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Three Approaches to Inverse Semigroups

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Abstract. I give a historical survey of the three main approaches to the study of the structure of inverse semigroups. The first is that via *inductive groupoids*, as studied by Charles Ehresmann. The second concerns the notion of a *fundamental* inverse semigroup and its *Munn representation*. Finally, the third centres upon the concept of an *E-unitary* or *proper* inverse semigroup and its representation (due to McAlister) by a so-called *P-semigroup*.

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1. Introduction

Since their introduction into the mathematical literature in the early 1950s, inverse semigroups have become one of the most-studied classes of semigroups, with entire monographs [51, 72] devoted to their understanding. Such important objects of study have naturally given rise to a number of different methods for their investigation. In this article, I provide a historical survey, together with an intuitive sketch, of the three main approaches, as identified by Fountain [20, p. 12]: those via inductive groupoids, the Munn representation, and proper inverse semigroups. The present article is a companion piece to my earlier articles [36, 40], and can also be read as a technical addendum to [41, Chapter 10].

The investigation of inverse semigroups by means of their corresponding inductive groupoids (on the history of which, I have already written in detail in [40]) goes back to the historical roots of the notion of an inverse semigroup. The problem arose in the 1930s of giving an abstract characterisation of systems of partial transformations of a set equipped with a partially-defined binary operation. Many mathematicians sought to 'complete' this operation to an everywhere-defined one before axiomatising, eventually giving rise to the abstract notion of an inverse semigroup. If, on the other hand, we are content to retain the original partial

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operation, then the abstract object that results is a so-called inductive groupoid. The fundamental connection between inverse semigroups and inductive groupoids is enshrined in the celebrated Ehresmann–Schein–Nambooripad Theorem, which, at its simplest, states that any inverse semigroup gives rise to a (unique) inductive groupoid, and vice versa, thus enabling us to use the structure of the corresponding groupoid to inform our investigation of the original inverse semigroup (or, indeed, to use the structure of the inverse semigroup to assist in the study of the inductive groupoid). The order relations in inverse semigroups and in inductive groupoids have a key role to play here.

The second approach to inverse semigroups is that via the so-called *Munn representation*. Here, an inverse semigroup S is represented not merely by partial bijections of an arbitrary set but by partial bijections of its semilattice of idempotents. Intimately connected with the notion of a Munn representation are those of *fundamental* inverse semigroups, and also *idempotent-separating congruences* and *morphisms*. As the name suggests, fundamental inverse semigroups may be used, in conjunction with the other notions mentioned here, as a basis for the characterisation of any given class of inverse semigroups.

The third and final of our three approaches to inverse semigroups — that via P-semigroups and related concepts, as espoused by McAlister — is, at its heart, an attempt to adapt to inverse semigroups certain methods that had proved particularly useful in the early theory of semigroups. In the early 1940s, both Rees [80] and Clifford [6] had characterised certain classes of regular semigroups in terms of (amongst other things) groups and semilattices — structures that, at least from the semigroup theorist's point of view, are simple and known (on the work of Rees and Clifford, see [37] or [41, Chapter 6]). McAlister thus attempted to characterise inverse semigroups in similar terms; his structure theorem (the so-called P-Theorem) does not in fact characterise *all* inverse semigroups, but an important subclass termed *E-unitary* (or *proper*) inverse semigroups. Just as Rees had given a simple recipe for what we now term *Rees matrix semigroups* and proved that any one of his semigroups of interest is isomorphic to an appropriate one of these, McAlister gave a slightly more involved recipe for a so-called *P-semigroup* and proved that any E-unitary inverse semigroup is isomorphic to one of these. Once again, the natural order relation in an inverse semigroup has an important role to play in McAlister's construction, as indeed it does in the other approaches to inverse semigroups that are discussed here. The philosophy of McAlister's approach is, however, rather different from that of Munn's, for whilst, via so-called 'covers' (see Section 5.2), McAlister compared his inverse semigroups to other semigroups that are, in a sense, 'larger', Munn took the opposite tack by considering quotients, hence semigroups that are 'smaller' than those under initial consideration.

The present article is structured as follows. In Section 2, I give a very brief sketch of those details of the development of the notion of an inverse semigroup that it will be useful for us to bear in mind as we proceed through the rest of the article. The three approaches to inverse semigroups outlined above are then dealt with in turn in Sections 3, 4 and 5. At the end of the article (in Section 6), I give a rough indication of the ways in which these approaches have been adapted to some classes of semigroups more general than inverse semigroups (specifically, those described in [36]). I assume that the reader is familiar with the basic concepts of the theories of inverse semigroups and partial bijections; further details of these may be found in

[44, Chapter 5] and [51].

