

At present, the most complete results have been obtained in the study of direct problems for equations of mixed type. For example, boundary-value problems for equations of mixed parabolic-hyperbolic type were studied in [2, 25]. In recent years, in [27, 28], a new approach, the spectral expansion method, was proposed for justifying the existence and uniqueness of solutions of direct problems for mixed-type equations. Via such a method, inverse problems for equations of mixed parabolic-hyperbolic type were solved in [25, 27]. The existence and uniqueness of the solution of these problems and more applications are discussed by several authors [12, 20, 26–29, 31]. However, the theory of the numerical solution of this problem is far from satisfactory.

In this paper, we proposed a numerical technique to solve this problem. This method is a combination of collocation method and GA-RBFs as a truly meshless method. The use of RBFs as a meshless method for numerical solution of partial differential equations is based on the collocation scheme. Due to the collocation technique, this method does not need to evaluate any integral. The main advantage of numerical procedures which use radial basis functions over traditional techniques is the meshless property of these methods.

RBFs are used actively for solving partial differential equations (PDEs) and ordinary differential equations (ODEs). For example see [10, 13, 21]. Also some applications of these functions in solving inverse problems can be found in [9, 17, 18, 23]. Our approach in the current paper is different. We introduce a direct computational method to solve the problem. This method consists of reducing the problem to a set of algebraic equations by expanding the candidate function as GA-RBFs with unknown coefficients.

2. Radial Basis Functions

For the last years, the RBFs method was known as a powerful tool for the scattered data interpolation problem. The main advantage of numerical methods which use radial basis functions is the meshless characteristic of these methods. The use of radial basis functions as a meshless method for the numerical solution of ODEs and PDEs is based on the collocation method.

Recently, RBFs was extended to solve various ODEs and PDEs including the nonlinear Klein-Gordon equation [4], high order ODEs [19], regularized long wave (RLW) equation [11], the case of heat transfer equations [22], Hirota-Satsuma coupled KdV equations [14] and second-order parabolic equation with nonlocal boundary conditions [5].

A radial basis function is a real-valued function whose value depends only on the distance from the origin, so that $\phi(\mathbf{x}) = \phi(\|\mathbf{x}\|)$; or alternatively on the distance from some other point $\mathbf{x}_i$, called a center, so that $\phi_i(x) = \phi(\mathbf{x}, \mathbf{x}_i) = \phi(\|\mathbf{x} - \mathbf{x}_i\|)$. Any function ϕ that satisfies the property $\phi(\mathbf{x}) = \phi(\|\mathbf{x}\|)$ is a radial function. The norm is usually Euclidean distance, although other distance functions are also possible [8]. Commonly used types of radial basis functions include (writing $r = \|\mathbf{x} - \mathbf{x}_i\|$):

- Gaussian (GA): $\phi(r) = e^{-(\varepsilon r)^2}$
- Multiquadric (MQ): $\phi(r) = \sqrt{\varepsilon^2 + r^2}$
- Inverse quadratic (IQ): $\phi(r) = \frac{1}{\varepsilon^2 + r^2}$
- Inverse multiquadric (IMQ): $\phi(r) = \frac{1}{\sqrt{\varepsilon^2 + r^2}}$
- Thin plate spline (TPS): $\phi(r) = r^2 \ln(r)$

where ε is a free positive parameter, often referred to as the shape parameter, to be specified by the user. Despite many research works which are done to finding algorithms for selecting the optimum values of ε [3, 6, 24], the optimal choice of shape parameter is an open problem which is still under intensive investigation.

2.1. Function Interpolation

Let $x_1, x_2, \cdots, x_N \in \Omega \subset \mathbb{R}$ be a given set of scattered data and $\phi_i(x) = \phi(\|x - x_i\|)$, $i = 1, \ldots, N$ be a set of RBFs. The function $s(x)$, $s : \mathbb{R} \to \mathbb{R}$, to be interpolated can be represented by RBFs as [16]:

$$s(x) \simeq \sum_{j=1}^{N} \lambda_j \phi_j(x) = \Lambda^T \Phi_N(x), \tag{10}$$

where the coefficient vector Λ and RBF vector $\Phi_N(x)$ are given by:

$$\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T, \tag{11}$$

$$\Phi_N(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_N(x)]^T, \tag{12}$$

respectively.

