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Generalized Hyers-Ulam-Rassias Stability of a System of Bi-Reciprocal Functional Equations

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Abstract. In this paper, we find the generalized Hyers-Ulam-Rassias stability of the system of bi-reciprocal functional equations

$$r(x+u,y)=\frac{r(x,y)r(u,y)}{r(x,y)+r(u,y)},$$
$$r(x,y+v)=\frac{r(x,y)r(x,v)}{r(x,y)+r(x,v)}$$

in the setting of Fréchet spaces.
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1. Introduction

In functional analysis and related areas of mathematics, Fréchet spaces, named after Maurice Fréchet, are special topological vector spaces. They are generalizations of Banach spaces (normed vector spaces which are complete with respect to the metric induced by the norm).

Many vector spaces of holomorphic, differentiable or continuous functions which arise in connection with various problems in analysis and its applications are defined by (at most) countably many conditions, whence they carry a natural Fréchet topology (if they are, in addition, complete). In particular, each Banach space is a Fréchet space and so has a countable basis of absolutely convex zero neighborhoods. A topological vector space X is a Fréchet space if and only if it satisfies the following three properties:

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(i) it is complete as a uniform space

(ii) it is locally convex

(iii) its topology can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, \in X$.

This means that a subset U of X is open if and only if for every u in U, there exists an $\epsilon > 0$ such that $\{v : d(u, v) < \epsilon\}$ is a subset of U. Note that there is no natural notion of distance between two points of a Fréchet space: many different translation-invariant metrics may induce the same topology.

The vector space $C^\infty([0,1])$ of all infinitely often differentiable functions $f : [0,1] \to \mathbb{R}$ becomes a Fréchet space with the seminorms $||f||_k = \sup\{|f^{(k)}(x)| : x \in [0,1]\}$ for every integer $k \geq 0$. Here, $f^{(k)}$ denotes the k^{th} derivative of f, and $f^{(0)} = f$. The spaces $C^\infty(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open, $\mathscr{D}(K)$, $C^\infty(K)$ for $K \subset \mathbb{R}^n$, $\mathscr{D}(\mathbb{R}^n)$, $\mathscr{H}(\Omega)$ for $\Omega \subset \mathbb{C}^n$ open are other examples of Fréchet spaces.

More generally, if M is a compact C^∞ manifold and B is a Banach space, then the set of all infinitely often differentiable functions $f : M \to B$ can be turned into a Fréchet space; the seminorms are given by the suprema of the norms of all partial derivatives.

The space ω of real valued sequences becomes a Fréchet space if we define the k^{th} seminorm of a sequence to be the absolute value of the k^{th} element of the sequence. Convergence in this Fréchet space is equivalent to element-wise convergence.

Not all vector spaces with complete translation-variant metrics are Fréchet spaces. An example is L_p with $p < 1$. Of course, such spaces fail to be locally convex.

The topology of a Fréchet space E can be given by a sequence of seminorms $\|\ \|_1 \leq \|\ \|_2 \leq \dots$ in the following way: a basis of neighborhoods of zero are the sets $U_{k,\epsilon} = \{x \in E : \|x\|_k \leq \epsilon\}$. Such a system is called a fundamental system of seminorms. It is by no means uniquely determined by the topology. In fact, two systems $\|\ \|_1 \leq \|\ \|_2 \leq \dots$ and $\|\ \|_1^\sim \leq \|\ \|_2^\sim \leq \dots$ give the same topology if and only if there exist constants C_k and $n(k) \in \mathbb{N}$ such that

$$\|\ \|_k \leq C_k \|\ \|_{n(k)}^\sim \quad \text{and} \quad \|\ \|_k^\sim \leq C_k \|\ \|_{n(k)}$$

for all k. In this case the systems of seminorms are called equivalent.

A Fréchet space equipped with a fixed fundamental system of seminorms is called a graded Fréchet space. This concept is important in connection with many problems in analysis, where the index of a norm indicates e.g. the order of derivatives involved.