2. A Sketch of the Development of Inverse Semigroups

In order to set the scene for what follows, it is necessary first to give a brief indication of the origins of the notion of an inverse semigroup. However, since I have already done this elsewhere, I endeavour to keep this section as short as possible. For a considerably more detailed treatment, see [41, Chapter 10]; the present account consists largely of a shortened version of [40, §2].

Although the fully-formed notion of an inverse semigroup did not emerge until the 1950s, the story of its development begins in the nineteenth century with the *Erlanger Programm*: the principle, advocated by Felix Klein, that every geometry can be viewed as the theory of invariants of a particular group of transformations, and, conversely, that any such group defines a corresponding geometry (see [2, 34]). The inextricable link between groups and the geometrical notion of symmetry, which we now take for granted, was thus forged. However, it was quickly realised that the scheme provided by the Erlanger Programm was not entirely useful in all cases, for there exist geometries (for example, Riemannian geometries) whose groups of automorphisms are trivial. Nevertheless, the earlier success of the Erlanger Programm meant that it was not simply discarded: efforts were made to extend it to these other geometries by using a different algebraic structure, necessarily more general than a group, to describe their symmetries. In the case of differential geometry, the problem was addressed by Oswald Veblen (1880–1960) and J. H. C. Whitehead (1904–1960), in their 1932 text *The foundations of differential geometry*, with the introduction of the notion of a *pseudogroup*. This was based, in part, upon the concept of a 'continuous transformation group', as introduced by Sophus Lie (1842–1899) [53] (on the history of these, see [7]).

Definition 1 ([95, p. 38])**.** *A* pseudogroup Γ *is a collection of partial homeomorphisms between open subsets of a topological space such that* Γ *is closed under composition and inverses, where we compose* $\alpha, \beta \in \Gamma$ *only if* $\mathrm{im}\ \alpha = \mathrm{dom}\ \beta$.

In the nineteenth century, the study of groups of permutations had gradually given rise to the notion of an *abstract* group. It was natural, therefore, for mathematicians next to seek the corresponding abstract structure for a pseudogroup. As I have discussed elsewhere (see, for example, [40]), there turned out to be two distinct (though closely connected) solutions to this problem: one where we retain the partial composition present in Definition 1, and one where we complete it to a fully-defined composition. In the case of a partial composition, we arrive at the notion of an *inductive groupoid*, which I will discuss further in Section 3. For a fully-defined composition, however, the problem took rather more effort to solve, for it was not immediately clear to researchers how they should go about 'completing' the partial composition above: a psychological bar appears to have existed with regard to the admission of the empty transformation into consideration (see [41, §10.2]). This obstacle was finally overcome in the early 1950s by the Russian mathematician V. V. Wagner (1908–1981), who defined the notion of what he called a 'generalised group' [96–98]: an axiomatisation of the collection of all partial bijections of a set, under the now-familiar (left to right) composition of such functions:

$$\operatorname{dom}\alpha\beta = \left[\operatorname{im}\alpha \cap \operatorname{dom}\beta\right]\alpha^{-1},\ x(\alpha\beta) = (x\alpha)\beta,\ \text{for any } x \in \operatorname{dom}\alpha\beta. \qquad (1)$$

The same notion was arrived at independently by the British mathematician G. B. Preston (1925–2015) at around the same time [74–77]; it was Preston who dubbed them 'inverse semigroups'. For a more detailed account of the development of inverse semigroups, see [41, Chapter 10]; other (shorter) accounts may be found in [79, 91, 92].

3. Inverse Semigroups and Inductive Groupoids

Our first approach to the study of inverse semigroups is that via *inductive groupoids*, which has its origins in the work of the French mathematician Charles Ehresmann (1905–1979). This approach has been exploited, and indeed championed, by Lawson [51] in particular. It is also the central theme of [40], although that earlier article focused largely upon the so-called Ehresmann–Schein–Nambooripad Theorem; the present article takes a slightly broader view.

As noted in Section 2, the drive to 'complete' the partial operation of Definition 1 and then axiomatise the resulting system led to the notion of an inverse semigroup. However, if we axiomatise with the partial operation still in place (and take into account the natural ordering possessed by a pseudogroup: that by restriction of mappings), the abstract description of a pseudogroup that we arrive at is an *inductive groupoid*: a special type of small ordered category in which all arrows are invertible. As one might expect, given their common origin, inverse semigroups and inductive groupoids are very closely connected; it is largely via the ordering on each that we may make the link.