2.2. The Operational Matrix of Derivative

Suppose $\phi_i(x)$ is the Gaussian radial basis function, e.i. $\phi_i(x) = e^{-\varepsilon^2(x-x_i)^2}$ and $x \in \mathbb{R}$. The differentiation of vectors $\Phi_N(x)$ in (12) can be expressed as [16]:

$$\Phi'_N(x) = D_N(x)\Phi_N(x), \tag{13}$$

where $D_N(x)$ is $N \times N$ operational matrix of derivative for radial basis function. The matrix $D_N(x)$ can be obtained as:

$$\Phi'_N(x) = [\phi'_1(x), \phi'_2(x), \ldots, \phi'_N(x)]^T = \begin{bmatrix} -2\varepsilon^2(x-x_1)\phi_1(x) \\ -2\varepsilon^2(x-x_2)\phi_2(x) \\ \vdots \\ -2\varepsilon^2(x-x_N)\phi_N(x) \end{bmatrix}. \tag{14}$$

Comparing (13) and (14), we can write:

$$D_N(x) = \begin{bmatrix} -2\varepsilon^2(x-x_1) & 0 & \cdots & 0 \\ 0 & -2\varepsilon^2(x-x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2\varepsilon^2(x-x_N) \end{bmatrix} \tag{15}$$

Also:

$$\Phi''_N(x) = \begin{bmatrix} -2\varepsilon^2 + 4\varepsilon^4(x-x_1)^2 & 0 & \cdots & 0 \\ 0 & -2\varepsilon^2 + 4\varepsilon^4(x-x_2)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2\varepsilon^2 + 4\varepsilon^4(x-x_N)^2 \end{bmatrix} \Phi_N(x).$$

So we have:

$$\Phi''_N(x) = (P_N + D_N^2(x)\Phi_N(x), \tag{16}$$

where:

$$P_N = \begin{bmatrix} -2\varepsilon^2 & 0 & \cdots & 0 \\ 0 & -2\varepsilon^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2\varepsilon^2 \end{bmatrix}. \tag{17}$$

2.3. Error Bound

Suppose that $H = L^2([0,1] \times [-\alpha, \beta])$ and $\{\phi_1(x,t), \phi_2(x,t), \ldots, \phi_N(x,t)\} \subset H$ be the set of Gaussian radial basis functions and $Y = span\{\phi_1(x,t), \phi_2(x,t), \ldots, \phi_N(x,t)\}$; and y be an arbitrary element in H. Since Y is a finite dimensional vector space, y has the unique best approximation out of Y such as $y_0 \in Y$, that is:

$$\forall g \in Y, \|y - y_0\| \leq \|y - g\|.$$

Since $y_0 \in Y$, there exist unique coefficients $\gamma_1, \gamma_2, \ldots, \gamma_N$ such that:

$$y \simeq y_0 = \sum_{i=1}^{N} \gamma_i \phi_i(x,t).$$

Theorem 1. *Let H be a Hilbert space and Y be a closed subspace of H such that $dimY < \infty$ and $\{y_1, y_2, \ldots, y_N\}$ is any basis for Y. Let y be an arbitrary element in H and y_0 be the unique best approximation to y out of Y. Then [15]*

$$\|y - y_0\|^2 = \frac{G(y, y_1, \ldots, y_N)}{G(y_1, y_2, \ldots, y_N)} \leq G(y) = \|y\|^2,$$

where

$$G(y_1, y_2, \cdots, y_N) = \begin{vmatrix} < y_1, y_1 > & < y_1, y_2 > & \cdots & < y_1, y_N > \\ < y_2, y_1 > & < y_2, y_2 > & \cdots & < y_2, y_N > \\ \vdots & \vdots & \ddots & \vdots \\ < y_N, y_1 > & < y_N, y_2 > & \cdots & < y_N, y_N > \end{vmatrix}.$$

The determinant $G(y_1, y_2, \cdots, y_N)$ is called the Gram determinant of $y_1, y_2, \ldots, y_N$.

3. The Employed Transformation

For solving the inverse problem (2)-(9), at first we transform it into a direct problem then use the proposed method. Let us once differentiate equations (4)-(9) with respect to t. Then for the function $v = u_t$ we obtain the equation:

$$Lv(x,t) = \begin{cases} v_t - v_{xx} + B^2 v = 0; & t > 0, \\ v_{tt} - v_{xx} + B^2 v = 0; & t < 0, \end{cases} \tag{18}$$

and from conditions (5) and (6) we get:

$$v(0,t) = u_t(0,t) = g'(t); \quad -\alpha \leq t \leq \beta, \tag{19}$$

$$v(1,t) = u_t(1,t) = h'(t); \quad -\alpha \leq t \leq \beta. \tag{20}$$

Using condition (8), we see that:

$$v(x,-\alpha) = u_t(x,-\alpha) = q(x); \quad 0 \leq x \leq 1. \tag{21}$$

The problem (2)-(9) may be divided into two separate problems. The first problem is the following parabolic problem:

$$v_t - v_{xx} + B^2 v = 0; \quad 0 < x < 1,\ 0 < t < \beta, \tag{22}$$

$$v(0,t) = g'(t); \quad 0 \le t \le \beta, \tag{23}$$

$$v(1,t) = h'(t); \quad 0 \le t \le \beta. \tag{24}$$

The second problem is a hyperbolic problem as follows:

$$v_{tt} - v_{xx} + B^2 v = 0; \quad 0 < x < 1,\ -\alpha < t < 0, \tag{25}$$

$$v(x,-\alpha) = q(x); \quad 0 \le x \le 1, \tag{26}$$

$$v(0,t) = g'(t); \quad -\alpha \le t \le 0, \tag{27}$$

$$v(1,t) = h'(t); \quad -\alpha \le t \le 0. \tag{28}$$

Therefore, for solving the inverse mixed parabolic-hyperbolic problem (2)-(9), we shall investigate the direct parabolic problem (22)-(24) and the direct hyperbolic problem (25)-(28).