It is a classical fact that the Fréchet spaces are characterized by the existence of a countable, sufficient and increasing family of semi-norms $\{p_i\}_{i\in\mathbb{N}}$ (that is $p_i(x) = 0$ implies $x = 0$ and $p_i(x) \leq p_{i+1}(x)$ for all $x \in X$ and $i \in \mathbb{N}$), which define the pseudo-norm

$$\Delta(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{p_i(x)}{1 + p_i(x)}$$

and the metric $d(x, y) = \Delta(x-y)$ invariant with respect to translations, such that d generates a complete topology equivalent to that of locally convex space. Also, notice that since $\frac{p_i(x)}{1+p_i(x)} \leq 1$ and $\sum_{i=0}^{\infty} \frac{1}{2^i} = 1$, it follows that $\Delta(x) \leq 1$ for all $x \in X$ (see [7]).

Moreover, d has the properties given by the following.

Theorem 1 ([7]). *Let $(X, \{p_i\}_{i\in\mathbb{N}}, d)$ be a Fréchet space, then*

(i) $d(cx, cy) \leq d(x, y)$ *for* $x, y \in X$, $|c| < 1$;

(ii) $d(x+u, y+v) \leq d(x, y) + d(u, v)$ *for* $x, y, u, v \in X$;

(iii) $d(kx, ky) \leq d(rx, ry)$ *if* $k, r \in \mathbb{R}$, $0 < k \leq r$;

(iv) $d(kx, ky) \leq kd(x, y)$ *for* $x, y \in X$, $k \in \mathbb{N}$, $k \geq 2$;

(v) $d(cx, cy) \leq (|c|+1)d(x, y)$ *for all* $x, y \in X$ *and* $c \in \mathbb{R}$.

Fréchet spaces are studied because even though their topological structure is more complicated due to the lack of a norm, many important results in functional analysis, like the open mapping theorem and the Banach-Steinhaus theorem, still hold. For further concepts on Fréchet spaces, one can refer to ([6, 14, 17]).

The stability theory of functional equations basically deals with the following question: "*Given an approximately linear mapping f, when does a linear mapping T estimating f exist?*" This problem was raised by S.M. Ulam [22] in the year 1940 and D.H. Hyers [11] in the year 1941, gave a first affirmative partial answer to the question of Ulam in the case of Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings in the year 1950 and by Th.M. Rassias [18] for linear mappings by considering an unbounded Cauchy difference in the year 1978. The type of stability investigated by Th.M. Rassias is known as "Hyers-Ulam-Rassias stability" of functional equation.

A generalized form of the theorem given by Th.M. Rassias was advocated by P. Gavruta [8] who replaced the unbounded Cauchy difference in Rassias' theorem by a general control function. This type of stability is called "Generalized Hyers-Ulam-Rassias Stability".

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 4, 5, 10, 12, 13, 15, 16, 21]).

Let A and B be vector spaces. A mapping $J : A \to B$ is called Jensen mapping if J satisfies the functional equation

$$2J\left(\frac{x+y}{2}\right) = J(x) + J(y).$$

Definition 1 ([3]). *Let A and B be vector spaces. A mapping $f : A \times A \to B$ is called a bi-Jensen mapping if f satisfies the system of functional equations*

$$\left.\begin{aligned} 2f\left(\frac{x+y}{2}, z\right) &= f(x,z) + f(y,z) \\ 2f\left(x, \frac{y+z}{2}\right) &= f(x,y) + f(x,z). \end{aligned}\right\} \tag{1}$$

When $A = B = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = axy + bx + cy + d$ is a solution of (1).

Definition 2 ([9])**.** *Let A and B be vector spaces. A mapping $f : A \times A \to B$ is called a bi-quadratic mapping if f satisfies the system of functional equations*

$$\left.\begin{aligned} f(x_1+x_2,y)+f(x_1-x_2,y) &= 2f(x_1,y)+2f(x_2,y) \\ f(x,y_1+y_2)+f(x,y_1-y_2) &= 2f(x,y_1)+2f(x,y_2). \end{aligned}\right\} \tag{2}$$

When $A = B = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) = x^2y^2$ is a solution of (2).