As indicated above, the leading light in the development of the inductive groupoid concept was Ehresmann, who realised that the *ordering* of a pseudogroup (that is, by restriction of mappings — just as in a symmetric inverse semigroup) has a crucial role to play. Lawson [51, p. 9] puts it succinctly when he observes that both Wagner and Preston axiomatised $(\mathscr{I}_X, \circ)$, where $\circ$ is the composition of (1), whilst Ehresmann axiomatised $(\mathscr{I}_X, \cdot, \subseteq)$, where $\cdot$ is Veblen and Whitehead's partial composition, and $\subseteq$ denotes the ordering of partial transformations by restriction.

The motivation for Ehresmann's work came from the study of so-called *local structures*: structures defined on topological spaces by using pseudogroups in a manner analogous to the way in which groups are used to define geometries. We therefore begin with a brief introduction to local structures. The discussion here is based upon that of [51, §1.2]; however, in contrast to [51], we will compose functions from left to right, for consistency with the rest of the article.

3.1. Local Structures

Let X and Y be topological spaces. We call X our *model space* — so-called because our goal is to build 'structures' on Y which look locally like pieces of X, and thereby use X to 'model' Y. We define a *chart* from X to Y (hereafter, $X \to Y$) to be a homeomorphism $\phi : U \to V$

between open subsets of X and Y. An *atlas* $\mathscr{A}$ $(X \to Y)$ is a collection of charts $X \to Y$ such that

$$1_Y = \bigcup_{\phi \in \mathscr{A}} \phi^{-1}\phi. \tag{2}$$

A *partial atlas* is a collection of charts which lacks property (2). Let Z be a third topological space and suppose that $\mathscr{A}$ is a partial atlas $X \to Y$ and that $\mathscr{B}$ is a partial atlas $Y \to Z$. Then we may compose atlases to obtain a new partial atlas $X \to Z$, given by

$$\mathscr{A}\mathscr{B} = \{\phi\psi : \phi \in \mathscr{A},\ \psi \in \mathscr{B}\}.$$

We may also 'invert' the partial atlas $\mathscr{A}$ to obtain $\mathscr{A}^{-1} = \{\phi^{-1} : \phi \in \mathscr{A}\}$: a partial atlas $Y \to X$. Lawson [51, pp. 10–11] comments:

> Intuitively, the existence of an atlas from X to Y means that Y can be described by a family of overlapping sets each of which looks like a piece of X. An atlas, in the geographical sense, provides a good example of an atlas in our sense from $\mathbb{R}^n$ to a sphere. ... The problem now arises of dealing with the overlaps between different charts, and it is here that pseudogroups get into the picture.

Let $\phi_i : U_i \to V_i$ and $\phi_j : U_j \to V_j$ be charts in a (partial) atlas $X \to Y$. We compose ϕ_i with ϕ_j^{-1} to obtain the partial homeomorphism

$$\phi_i\phi_j^{-1} : (V_i \cap V_j)\phi_i^{-1} \to (V_i \cap V_j)\phi_j^{-1}.$$

Lawson calls this a *transition function* of the atlas $\mathscr{A}$. We see that the collection of all such transition functions of a (partial) atlas $\mathscr{A}$ is contained in the pseudogroup $\Gamma(X)$ of all partial homeomorphisms between open subsets of X, that is, $\mathscr{A}\mathscr{A}^{-1} \subseteq \Gamma(X)$. In the specific case of differential geometry, we would take X to be $\mathbb{R}^n$ and Y to be some n-dimensional differentiable manifold M. We would then be able to use pieces of $\mathbb{R}^n$ to 'model' pieces of M, hence the following comment from Lawson [51, p. 10]:

> [a]t its simplest, differential geometry concerns spaces which look locally like pieces of $\mathbb{R}^n$ and pseudogroups provide the glue to hold these pieces together.

Remaining in the general setting, we now let f, g be arbitrary partial bijections on a set A. We attempt to define a new partial bijection $f \cup g$ on $\operatorname{dom} f \cup \operatorname{dom} g \subseteq A$. However, $f \cup g$ may fail to be a partial bijection for two reasons:

(1) if $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, then f and g may differ on this set, in which case '$f \cup g$' simply does not make sense;

(2) f may map $x \in \operatorname{dom} f \setminus (\operatorname{dom} f \cap \operatorname{dom} g)$ to the same value as g maps $y \in \operatorname{dom} g \setminus (\operatorname{dom} f \cap \operatorname{dom} g)$, in which case $f \cup g$ fails to be one-one.

If, however, $f \cup g$ *does* form a partial bijection, we say that f and g are *compatible*, and denote the fact by $f \sim g$. A set of partial bijections is said to be *compatible* if its elements are pairwise compatible. The compatibility relation will appear again in Section 5. Returning to considerations of pseudogroups, we define a *complete pseudogroup* on a set X to be an inverse subsemigroup Γ' of $\Gamma(X)$ such that the union of every non-empty compatible subset of Γ' belongs to Γ'.