4. Numerical Procedures

In this section, the GA-RBFs method are used for solving the mixed parabolic-hyperbolic problem (2)-(9). In order to use the GA-RBFs for solving this problem, we shall investigate the direct parabolic problem (22)-(24) and direct hyperbolic problem (25)-(28).

4.1. Application of CA-RBFs in the Parabolic Problem (22)-(24)

For this problem, let $x_i = \frac{i-1}{N-1}$, $i = 1,2,\ldots,N$ and $t_j = \beta\frac{j}{M}$, $j = 1,2,\ldots,M$. And supposed that $\{(x_i,t_j), i = 1,2,\ldots,N, j = 1,\ldots,M\}$ be a set of scattered nodes. Note that $u(x,0)$ is unknown so that boundary $t = 0$ does not discretization. Then the solution of the problem (22)-(24) by using GA-RBFs is considered as follows:

$$\begin{aligned} v(x,t) \simeq \sum_{i=1}^{N}\sum_{j=1}^{M} v_{ij}\phi_{ij}(x,t) &= \sum_{i=1}^{N}\sum_{j=1}^{M} v_{ij} e^{-\varepsilon^2((x-x_i)^2)+(t-t_j)^2)} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{M} v_{ij}\phi_i(x)\phi_j(t) \\ &= \Phi_N^T(x) V \Phi_M(t), \end{aligned} \tag{29}$$

where $\phi_i(x)$ is the GA-RBF on $[0,1]$, e.i. $\phi_i(x) = e^{-\varepsilon^2(x-x_i)^2}$, $\phi_j(t)$ is the GA-RBF on $[0,\beta]$ and the unknown matrix V is $N \times M$ and can be shown as:

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1M} \\ v_{21} & v_{22} & \cdots & v_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & \cdots & v_{NM} \end{bmatrix}$$

By using (13) we have:

$$v_t(x,t) = \frac{\partial}{\partial t}\Phi_N^T(x)V\Phi_M(t) = \Phi_N^T(x)V\Phi'_M(t) = \Phi_N^T(x)VD_M(t)\Phi_M(t). \tag{30}$$

Also, from (16) we obtain:

$$v_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\Phi_N^T(x)V\Phi_M(t) = \Phi_N''^T(x)V\Phi_M(t) = \Phi_N^T(x)(P_N + D_N^2(x))^T V\Phi_M(t). \tag{31}$$

Using (29)-(31) in (22), we obtain:

$$\Phi_N^T(x)VD_M(t)\Phi_M(t) - \Phi_N^T(x)(P_N + D_N^2(x))^T V\Phi_M(t) + B^2\Phi_N^T(x)V\Phi_M(t) = 0. \tag{32}$$

And using (29) in (23)-(24) yields:

$$\Phi_N^T(0)V\Phi_M(t) - g'(t) = 0, \tag{33}$$

$$\Phi_N^T(1)V\Phi_M(t) - h'(t) = 0. \tag{34}$$

The collocation technique is used for finding the unknown matrix V. We collocate (32) in $(N-2)\times M$ points $\Omega_1 = \{(x_k,t_s)|k = 2,3,\dots,N-1, s = 1,2,\dots,M\}$, we get:

$$\Phi_N^T(x_k)VD_M(t_s)\Phi_M(t_s) - \Phi_N^T(x_k)(P_N + D_N^2(x_k))^T V\Phi_M(t_s) + B^2\Phi_N^T(x_k)V\Phi_M(t_s) = 0;\ (x_k,t_s)\in\Omega_1. \tag{35}$$

Now, collocation (33) and(34) in M points
$\Omega_2 = \{(x_k,t_s)|k = 1, s = 1,2,\dots,M\}$ and $\Omega_3 = \{(x_k,t_s)|k = N, s = 1,2,\dots,M\}$, respectively, yields:

$$\Phi_N^T(0)V\Phi_M(t_s) - g'(t_s) = 0; \quad (x_k,t_s)\in\Omega_2 \tag{36}$$

$$\Phi_N^T(1)V\Phi_M(t_s) - h'(t_s) = 0; \quad (x_k,t_s)\in\Omega_3 \tag{37}$$

Equations (35)-(37) give a $N\times M$ system of linear algebraic equations with the $N\times M$ unknown coefficients v_{ij}. Solving this system, the unknown function of $v(x,t)$ on $t\in[0,\beta]$ can be found.