In the year 2010, K. Ravi and B.V. Senthil Kumar [19] investigated the generalized Hyers-Ulam-Rassias stability for the reciprocal functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{3}$$

where $r : \mathbb{R}^* \to \mathbb{R}$ is a mapping with R^* as the space of non-zero real numbers and with the assumptions $x+y \neq 0$, $r(x)+r(y) \neq 0$ and $r(x) \neq 0$, for all $x, y \in \mathbb{R}^*$. The reciprocal function $r(x) = \frac{1}{x}$ is a solution of the functional equation (3).

K.Ravi, J.M. Rassias and B.V. Senthil Kumar [20] obtained the general solution and investigated the generalized Hyers-Ulam-Rassias stability of a 2-variable reciprocal functional equation

$$F(x+u, y+v) = \frac{F(x,y)F(u,v)}{F(x,y)+F(u,v)} \tag{4}$$

where $F : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}$ is a mapping with R^* as the space of non-zero real numbers and with the conditions $x+y \neq 0$, $u+v \neq 0$, $x+u \neq 0$, $y+v \neq 0$, $F(x,y) \neq 0$ and $F(x,y)+F(u,v) \neq 0$ for all $x,u,y,v \in \mathbb{R}^*$. The 2-variable reciprocal function $F(x,y) = \frac{1}{x+y}$ is a solution of the functional equation (4).

Motivated by the system of functional equations (1) and (2), we say that a mapping $r : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is bi-reciprocal if r satisfies the system of functional equations

$$\left.\begin{aligned} r(x+u,y) &= \frac{r(x,y)r(u,y)}{r(x,y)+r(u,y)} \\ r(x,y+v) &= \frac{r(x,y)r(x,v)}{r(x,y)+r(x,v)}. \end{aligned}\right\} \tag{5}$$

It is easy to see that $r(x,y) = \frac{1}{xy}$ is a solution of the system of functional equations (5).

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability problem for the system of functional equations (5). Throughout this paper, we assume that E is a real normed space and F is a real Banach space. We also assume that X is the space of non-zero real numbers and Y is a real Fréchet space with metric d with the conditions $x+u \neq 0$, $y+v \neq 0$, $r(x,y) \neq 0$, $r(x,y)+r(u,y) \neq 0$ and $r(x,y)+r(x,v) \neq 0$ for all $x,u,y,v \in X$.

For notational convenience, let us denote for a given mapping $r : X \to Y$, the difference operators $\delta_r : X \times X \times X \to Y$ and $\Delta_r : X \times X \times X \to Y$ by

$$\delta_r(x,u,y) = d\left(r(x+u,y), \frac{r(x,y)r(u,y)}{r(x,y)+r(u,y)}\right),$$

$$\Delta_r(x,y,v) = d\left(r(x,y+v), \frac{r(x,y)r(x,v)}{r(x,y)+r(x,v)}\right)$$

for all $x,u,y,v \in X$.

2. Generalized Hyers-Ulam-Rassias Stability of the System of Functional Equations (5)

Theorem 2. *Let $G,H : X \times X \times X \to [0,\infty)$ be mappings satisfying*

$$\left.\begin{aligned} &\sum_{i=0}^{\infty} 4^i G(2^i x, 2^i x, 2^i y) < \infty, \\ &\sum_{i=0}^{\infty} 4^i H(2^{i+1} x, 2^i y, 2^i y) < \infty \end{aligned}\right\} \tag{6}$$

for all $x,y \in X$. Let $r : X \times X \to Y$ be a mapping such that

$$\delta_r(x,u,y) \leq G(x,u,y) \tag{7}$$

$$\Delta_r(x,y,v) \leq H(x,y,v) \tag{8}$$

for all $x,u,y,v \in X$. Then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ satisfying (5) *and*

$$d(R(x,y), r(x,y)) \leq 2\sum_{i=0}^{\infty} 4^i G(2^i x, 2^i x, 2^i y) + 4\sum_{i=0}^{\infty} 4^i H(2^{i+1} x, 2^i y, 2^i y) \tag{9}$$