We are now finally in a position to define the notion of a *local structure*. Let X and Y be topological spaces, with X our model space, and let Γ' be a complete pseudogroup on X. An atlas $\mathscr{A}$ $(X \to Y)$ is *compatible* with Γ' if $\mathscr{A}\mathscr{A}^{-1} \subseteq \Gamma'$, that is, we require all transition functions to belong to Γ'. Let $\mathscr{A}, \mathscr{B}$ be atlases $X \to Y$ which are both compatible with Γ'. If $\mathscr{A}\mathscr{B}^{-1} \subseteq \Gamma'$, we say that $\mathscr{A}$ and $\mathscr{B}$ are *compatible modulo* Γ'. Let $\Gamma'(X,Y)$ denote the collection of all atlases $X \to Y$ which are compatible with Γ'. It is possible to show that 'compatibility modulo Γ'' is an equivalence relation on $\Gamma'(X,Y)$, and that every equivalence class has a maximum element [51, Proposition 1.2.2]. This maximum element is called a *complete atlas compatible with* Γ' and is said, finally, to define a Γ'-*structure* (or *local structure*) on Y. Any atlas $\mathscr{A}$ $(X \to Y)$ determines a local structure on Y in this way, namely, that determined by the maximum element of the equivalence class containing $\mathscr{A}$. Lawson [51, p. 14] says of local structures that they

> are analogues of the geometries in the Erlanger Programm and the pseudogroup Γ' replaces the group.

Lawson gives two examples of local structures on the same page. It was the abstract study of such notions that led Ehresmann to the concept of an inductive groupoid, beginning in around 1947 (see [14], together with [3–5] on the history of groupoids).

3.2. Inductive Groupoids

I have so far made repeated use of the term 'inductive groupoid' without actually defining it. In fact, I am not going to give a precise definition (since this may be found elsewhere; see, for example, [51] and [40]), but attempt rather to give a more intuitive idea of what an inductive groupoid is.

As I stated earlier, an inductive groupoid is a special type of small ordered category in which all arrows are invertible. Let us start with the notion of a category. There are a couple of different ways of viewing a category (contrast those of [45] and [51], for example), but we will adopt a particularly algebraic point of view, which also enables us to view a category as a directed graph. From this viewpoint, a category is a class upon which there is given a partially-defined binary operation. A category which is based upon a set rather than a class is termed a *small* category. All categories considered from here on will be small categories. Within a category, we distinguish two types of elements: *identities* (or *objects*) on the one hand, and *arrows* (or *morphisms*) on the other. In the representation of a category as a directed graph, the identities become the vertices, whilst the arrows become the edges. Thus, there are two identities associated with each arrow, namely, its initial and terminal vertices. For an arrow x, the initial vertex/identity is denoted by $\mathbf{d}(x)$ ('**d**' for 'domain'), whilst the terminal vertex/identity is denoted by $\mathbf{r}(x)$ ('**r**' for 'range'). We see that in Figure 1, $f = \mathbf{d}(y)$, $g = \mathbf{r}(y)$, and so on. We note that the composition of two arrows is defined only when the range of the first coincides with the domain of the second (cf. Definition 1). Thus, for example, in

Figure 1, we may compose y with z and z with u, but we may not compose y with u. The pictorial representation of a category breaks down slightly when we consider the composition of identities. That we may not compose distinct identities is reasonably clear from the diagram, but what is not obvious is that we may compose any identity j with itself, and that $j \cdot j = j$. Furthermore, we may compose any arrow with its domain on the left, and with its range on the right; the domain serves as a left identity for the arrow, and the range as a right identity. Thus, again referring to Figure 1, we have that the composition $f \cdot y$ exists and is equal to y; similarly, we may compose y with g, with y as the result, and so on — by the above comments, $e \cdot e$ exists and is equal to e. Furthermore, one of the features of composition in a category is that it must be associative wherever it is defined, such as in the case of the composition $y \cdot z \cdot u$ in Figure 1; it is also clear from the diagram that a composition $a \cdot b \cdot c$ is defined precisely when the compositions $a \cdot b$ and $b \cdot c$ are defined.