4.2. Application of GA-RBFs in the Hyperbolic Problem (25)-(28)

Now, let $x_i = \frac{i-1}{N-1}; i = 1,2,\dots,N$, and $t_j = -\alpha\frac{j-1}{M-1}$, $j = 1,2,\dots,M$. The unknown function $v(x,t)$ in (25)-(28) can be approximated as:

$$\begin{aligned} v(x,t) \simeq \sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}\phi_{ij}(x,t) &= \sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}e^{-\varepsilon^2((x-x_i)^2)+(t-t_j)^2)} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}\phi_i(x)\phi_j(t) \\ &= \Phi_N^T(x)W\Phi_M(t), \end{aligned} \tag{38}$$

where $\phi_i(x)$ is the GA-RBF on $[0,1]$, $\phi_j(t)$ is the GA-RBF on $[-\alpha,0]$ and the unknown matrix W is $N\times M$ and can be shown as:

$$\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1M} \\ w_{21} & w_{22} & \cdots & w_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N1} & w_{N2} & \cdots & w_{NM} \end{bmatrix}$$

By using (16) and (30) we have:

$$v_{tt}(x,t)=\frac{\partial^2}{\partial t^2}\Phi_N^T(x)W\Phi_M(t)=\Phi_N^T(x)W\Phi_M''(t)=\Phi_N^T(x)W(P_M+D_M^2(t))\Phi_M(t). \tag{39}$$

Using (39), (30) and (31) in (25), we can write:

$$\Phi_N^T(x)W(P_M+D_M^2(t))\Phi_M(t)-\Phi_N^T(x)(P_N+D_N^2(x))^T W\Phi_M(t)+B^2\Phi_N^T(x)W\Phi_M(t)=0. \tag{40}$$

And using (39) in (26)-(28) yields:

$$\Phi_N^T(x)V\Phi_M(-\alpha)-q(x)=0, \tag{41}$$

$$\Phi_N^T(0)V\Phi_M(t)-g'(t)=0, \tag{42}$$

$$\Phi_N^T(1)V\Phi_M(t)-h'(t)=0. \tag{43}$$

The collocation technique is used for finding unknown matrix W. We collocate Equation (40) in $(N-2)\times(M-1)$ points $\gamma_1=\{(x_k,t_s)|k=2,3,\dots,N-1,s=2,3,\dots,M\}$, we get:

$$\Phi_N^T(x_k)W(P_M+D_M^2(t_s))\Phi_M(t_s)-\Phi_N^T(x_k)(P_N+D_N^2(x_k))^T W\Phi_M(t_s)+B^2\Phi_N^T(x_k)W\Phi_M(t_s)=0, \tag{44}$$

for $(x_k,t_s)\in\gamma_1$. By collocation (41) in N points $\gamma_2=\{(x_k,t_s)|k=1,2,\dots,N,s=1\}$, we have:

$$\Phi_N^T(x_k)V\Phi_M(-\alpha)-q(x_k)=0;\ (x_k,t_s)\in\gamma_2. \tag{45}$$

And collocation (42) and(43) in $(M-1)$ points $\gamma_3=\{(x_k,t_s)|k=1,s=2,3,\dots,M\}$ and $\gamma_4=\{(x_k,t_s)|k=N,s=2,3,\dots,M\}$, respectively, yields:

$$\Phi_N^T(0)V\Phi_M(t_s)-g'(t_s)=0;\quad (x_k,t_s)\in\gamma_3, \tag{46}$$

$$\Phi_N^T(1)V\Phi_M(t_s)-h'(t_s)=0;\quad (x_k,t_s)\in\gamma_3. \tag{47}$$

Equations (44)-(47) give a $N\times M$ system of linear algebraic equations with the $N\times M$ unknown coefficients w_{ij}. Solving this system, the unknown function of $v(x,t)$ on $t\in[-\alpha,0]$ can be found.

4.3. Solve $u(x,t)$, $f_1(x)$ and $f_2(x)$ from $v(x,t)$

In order to recover u from v in $[0,\beta]$, by integration both sides of $v(x,t) = u_t(x,t)$ with respect to the variable t from t to β, we obtain:

$$u(x,t) = u(x,\beta) - \int_t^{\beta} v(x,s)ds = \varphi(x) - \int_t^{\beta} v(x,s)ds. \tag{48}$$

And for recover u in $[-\alpha,0]$, by integration both sides of $v(x,t) = u_t(x,t)$ with respect to the variable t from $-\alpha$ to t, we can write:

$$u(x,t) = u(x,-\alpha) + \int_{-\alpha}^{t} v(x,s)ds = \psi(x) + \int_{-\alpha}^{t} v(x,s)ds. \tag{49}$$