for all $x,y \in X$. The mapping $R(x,y)$ is defined by

$$R(x,y) = \lim_{n\to\infty} 4^n r(2^n x, 2^n y), \text{ for all } x,y \in X.$$

Proof. Setting $u = x$ in (7) and then multiplying by 2 on both sides, we get

$$d(2r(2x,y), r(x,y)) \leq 2G(x,x,y) \tag{10}$$

for all $x,y \in X$. Putting $v = y$ in (8) and then multiplying by 2 on both sides, we obtain

$$d(2r(x,2y), r(x,y)) \leq 2H(x,y,y) \tag{11}$$

for all $x,y \in X$. Replacing x by $2x$ in (11) and then multiplying by 2 on both sides, yields

$$d(4r(2x,2y), 2r(2x,y)) \leq 4H(2x,y,y) \tag{12}$$

for all $x,y \in X$. Combining (10) and (12) and using triangle inequality, we get

$$d(4r(2x,2y), r(x,y)) \leq 2G(x,x,y) + 4H(2x,y,y) \tag{13}$$

for all $x, y \in X$. Now, substituting (x, y) by $(2x, 2y)$ in (13) and then multiplying by 4 on both sides, we have

$$d(16r(4x,4y),4r(2x,2y)) \leq 8G(2x,2x,2y) + 16H(4x,2y,2y) \tag{14}$$

for all $x, y \in X$. Combining (13) and (14), we see that

$$d\left(4^2 r(2^2 x, 2^2 y), r(x,y)\right) \leq 2\sum_{i=0}^{1} 4^i G(2^i x, 2^i x, 2^i y) + 4\sum_{i=0}^{1} 4^i H(2^{i+1}x, 2^i y, 2^i y)$$

for all $x, y \in X$. Using induction arguments, we conclude that

$$\begin{aligned} &d(4^n r(2^n x, 2^n y), r(x,y)) \\ &\leq 2\sum_{i=0}^{n-1} 4^i G(2^i x, 2^i x, 2^i y) + 4\sum_{i=0}^{n-1} 4^i H(2^{i+1}x, 2^i y, 2^i y) \\ &\leq 2\sum_{i=0}^{\infty} 4^i G(2^i x, 2^i x, 2^i y) + 4\sum_{i=0}^{\infty} 4^i H(2^{i+1}x, 2^i y, 2^i y) \end{aligned} \tag{15}$$

for all $x, y \in X$. In order to prove the convergence of the sequence $\{4^n r(2^n x, 2^n y)\}$, replace (x, y) by $(2^m x, 2^m y)$ in (15) and multiply by 4^m to get

$$\begin{aligned} d(4^{n+m} r(2^{n+m}, 2^{n+m} y), 4^m r(2^m x, 2^m y)) =& 4^m d(4^n r(2^{n+m} x, 2^{n+m} y), r(2^m x, 2^m y)) \\ \leq& 2\sum_{i=0}^{\infty} 4^{m+i} G(2^{m+i} x, 2^{m+i} x, 2^{m+i} y) \\ &+ 4\sum_{i=0}^{\infty} 4^{m+i} H(2^{m+i+1} x, 2^{m+i} y, 2^{m+i} y). \end{aligned}$$

Using (6), the right-hand side of the above inequality tends to zero as $m \to \infty$. This shows that $\{4^n r(2^n x, 2^n y)\}$ is a Cauchy sequence in Y. Since Y is a Fréchet space, it follows that the sequence $\{4^n r(2^n x, 2^n y)\}$ converges. Define $R : X \times X \to Y$ by

$$R(x,y) = \lim_{n\to\infty} 4^n r(2^n x, 2^n y)$$

for all $x, y \in X$. It follows from (7) that

$$\begin{aligned} \delta_R(x,u,y) &= \lim_{n\to\infty} 4^n \delta_r(2^n x, 2^n u, 2^n y) \\ &\leq \lim_{n\to\infty} 4^n G(2^n x, 2^n u, 2^n y) = 0 \end{aligned}$$