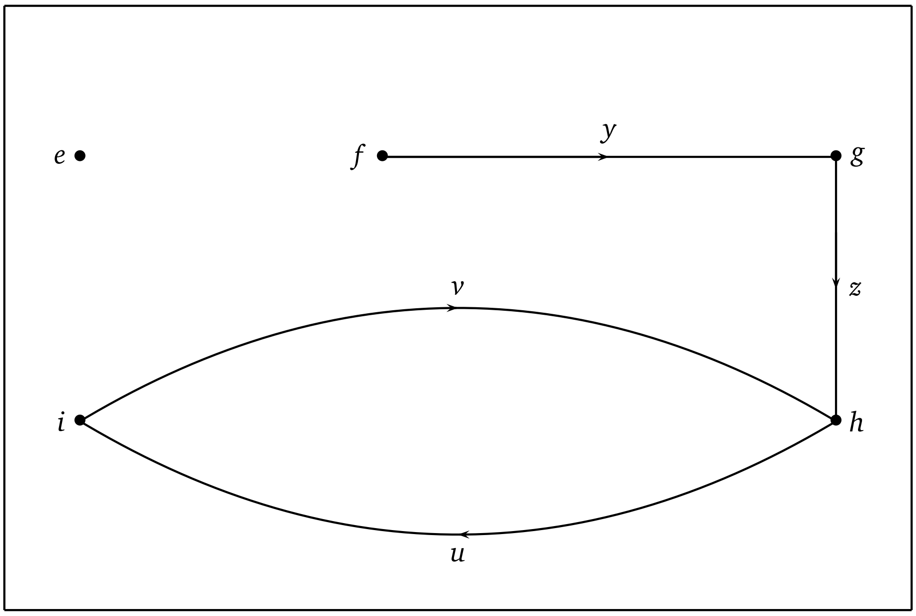

Figure 1: Part of a small category represented as a directed graph, featuring identities e, f, g, h, i and arrows y, z, u, v

Now that we have an intuitive idea of what constitutes a category, it is an easy next step to grasp the notion of a groupoid, for a groupoid is simply a small category in which all arrows are invertible. More precisely, for any arrow a with domain d and range r, there must exist an arrow a^{-1} with domain r and range d such that the compositions $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are defined, and are equal to d and r, respectively.* We see that in Figure 1, u and v may be inverses for each other (we require additional information in order to say whether or not they are in fact inverses), but y and z do not have inverses; e and all the other identities are self-inverse.

An *inductive* groupoid is a special type of *ordered* groupoid, that is, a groupoid whose underlying set is endowed with a partial ordering which satisfies certain conditions. For example,

*Note that we differ here from Lawson [51]: the fact that he composes functions from right to left means that, for him, $\mathbf{d}(a) = a^{-1} \cdot a$ and $\mathbf{r}(a) = a \cdot a^{-1}$.

we demand that the ordering be compatible with composition wherever the latter is defined. Another condition on the ordering deals, intuitively, with 'restriction' of domains. Suppose that a is an arrow in a category and that a has domain d. Suppose also that e is some other identity in the category, with $e \leq d$. Then there exists a unique arrow $b \leq a$ whose domain is e; such an arrow b is usually denoted by $e|a$ (see Figure 2).

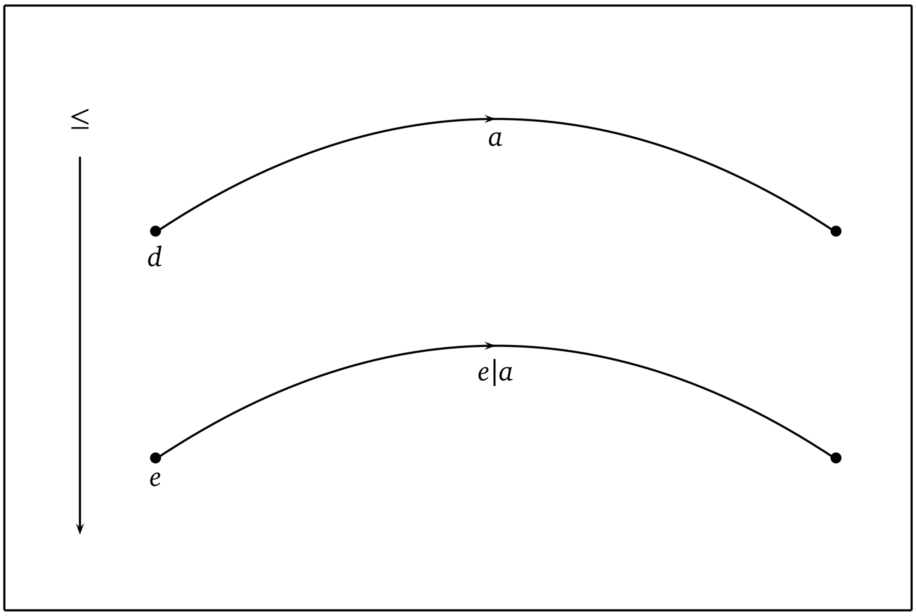

Figure 2: Illustration of the 'restriction' of domains in an ordered category

Other similarly natural conditions are also required for an ordered groupoid. An *inductive* groupoid is an ordered groupoid in which every pair of identities has a greatest lower bound (which is also an identity).