By using (4), we get:

$$v(x,\beta) = u_t(x,\beta) = u_{xx}(x,\beta) - B^2u(x,\beta) + f_1(x), \tag{50}$$

therefore, from (9), we have:

$$f_1(x) = v(x,\beta) - \varphi''(x) + B^2\varphi(x). \tag{51}$$

Also, using (4) and (7), we get:

$$f_2(x) = v_t(x,-\alpha) - \psi''(x) + B^2\psi(x). \tag{52}$$

Therefore, having approximation solution of v $(\tilde{v})$ is determined, then approximation solutions of u, f_1 and f_2 can be obtain as follow:

$$\tilde{u}(x,t) = \varphi(x) - \int_t^{\beta} \tilde{v}(x,s)ds; \quad t > 0, \tag{53}$$

$$\tilde{u}(x,t) = \psi(x) + \int_{-\alpha}^{t} \tilde{v}(x,s)ds; \quad t < 0, \tag{54}$$

$$\tilde{f}_1(x) = \tilde{v}(x,\beta) - \varphi''(x). \tag{55}$$

$$\tilde{f}_2(x) = \tilde{v}_t(x,-\alpha) - \psi''(x). \tag{56}$$

5. Test Examples

In this section two examples are presented to demonstrate the applicability and accuracy of the method. These tests are chosen such that their analytical solutions are known. But the method developed in this research can be applied to more complicated problems. The numerical implementation is carried out in Maple 13.

We tested the accuracy and stability of the method presented in this paper by performing the mentioned method for different values of N and M. To study the convergence behavior of the RBFs method, we applied the following law:

The root mean square (RMS) is described using:

$$RMS(u) = \sqrt{\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}|u(x_i,t_j) - \tilde{u}(x_i,t_j)|^2},$$

$$RMS(f_1) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}|f_1(x_i) - \tilde{f}_1(x_i)|^2},$$

$$RMS(f_2) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}|f_2(x_i) - \tilde{f}_2(x_i)|^2}.$$

For simplicity, we set $\alpha = \beta = 1$ in all following examples

5.1. Example 1

We solve the problem (2)-(9) with $B = 1$ and:

$$g(t) = \begin{cases} 1 + \exp(-2t); & 0 < t < 1, \\ \cos(\sqrt{2}t); & -1 < t < 0, \end{cases}$$

$$h(t) = \begin{cases} (1 + \exp(-2t))\cos(1); & 0 < t < 1, \\ \cos(\sqrt{2}t)\cos(1); & -1 < t < 0, \end{cases}$$

$$q(x) = -\sqrt{2}\sin(\sqrt{2})\cos(x),$$

$$\varphi(x) = (1 + \exp(-2))\cos(x),$$

$$\psi(x) = \cos(\sqrt{2})\cos(x).$$

The exact solution of this problem is:

$$u(x,t) = \begin{cases} (1 + \exp(-2t))\cos(x); & 0 < t < 1, \\ \cos(\sqrt{2}t)\cos(x); & -1 < t < 0, \end{cases}, \quad \begin{cases} f_1(x) = 2\cos(x), \\ f_2(x) = 0. \end{cases}$$

Tables 1 and 2 show the absolute values of error for u at $t = 0.5$ and $t = -0.5$ for different values of M and N, using the method presented in Section 3, respectively. The corresponding results obtained for $f_1(x)$ and $f_2(x)$ are presented in Tables 3 and 4, respectively. Also, Table 5 shows the RMS error values for $f_1(x)$, $u(x,t)$ on the interval $t \in [0,1]$ and $x \in [0,1]$, $f_2(x)$ and $u(x,t)$ on the interval $t \in [-1,0]$ and $x \in [0,1]$ for various values of N and M. It can be obtained from results obtained that the accuracy increases with the increase of the number of collocation points. In addition, the graphs of the error functions $|u(x,t) - \tilde{u}(x,t)|$ on the interval $t \in [0,1]$ and $t \in [-1,0]$, $|f_1(x) - \tilde{f}_1(x)|$ and $|f_2(x) - \tilde{f}_2(x)|$ are plotted in Figure 1.

Table 1: Absolute Values of Error for u from Example 1 with $t = 0.5$ and $\varepsilon = 0.1$.

x	$N = 6, M = 5$	$N = 7, M = 7$	$N = 8, M = 10$	$N = 11, M = 11$
0.0	2.8×10^{-5}	2.1×10^{-7}	7.0×10^{-9}	7.9×10^{-10}
0.2	5.0×10^{-5}	2.8×10^{-7}	1.2×10^{-8}	1.3×10^{-10}
0.4	6.6×10^{-5}	4.9×10^{-7}	1.7×10^{-8}	1.7×10^{-10}
0.6	7.5×10^{-5}	5.5×10^{-7}	2.0×10^{-7}	1.9×10^{-10}
0.8	7.8×10^{-5}	5.6×10^{-7}	2.1×10^{-9}	1.9×10^{-10}
1.0	7.4×10^{-5}	5.3×10^{-7}	2.0×10^{-9}	1.8×10^{-10}

Table 2: Absolute Values of Error for u from Example 1 with $t = -0.5$ and $\varepsilon = 0.1$.

x	$N = 6, M = 5$	$N = 7, M = 7$	$N = 8, M = 10$	$N = 11, M = 11$
0.0	1.2×10^{-5}	5.9×10^{-8}	6.8×10^{-10}	2.5×10^{-10}
0.2	7.8×10^{-3}	1.6×10^{-3}	3.5×10^{-5}	2.1×10^{-5}
0.4	1.2×10^{-2}	2.9×10^{-3}	5.7×10^{-5}	3.4×10^{-5}
0.6	1.2×10^{-2}	2.6×10^{-3}	5.7×10^{-5}	3.4×10^{-5}
0.8	7.8×10^{-3}	1.6×10^{-3}	3.5×10^{-5}	2.1×10^{-5}
1.0	6.9×10^{-6}	3.4×10^{-8}	7.3×10^{-10}	1.5×10^{-10}

Table 3: Absolute Values of Error for $f_1(x)$ from Example 1 with $\varepsilon = 0.1$.