for all $x, u, y, v \in X$. Also it follows from (8) that

$$\begin{aligned} \Delta_R(x,y,v) &= \lim_{n\to\infty} 4^n \Delta_r(2^n x, 2^n y, 2^n v) \\ &\leq \lim_{n\to\infty} 4^n H(2^n x, 2^n y, 2^n v) = 0 \end{aligned}$$

for all $x, y, v \in X$, which shows that R is bi-reciprocal. To prove R is unique bi-reciprocal mapping, let us consider another bi-reciprocal mapping $R' : X \times X \to Y$ which satisfies (5) and (9). Since $4^n R(2^n x, 2^n y) = R(x, y)$ and $4^n R'(2^n x, 2^n y) = R'(x, y)$ for all $x, y \in X$, we conclude that

$$\begin{aligned} d(R(x,y),R'(x,y)) =& 4^n d(R(2^n x,2^n y),R'(2^n x,2^n y)) \\ \leq& 4^n \left\{ d(R(2^n x,2^n y),r(2^n x,2^n y)) + d(r(2^n x,2^n y),R'(2^n x,2^n y)) \right\} \\ \leq& 4\sum_{i=0}^{\infty} 4^{n+i} G(2^{n+i}x,2^{n+i}x,2^{n+i}x) + 8\sum_{i=0}^{\infty} 4^{n+i} H(2^{n+i+1}x,2^{n+i}y,2^{n+i}y) \end{aligned}$$

for all $x, y \in X$. Using (6), letting $n \to \infty$ in the right-hand side of the above inequality, it follows that $R(x,y) = R'(x,y)$ for all $x, y \in X$, which completes the proof of the theorem. □

Theorem 3. *Suppose the mapping $G, H : E \times E \times E \to [0, \infty)$ satisfy (6) for all $x, y \in E$. If $r : E \times E \to F$ is a mapping such that*

$$\left\| r(x+u,y) - \frac{r(x,y)r(u,y)}{r(x,y)+r(u,y)} \right\| \leq G(x,u,y), \tag{16}$$

$$\left\| r(x,y+v) - \frac{r(x,y)r(x,v)}{r(x,y)+r(x,v)} \right\| \leq H(x,y,v) \tag{17}$$

for all $x, u, y, v \in E$, then there exists a unique bi-reciprocal mapping $R : E \times E \to F$ satisfying (5) and

$$\left\| R(x,y) - r(x,y) \right\| \leq 2\sum_{i=0}^{\infty} 4^i G(2^i x, 2^i x, 2^i y) + 4\sum_{i=0}^{\infty} 4^i H(2^{i+1}x, 2^i y, 2^i y)$$

for all $x, y \in E$.

Proof. Putting $d(a,b) = \|a - b\|$, for all $a, b \in E$ in Theorem 2, the proof follows immediately. □

We investigate the Hyers-Ulam-Rassias stability of the system of functional equations (5) in the corollary presented below.

Corollary 1. *Let $c_1 > 0$ be fixed and $p < -2$. If a mapping $r : E \times E \to F$ satisfies the inequalities*

$$\left.\begin{aligned} \left\| r(x+u,y) - \frac{r(x,y)r(u,y)}{r(x,y)+r(u,y)} \right\| &\leq c_1 \left(\|x\|^p + \|u\|^p + \|y\|^p \right), \\ \left\| r(x,y+v) - \frac{r(x,y)r(x,v)}{r(x,y)+r(x,v)} \right\| &\leq c_1 \left(\|x\|^p + \|y\|^p + \|v\|^p \right) \end{aligned}\right\} \tag{18}$$

for all $x, u, y, v \in E$, then there exists a unique bi-reciprocal mapping $R : E \times E \to F$ satisfying (5) and

$$\left\| R(x,y) - r(x,y) \right\| \leq \left(\frac{2c_1}{1 - 2^{p+2}} \right) \left\{ 2(2^p + 1)\|x\|^p + \|y\|^p \right\} \tag{19}$$

for all $x, y \in E$.