The intuitive definition that we have given for an inductive groupoid may seem a little contrived upon first glance. However, when we remember where this idea comes from, it begins to seem much more natural. Recall Lawson's observation that inductive groupoids emerged as a result of Ehresmann's axiomatisation of $(\mathscr{I}_X, \cdot, \subseteq)$. It is not too difficult to see that $(\mathscr{I}_X, \cdot, \subseteq)$ does indeed form an inductive groupoid, often denoted by $\mathscr{G}_X$ and termed the *symmetric groupoid* on X — see [22], for example. The identities in $\mathscr{G}_X$ are none other than the partial identity transformations I_A, for $A \subseteq X$. Thus, the fact that the composition $\alpha \cdot \beta$ is defined only if $\operatorname{im} \alpha = \operatorname{dom} \beta$ corresponds to the abstract condition $\mathbf{r}(\alpha) = \mathbf{d}(\beta)$, where $\mathbf{r}(\alpha) = I_{\operatorname{im} \alpha}$ and $\mathbf{d}(\beta) = I_{\operatorname{dom} \beta}$. Moreover, restriction of mappings can be shown to satisfy the various properties required of the partial ordering in the abstract definition. The 'inductive' condition holds because $I_{A \cap B}$ is the greatest lower bound of I_A and I_B. Under the Ehresmann–Schein–Nambooripad Theorem (see below), $\mathscr{G}_X$ corresponds to the symmetric inverse semigroup $\mathscr{I}_X$.

The development of this abstract, category-theoretic description of partial bijections of a set has its origins in a 1957 paper by Ehresmann [13], in which the notion of a local structure was treated rigorously: Schein [90, p. 194] comments that

> Ehresmann was probably the first to recognize the importance of category theory for differential geometry.

However, Ehresmann's 'inductive' condition was slightly more stringent than that given above (see [51, §§1.6 and 4.4] for more details on Ehresmann's publications). Nevertheless, Ehresmann achieved an axiomatisation of a pseudogroup of local transformations and it is in this abstract form that he employed a pseudogroup to define the local structures described in Section 3.1. Further work on the axiomatics of pseudogroups was carried out in [11, 12, 55], for example.

3.3. The Ehresmann–Schein–Nambooripad Theorem

The theories of inverse semigroups and inductive groupoids developed separately for some time following the definition of the relevant concepts. It seems that Ehresmann was aware of the connection between his work and that of Wagner (see [51, p. 131]). However, it was left to Schein to make the connection explicit [87, 89]. By relaxing Ehresmann's original conditions for 'inductivity', Schein proved that to any inverse semigroup there corresponds an inductive groupoid and vice versa [89, p. 109 and Theorem 3.4]. Schein's name for the groupoids obtained from inverse semigroups was *replenishable Croisot groupoids*. In essence,[†] the 'replenishable' part of the name expresses the inductive and ordering conditions; 'Croisot groupoid' (in fact, simply a groupoid in our sense) was used for the underlying unordered structure since such objects had appeared previously in a brief paper by Robert Croisot [10], whose own name for them had been *partial groups* (*groupes partiels*). Croisot had arrived at these objects by generalising Brandt groupoids (on which, see [37, §4] or [41, §6.2]).

As Schein observed, constructing an inductive groupoid from an inverse semigroup is reasonably straightforward; we simply need to 'restrict' the semigroup's multiplication in such a way that we obtain an inductive groupoid from the same underlying set. The natural partial order of the inverse semigroup (as defined, for example, in [44, §5.2]) then serves as the ordering in the inductive groupoid. The opposite construction is slightly trickier: we somehow have to take a partial product and construct a fully-defined one. The key to this, however, had appeared earlier in Ehresmann's work, where the partial ordering in an inductive groupoid had always played a central role. Indeed, Ehresmann observed that the partial order contains the information lacking in the partial multiplication; together, they may be used to construct a fully-defined, associative multiplication, and thereby construct an inverse semigroup from an inductive groupoid.

The results of Schein were subsequently generalised to the regular case by K. S. S. Nambooripad, whereby regular semigroups may be associated with more general types of ordered groupoids. Nambooripad [68] also placed the correspondence between inverse semigroups and inductive groupoids in a more technical, category-theoretic setting. This new formulation, together with a subsequent extension due to Nambooripad and Veeramony [69], were

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[†]I say 'in essence' here because Schein did not define a replenishable Croisot groupoid to be an ordered object. On the contrary, he defined a replenishable Croisot groupoid to be a Croisot groupoid which may be obtained from an inverse semigroup in a specified way (see [40]). Nevertheless, he then proved that a Croisot groupoid is replenishable if and only if it can be ordered in such a way as to make it inductive.

gathered together into a single theorem by Lawson [51, Theorem 4.1.8], who named it the *Ehresmann–Schein–Nambooripad Theorem* to reflect its disparate origins. As noted in the Introduction, the Ehresmann–Schein–Nambooripad Theorem, at its simplest, states that we may always construct an inverse semigroup from an inductive groupoid and vice versa (see [51] or [40]).