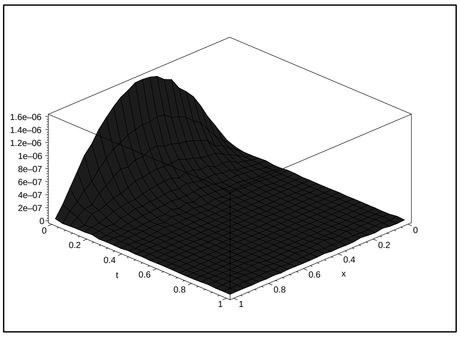
x	$N = 6, M = 5$	$N = 7, M = 7$	$N = 8, M = 10$	$N = 11, M = 11$
0.2	2.8×10^{-4}	2.9×10^{-6}	8.8×10^{-9}	1.5×10^{-9}
0.4	3.1×10^{-4}	2.6×10^{-6}	1.4×10^{-8}	8.3×10^{-10}
0.6	2.8×10^{-4}	2.2×10^{-6}	1.1×10^{-8}	7.5×10^{-10}
0.8	2.4×10^{-4}	1.8×10^{-6}	3.9×10^{-9}	4.4×10^{-10}

Table 4: Absolute Values of Error for $f_2(x)$ from Example 1 with $\varepsilon = 0.1$.

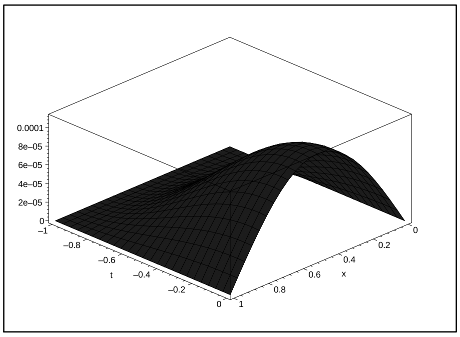
x	$N = 6, M = 5$	$N = 7, M = 7$	$N = 8, M = 10$	$N = 11, M = 11$
0.2	6.9×10^{-2}	1.6×10^{-2}	2.1×10^{-3}	3.5×10^{-4}
0.4	1.1×10^{-1}	2.7×10^{-2}	3.5×10^{-3}	5.7×10^{-4}
0.6	1.1×10^{-1}	2.7×10^{-2}	3.5×10^{-3}	5.7×10^{-4}
0.8	6.9×10^{-2}	1.6×10^{-2}	2.1×10^{-3}	3.5×10^{-4}

Table 5: RMS errors for u, f_1 and f_2 for Example 1 with $\varepsilon = 0.1$.

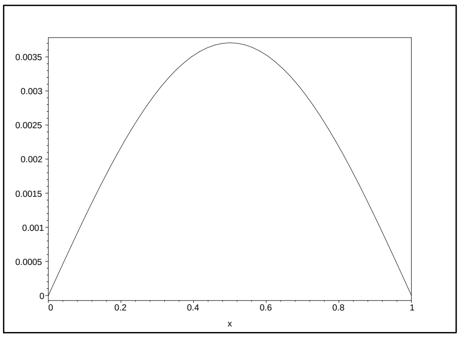
Error	$N=6, M=5$	$N=7, M=7$	$N=8, M=10$	$N=11, M=11$
$RMS(u(x,t)),\ t \in [0,1]$	2.402E–04	1.256E–05	3.511E–07	1.501E–08
$RMS(u(x,t)),\ t \in [-1,0]$	1.104E–02	2.220E–03	4.749E–04	2.951E–05
$RMS(f_1(x))$	2.248E–04	2.063E–06	1.600E–08	7.129E–10
$RMS(f_2(x))$	8.004E–02	1.877E–02	4.017E–03	2.499E–04



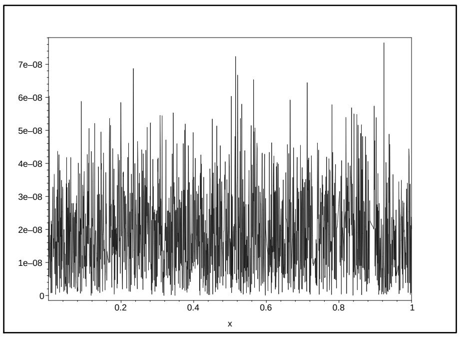
(a) Graph of $|u(x,t) - \tilde{u}(x,t)|$ on $t \in [0,1]$.



(b) Graph of $|u(x,t) - \tilde{u}(x,t)|$ on $t \in [-1,0]$.