Proof. Considering $G(x,y,z) = H(x,y,z) = c_1\left(\|x\|^p + \left\|y\right\|^p + \|z\|^p\right)$, for all $x,y,z \in E$ in Theorem 3, we get

$$\begin{aligned}
&\left\|R(x,y) - r(x,y)\right\| \\
\leq& 2c_1\sum_{i=0}^{\infty} 4^i\left(\left\|2^i x\right\|^p + \left\|2^i x\right\|^p + \left\|2^i y\right\|^p\right) + 4c_1\sum_{i=0}^{\infty} 4^i\left(\left\|2^{i+1} x\right\|^p + \left\|2^i y\right\|^p + \left\|2^i y\right\|^p\right) \\
\leq& 2c_1\sum_{i=0}^{\infty} 4^i 2^{pi}\left(2\|x\|^p + \left\|y\right\|^p\right) + 4c_1\sum_{i=0}^{\infty} 4^i 2^{pi}\left(2^p\|x\|^p + 2\left\|y\right\|^p\right) \\
\leq& 2c_1\sum_{i=0}^{\infty} 2^{(p+2)i}\left(2\|x\|^p + \left\|y\right\|^p\right) + 4c_1\sum_{i=0}^{\infty} 2^{(p+2)i}\left(2^p\|x\|^p + 2\left\|y\right\|^p\right) \\
\leq& 2c_1\left(\frac{1}{1-2^{p+2}}\right)\left(2\|x\|^p + \left\|y\right\|^p\right) + \left(\frac{4c_1}{1-2^{p+2}}\right)\left(2^p\|x\|^p + 2\left\|y\right\|^p\right) \\
\leq& \left(\frac{2c_1}{1-2^{p+2}}\right)\left\{2\left(2^p+1\right)\|x\|^p + 5\left\|y\right\|^p\right\}, \text{ for all } x,y \in E.
\end{aligned}$$

□

Theorem 4. *Let $G,H : X \times X \times X \to [0,\infty)$ be mappings satisfying*

$$\left.\begin{aligned}
\sum_{i=0}^{\infty}\frac{1}{4^i}G\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{2^{i+1}}\right) < \infty, \\
\sum_{i=0}^{\infty}\frac{1}{4^i}H\left(\frac{x}{2^{i}},\frac{y}{2^{i+1}},\frac{y}{2^{i+1}}\right) < \infty \quad
\end{aligned}\right\} \tag{20}$$

for all $x,y \in X$. Let $r : X \times X \to Y$ be a mapping such that (7) *and* (8) *hold for all $x,u,y,z \in X$. Then there exists a unique bi-reciprocal mapping $R : X \times X \to Y$ satisfying* (5) *and*

$$d(r(x,y),R(x,y)) \leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{4^i}G\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{y}{2^{i+1}}\right) + \sum_{i=0}^{\infty}\frac{1}{4^i}H\left(\frac{x}{2^{i}},\frac{y}{2^{i+1}},\frac{y}{2^{i+1}}\right) \tag{21}$$

for all $x,y \in X$. The mapping $R(x,y)$ is defined by

$$R(x,y) = \lim_{n\to\infty}\frac{1}{4^n}r\left(\frac{x}{2^n},\frac{y}{2^n}\right), \textit{for all} x,y \in X.$$

Proof. Replacing (x,u,y) by $\left(\frac{x}{2},\frac{x}{2},\frac{y}{2}\right)$ in (7) and then dividing by 2, we obtain

$$d\left(\frac{1}{2}r\left(x,\frac{y}{2}\right),\frac{1}{4}r\left(\frac{x}{2},\frac{y}{2}\right)\right) \leq \frac{1}{2}G\left(\frac{x}{2},\frac{x}{2},\frac{y}{2}\right) \tag{22}$$

for all $x,y \in X$. Now, replacing (x,y,v) by $\left(\frac{x}{2},\frac{y}{2},\frac{y}{2}\right)$ in (8), we get

$$d\left(r\left(\frac{x}{2},y\right),\frac{1}{2}r\left(\frac{x}{2},\frac{y}{2}\right)\right) \leq H\left(\frac{x}{2},\frac{y}{2},\frac{y}{2}\right) \tag{23}$$