The Ehresmann–Schein–Nambooripad Theorem has provided a useful tool for inverse semigroup theorists, who may study inductive groupoids as a way of informing them about inverse semigroups: for example, Gilbert [22] studied the partial actions of inductive groupoids and linked these with the partial actions of inverse semigroups. The theorem is tied very closely to the development of the inverse semigroup concept and gives a nice expression of the fact that inverse semigroups and inductive groupoids are two distinct, yet closely related, solutions to the same problem.

4. The Munn Representation

In this section, we turn to the second of the major approaches to the structure of inverse semigroups: the notion of a *fundamental inverse semigroup* and the *Munn representation*, both due largely to W. D. Munn. Fountain describes these as Munn's "most important and influential contributions to semigroup theory" [20, p. 11].

4.1. Separation of Idempotents

To begin, we need the notion of an *idempotent-separating morphism*: a (homo)morphism which restricts to an isomorphism on idempotents, that is to say, it 'separates' idempotents. Such morphisms made an early appearance in Preston's DPhil thesis [74], where they appeared amongst material that was seemingly inspired by group-theoretic considerations, such as the notion of a 'normal' subsemigroup of an inverse semigroup (for further comments on Preston's thesis, see [41, §10.6]). Idempotent-separating morphisms went on to be studied by other authors (indeed, see Section 5). Allied to the notion of an idempotent-separating morphism is that of an *idempotent-separating congruence*: a congruence which has at most one idempotent in each congruence class. In particular, Howie [43] showed that any inverse semigroup has a *maximum* idempotent-separating congruence: an idempotent-separating congruence μ which contains every other idempotent-separating congruence on the semigroup, that is, for any idempotent-separating congruence ρ, if $x \,\rho\, y$, then $x \,\mu\, y$. Howie gave the following characterisation of μ:

$$a \,\mu\, b \Longleftrightarrow a^{-1}ea = b^{-1}eb, \text{ for all } e \in E(S). \tag{3}$$

There may indeed also have been some group-theoretic inspiration behind the study of idempotent-separating congruences; Lawson [51, p. 138] comments:

> Of all the types of congruences on inverse semigroups, it is the idempotent-separating congruences which behave most like group congruences, in that they are entirely determined by their Kernels [*sic*] [namely, the unions of all congruence classes that contain an idempotent].

To return to μ, we note that, as Munn [63] showed, this is also the largest congruence contained in Green's relation $\mathscr{H}$. Further early results on μ are due to Scheiblich [86] and D. G. Green [31]. We observe also that the congruence μ had earlier appeared in work of Wagner [99], who had described various properties of it: the 'maximum idempotent-separating' property, for example, is implicit in Wagner's Theorem 28. The characterisation (3) was later obtained independently by Shiryaev [93].

4.2. Fundamental Inverse Semigroups

The congruence μ went on to play an important role in a 1970 paper by Munn [65]. In this paper, an inverse semigroup S is said to be *fundamental* if μ is the equality relation, that is, $s \mathrel{\mu} t$ if and only if $s = t$. Since equality is the smallest congruence on a semigroup (it is contained in every other congruence), and since μ is the maximum idempotent-separating congruence on an inverse semigroup, this means that a fundamental inverse semigroup has no idempotent-separating congruences other than equality. Thus, fundamental inverse semigroups represent one extreme in the study of idempotent-separating congruences. Munn's goal was to describe the structure of fundamental inverse semigroups. The principal tool for this turned out to be a particular semigroup introduced by Munn in a 1966 paper.

Let E be a semilattice; $\mathscr{I}_E$ denotes the symmetric inverse semigroup on E. We define a semigroup T_E to be the inverse subsemigroup of $\mathscr{I}_E$ consisting of all isomorphisms between principal ideals of E. The semigroup T_E is called the *Munn semigroup (of E)*. It is possible to show that the semilattice of idempotents of T_E is isomorphic to E (see, for example, [44, Theorem 5.4.1]). Furthermore, Munn [64] showed that for any inverse semigroup S there is a morphism $S \to T_{E(S)}$ which maps $E(S)$ isomorphically onto $E(T_{E(S)})$ and which induces the maximum idempotent-separating congruence on S. Lawson [51, p. 141] leaves us in no doubt about the significance of the Munn semigroup:

> This inverse semigroup is second only to the symmetric inverse monoid in its importance in inverse semigroup theory. It is particularly useful in constructing examples of inverse semigroups with a given semilattice of idempotents.