(c) Graph of $|f_1(x) - \tilde{f}_1(x)|$.



(d) Graph of $|f_2(x) - \tilde{f}_2(x)|$.

Figure 1: Graph of absolute error for u, f_1 and f_2 by using GA-RBF for Example 1 with $N = 8, M = 10$ and $\varepsilon = 0.1$.

5.2. Example 2

We solve the problem (2)-(9) with $B = 0$ and:

$$g(t) = \begin{cases} 0; & 0 < t < 1, \\ 0; & -1 < t < 0, \end{cases}$$

$$h(t) = \begin{cases} -3; & 0 < t < 1, \\ 2; & -1 < t < 0, \end{cases}$$

$$q(x) = 2x,$$

$$\varphi(x) = x^3 - 3x,$$
$$\psi(x) = \sin(x) + 2x,$$

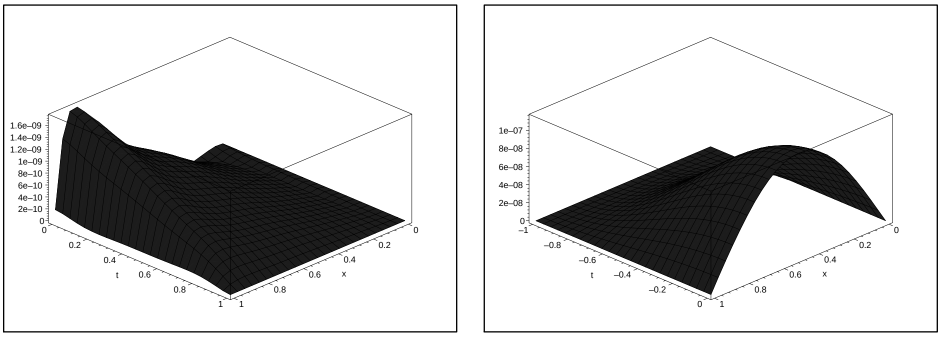
The exact solution of this problem is:

$$u(x,t) = \begin{cases} x^3 - 3xt; & 0 < t < 1, \\ \sin(x) + 2xt; & -1 < t < 0, \end{cases} \quad \begin{cases} f_1(x) = -9x, \\ f_2(x) = \sin(x). \end{cases}$$

Table 6 shows the RMS error values for $u(x,t)$ $f_1(x)$ and $f_2(x)$ for various values of N and M. Also, the graphs of the error functions $|u(x,t) - \tilde{u}(x,t)|$ on the interval $t \in [0,1]$ and $t \in [-1,0]$ are plotted in Figure 2.

Table 6: RMS Errors for u, f_1 and f_2 for Example 2 with $\varepsilon = 0.1$.

Error	$N=6, M=5$	$N=7, M=7$	$N=8, M=10$	$N=11, M=11$
$RMS(u(x,t)),\ t \in [0,1]$	4.698E–10	1.541E–10	1.835E–13	4.716E–15
$RMS(u(x,t)),\ t \in [-1,0]$	4.704E–08	1.112E–08	2.650E–10	1.111E–12
$RMS(f_1(x))$	1.222E–09	2.565E–10	4.910E–13	2.001E–14
$RMS(f_2(x))$	3.697E–007	1.091E–07	7.141E–10	8.304E–11



(a) Graph of $|u(x,t) - \tilde{u}(x,t)|$ on $t \in [0,1]$. (b) Graph of $|u(x,t) - \tilde{u}(x,t)|$ on $t \in [-1,0]$.

Figure 2: Graph of absolute error for u by using GA-RBF for Example 2 with $N = 6, M = 5$ and $\varepsilon = 0.1$.

6. Conclusion

In this paper, we presented a numerical scheme for solving an inverse problem to a class of mixed parabolic-hyperbolic equation. This inverse problem related to finding the unknown right-hand side of the equation of mixed parabolic-hyperbolic type in a rectangular domain. The GA-RBFs were employed. This method is meshless, so that unknown boundary does not require to discretization. Using this method, a rapid convergent solution is produced which tends to the exact solution of the problem. Illustrative examples are included to demonstrate the validity and applicability of the technique.

ACKNOWLEDGEMENTS We would like to thank the anonymous referees for their valuable comments and helpful suggestions to improve the final manuscript.

References

[1] A S Berdyshev and E T Karimov. Some non-local problems for the parabolic-hyperbolic type equation with non-characteristic line of changing type. *Central European Journal of Mathematics*, 4(2):183–193, 2006.

[2] Kh G Bzhikhatlov and A M Nakhushev. A boundary value problem for a mixed equation of parabolic-hyperbolic type. *Doklady Akademii Nauk*, 183(2):261–264, 1968.

[3] A H D Cheng, M A Golberg, E J Kansa, and Q Zammito. Exponential convergence and H-c multiquadric collocation method for partial differential equations. *Numerical Methods Partial Differential Equations*, 19:571–594, 2003.

[4] M Dehghan and A Shokri. Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions. *Journal of Computational and Applied Mathematic*, 230:400–410, 2009.