for all $x, y \in X$. Putting $x = 2x$ in (23), we lead to

$$d\left(r(x,y), \frac{1}{2}r\left(x, \frac{y}{2}\right)\right) \leq H\left(x, \frac{y}{2}, \frac{y}{2}\right) \tag{24}$$

for all $x, y \in X$. Combining (22) and (24), applying triangle inequality, yields

$$d\left(r(x,y), \frac{1}{4}r\left(\frac{x}{2}, \frac{y}{2}\right)\right) \leq \frac{1}{2}G\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}\right) + H\left(x, \frac{y}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Proceeding further and using induction arguments on a positive integer n, we have

$$\begin{aligned} d\left(r(x,y), \frac{1}{4^n}r\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) &\leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{4^i}G\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) + \sum_{i=0}^{n-1}\frac{1}{4^i}H\left(\frac{x}{2^i}, \frac{y}{2^{i+1}}, \frac{y}{2^{i+1}}\right) \\ &\leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{4^i}G\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) + \sum_{i=0}^{\infty}\frac{1}{4^i}H\left(\frac{x}{2^i}, \frac{y}{2^{i+1}}, \frac{y}{2^{i+1}}\right) \end{aligned}$$

for all $x, y \in X$. The rest of the proof is obtained by similar arguments as in Theorem 2. □

Theorem 5. *Suppose the mappings $G, H : E \times E \times E \to [0, \infty)$ satisfy (23) and (24) for all $x, y \in E$. If $r : E \times E \to F$ is a mapping such that (16) and (17) hold for all $x, u, y, v \in E$, then there exists a unique bi-reciprocal mapping $R : E \times E \to F$ satisfying (5) and*

$$\left\|r(x,y) - R(x,y)\right\| \leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{4^i}G\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) + \sum_{i=0}^{\infty}\frac{1}{4^i}H\left(\frac{x}{2^i}, \frac{y}{2^{i+1}}, \frac{y}{2^{i+1}}\right)$$

for all $x, y \in E$.

Proof. By taking $d(a, b) = \|a - b\|$, for all $a, b \in E$ in Theorem 4, we arrive at the desired result. □

Corollary 2. *Let $\epsilon > 0$ be fixed. If $r : E \times E \to F$ satisfies*

$$\left\|r(x+u, y) - \frac{r(x,y)r(u,y)}{r(x,y) + r(u,y)}\right\| \leq \frac{\epsilon}{2},$$
$$\left\|r(x, y+v) - \frac{r(x,y)r(x,v)}{r(x,y) + r(x,v)}\right\| \leq \frac{\epsilon}{2}$$

for all $x, u, y, v \in E$, then there exists a unique bi-reciprocal mapping $R : E \times E \to F$ such that

$$\left\|r(x,y) - R(x,y)\right\| \leq \epsilon, \text{ for all } x, y \in E.$$

Proof. Letting $G(x,y,z) = H(x,y,z) = \frac{\epsilon}{2}$, for all $x,y,z \in E$ in Theorem 5, we lead to

$$\begin{aligned}\left\|r(x,y) - R(x,y)\right\| \leq & \frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{4^i}\frac{\epsilon}{2} + \sum_{i=0}^{\infty}\frac{1}{4^i}\frac{\epsilon}{2} \\ \leq & \frac{3\epsilon}{4}\sum_{i=0}^{\infty}\frac{1}{4^i} = \frac{3\epsilon}{4}\left(\frac{4}{3}\right) = \epsilon, \text{ for all } x, y \in E.\end{aligned}$$

□

Corollary 3. *Let $c_1 > 0$ be fixed and $p > -2$. If a mapping $r : E \times E \to F$ satisfies the inequalities* (18)*, for all $x,u,y,v \in E$, then there exists a unique bi-reciprocal mapping $R : E \times E \to F$ satisfying* (5) *and*

$$\left\|r(x,y) - R(x,y)\right\| \leq \left(\frac{2c_1}{2^{p+2}-1}\right)\left\{2\left(2^p+1\right)\|x\|^p + 5\left\|y\right\|^p\right\}, \text{ for all } x, y \in E.$$

Proof. The proof is similar to that of Corollary 1. □

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References

[1] C. Alsina. On the stability of a functional equation. *General Inequalities, Oberwolfach, Birkhauser, Basel*, 5:263–271, 1987.

[2] T. Aoki. On the stability of the linear transformation in banach spaces. *Journal of Mathematical Society of Japan*, 2:64–66, 1950.

[3] J.H. Bae and W.G. Park. On the solution of a bi-Jensen functional equation and its stability. *Bulletin of the Korean Mathematical Society 3*, 43(3):499–507, 2006.

[4] I.S. Chang and Y.S. Jung. Stability of functional equations deriving from cubic and quadratic functions. *Journal of Mathematical Analysis and Applications*, 283:491–500, 2003.

[5] I.S. Chang and H.M. Kim. On the Hyers-Ulam stability of quadratic functional equations. *Journal of Inequalities in Applied Mathematics*, 33:1–12, 2002.

[6] E. Dubinsky. *The structure of nuclear Fréchet spaces, Lecture Notes in Mathematics*. Springer, 1979.

[7] C.S. Gal, S.G. Gal, and G.M. N'Giérékata. Almost automorphic functions in Fréchet spaces and applications to differential equations. *Semigroup Form*, 72(10):23–48, 2005.

[8] P. Gavruta. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *Journal of Mathematical Analysis and Applications*, 184:431–436, 1994.

[9] M.E. Gordji, A. Javadian, and J.M. Rassias. Stability of systems of bi-quadratic and additive-cubic functional equations in Fréchet spaces. *Functional Analysis, Approximation and Computation*, 3(2):57–68, 2011.

[10] M.E. Gordji, S. Zolfaghari, J.M. Rassias, and M.B. Savadkouhi. Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces. *Abstract and Applied Analysis*, Article ID 417473:1–14, 2009.

[11] D.H. Hyers. On the stability of the linear functional equation. *Proceedings of the National Academy of Sciences*, 27:222–224, 1941.

[12] D.H. Hyers, G. Isac, and Th.M. Rassias. *Stability of Functional Equations in Several Variables*. Birkhauser, Basel, 1998.

[13] S.M. Jung. *Hyers-Ulam-Rassias stability of functional equations in Mathematical Analysis*. Hardonic press, Palm Harbor, 2001.

[14] G. Köthe. *Topologische lineare Räume*. Springer, 1960.

[15] J.R. Lee, D.Y. Shin, and C. Park. Hyers-Ulam stability of functional equations in matrix normed spaces. *Journal of Inequalities and Applications*, 22, 2013.

[16] E. Movahednia. Fixed point and generalized Hyers-Ulam-Rassias stability of a quadratic functional equation. *Journal of Mathematics and Computer Science*, 6:72–78, 2013.

[17] A. Pietsch. *Nuclear locally convex spaces*. Ergebnisse der Mathematik 66, Springer, 1972.

[18] Th.M. Rassias. On the stability of the linear mapping in banach spaces. *Proceedings of American Mathematical Society*, 72:297–300, 1978.

[19] K. Ravi and B.V. Senthil Kumar. Ulam-Gavruta-Rassias stability of Rassias Reciprocal functional equation. *Global Journal of Applied Mathematics and Mathematical Sciences*, 3(1-2):57–79, 2010.

[20] K. Ravi, J.M. Rassias, and B.V. Senthil Kumar. Generalized Hyers-Ulam stability of a 2-variable reciprocal functional equation. *Bulletin of Mathematical Analysis and Applications*, 2(2):84–92, 2010.

[21] F. Skof. Proprieta locali e approssimazione di operatori. *Rendiconti del Seminario Matematico e Fisico di Milano*, 53:113–129, 1983.

[22] S.M. Ulam. *Problems in Modern Mathematics, Chapter VI*. Wiley-Interscience, New York, 1964.