[5] M Dehghan and M Tatari. Use of radial basis functions for solving the second-order parabolic equation with nonlocal boundary conditions. *Numerical Methods for Partial Differential Equations*, 24:924–938, 2008.

[6] G Fasshauer and J Zhang. On choosing "optimal" shape parameters for RBF approximation. *Numerical Algorithms*, 45:346–368, 2007.

[7] T M Gelfand. Some questions of analysis and differential equations. *Uspekhi Mat. Nauk*, 3(87):3–19, 1959.

[8] M A Golberg. Some recent results and proposals for the use of radial basis functions in the BEM. *Engineering Analysis with Boundary Elements*, 23:285–296, 1999.

[9] Y C Hon and T Wei. A meshless computational method for solving inverse heat conduction problem. *International Series on Advances in Boundary Elements*, 13:135–144, 2002.

[10] Y C Hon and T Wei. A fundamental solution method for inverse heat conduction problem. Engineering Analysis with Boundary Elements. *Journal of Computational and Applied Mathematics*, 28:489–495, 2004.

[11] S U Islam, S Haqb, and A Ali. A meshfree method for the numerical solution of the RLW equation. *Journal of Computational and Applied Mathematics*, 223:997–1012, 2009.

[12] N Yu Kapustin. Tricomi problem for a parabolic-hyperbolic equation with degeneracy in the hyperbolic part. *Differentsial'nye Uravneniya*, 23(1):72–78, 1987.

[13] S Kazem, J A Rad, and K Parand. Radial basis functions methods for solving Fokker Planck equation. *Engineering Analysis with Boundary Elements*, 36:181–189, 2012.

[14] A J Khattak, S I A Tirmizi, and S U Islam. Application of meshfree collocation method to a class of nonlinear partial differential equations. *Engineering Analysis with Boundary Elements*, 33:661–667, 2009.

[15] E Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley and Sons Press, New York, 1978.

[16] M Lakestani and M Dehghan. The use of chebyshev cardinal functions for the solution of a partial differential equation with an unknown time-dependent coefficient subject to an extra measurement. *Journal of Computational and Applied Mathematics*, 235(3):669–678, 2010.

[17] J Li. A radial basis meshless method for solving inverse boundary value problem. *Communications in Numerical Methods in Engineering*, 20:51–61, 2004.

[18] J Li. Application of radial basis meshless methods to direct and inverse biharmonic boundary value problems. *Communications in Numerical Methods in Engineering*, 21, 2005.

[19] N Mai-Duy. Solving high order ordinary differential equations with radial basis function networks. *International Journal for Numerical Methods in Engineering*, 62:824–852, 2005.

[20] A M Nakhushev and Kh G Bzhikhatlov. One boundary-value problem for a mixed-type equation. *Soviet physics Doklady*, 183(2):261–264, 1968.

[21] K Parand, S Abbasbandy, S Kazem, and J A Rad. A novel application of radial basis functions for solving a model of first-order integro-ordinary differential equation. *Communications in Nonlinear Science and Numerical Simulation*, 16:4250–4258, 2011.

[22] K Parand, S Abbasbandy, S Kazem, and A R Rezaei. Comparison between two common collocation approaches based on radial basis functions for the case of heat transfer equations arising in porous medium. *Communications in Nonlinear Science and Numerical Simulation*, 16:1396–1407, 2011.

[23] F Parzlivand and A M Shahrezaee. The use of radial basis functions for the solution of a partial differential equation with an unknown time-dependent coefficient. *Journal of Information and Computing Science.*, 9(4):298–309, 2014.

[24] S Rippa. An algorithm for selecting a good parameter c in radial basis function interpolation. *Advances in Computational Mathematics*, 11:193–210, 1999.

[25] K B Sabitov. Contribution to the theory of equations of mixed parabolic-hyperbolic type with spectral parameters. *Differentsial'nye Uravneniya*, 25(1):117–126, 1989.

[26] K B Sabitov. On the theory of equations of mixed parabolic-hyperbolic type with a spectral parameter. *Differentsial'nye Uravneniya*, 25(1):117–126, 1989.

[27] K B Sabitov. Dirichlet problem for mixed-type equations in a rectangular domain. *Doklady Russian Akademii Nauk*, 413(1):23–26, 2007.

[28] K B Sabitov. On a certain boundary-value problem for a third-order mixed-type equation. *Doklady Russian Akademii Nauk*, 427(5):593–596, 2009.

[29] K B Sabitov and E M Safin. Inverse problem for a parabolic-hyperbolic-type equation in a rectangular domain. *Doklady Russian Akademii Nauk*, 429(4):451–454, 2009.

[30] K B Sabitov and E M Safin. The inverse problem for an equation of mixed parabolic-hyperbolic type. *Matematicheskie Zametki*, 87(6):908–919, 2010.

[31] K B Sabitov and E M Safin. Some questions of analysis and differential equations. *Uspekhi Matematicheskikh Nauk*, 4:55–62, 2010.