The Munn semigroup plays a central role in Munn's description of fundamental inverse semigroups: the *fundamental representation*, nowadays termed the *Munn representation*. This is a morphism $\alpha : S \to \mathscr{I}_{E(S)}$, where the partial bijection $a\alpha = \alpha_a$ is given by $e\alpha_a = a^{-1}ea$ on the domain $E(S)aa^{-1}$. The Munn representation thus gives a rather different representation of an inverse semigroup from that of Wagner and Preston;[‡] unlike the Wagner–Preston representation, the Munn representation is not necessarily faithful: it is faithful only if S is fundamental (see [82] for efforts to make the Munn representation faithful for any S). The Munn representation has also become a useful tool in the study of congruence-free inverse semigroups (inverse semigroups with no non-trivial congruences) — see [72, §IV.3].

[‡]We note that Preston had in fact carried out some preliminary investigations on the representation of inverse semigroups by partial bijections of their semilattices of idempotents in his early work; see, for example, [77, Theorem 2].

Fountain [20, Theorem 2.5] presents the following theorem, which combines results of Munn's 1966 and 1970 papers:

Theorem 1. *Let S be an inverse semigroup with semilattice E of idempotents. Then*

(1) T_E is an inverse subsemigroup of $\mathscr{I}_E$;

(2) if $\alpha : S \to \mathscr{I}_{E(S)}$ is the fundamental representation, then $\operatorname{im}\alpha$ is a full subsemigroup of $T_{E(S)}$ (that is, it contains all the idempotents of $T_{E(S)}$), $\operatorname{im}\alpha \cong S/\mu$ and $\operatorname{im}\alpha$ is fundamental;

(3) S is fundamental if and only if it is isomorphic to a full inverse subsemigroup of $T_{E(S)}$.

Note, in particular, that T_E is itself fundamental. Besides those results listed above, Munn also gave canonical constructions (by means of partial bijections) for bisimple inverse semigroups, simple inverse semigroups, and inverse semigroups with no non-trivial groups as homomorphic images. Each of these was characterised using a special type of subsemigroup of T_E. Fundamental inverse semigroups thus live up to their name by providing a foundation on which to construct various different classes of inverse semigroups.

Although we have focused on Munn's contributions to this aspect of semigroup theory which bears his name, we note that he was not the only person to consider such problems. Indeed, Wagner [100–102] studied fundamental inverse semigroups independently around the same time, under the name of *antigroups*.[§] As in his earlier investigation of inverse semigroups, Wagner's work on fundamental inverse semigroups was based heavily on the theory of binary relations: in [100], for example, the majority of the results are theorems on these, which are then used to prove results on fundamental inverse semigroups. The main result of this paper is the following extension of Theorem 1 [100, Theorem 11]:

Theorem 2. *An inverse semigroup S is fundamental if and only if it is isomorphic to a semigroup of homeomorphisms between open sets of a T_0-topological space.*

In the years since its introduction, a considerable theory has grown up around the notion of a fundamental inverse semigroup — far too much to cover here, so I refer the reader to the standard texts on inverse semigroups: [44, §5.4], [72, §IV.2] and [51, §5.2]. One appearance of fundamental inverse semigroups in the literature that is worth noting, however, is Mills' use of them in the study of partial symmetries of a convex polygon in the plane [61]. Like certain earlier techniques of Rees and Clifford (discussed in [37] and [41, Chapter 6]), Munn's general approach to the study of fundamental inverse semigroups has provided a model for many subsequent results, a flavour of which will be given (in a specific context) in Section 6.2.

5. Proper Inverse Semigroups and the *P*-Theorem

Our third and final approach to the study of the structure of inverse semigroups is derived in large part from the work of McAlister [57] (though some elements were also present in a

[§]Shiryaev [93] called them *rigid inverse semigroups*, whilst Petrich [72, p. 135] suggested the name *E-faithful inverse semigroup*, in view of the fact that the Munn representation is faithful whenever S is fundamental.

1939 paper by Stanisław Gołąb, as we will see). This is the study of so-called *E-unitary* (or *proper*) inverse semigroups, culminating in the celebrated *P-Theorem*, which gave a complete description of these semigroups.

5.1. The Minimum Group Congruence

To begin, we should define the notion of a *proper* inverse semigroup, and in order to do this, we need that of the 'minimum group congruence'. Any congruence on a semigroup for which the corresponding factor semigroup is a group is called a *group congruence*. Munn [62, Theorem 1] showed that if S is an inverse semigroup, then the congruence σ on S given by

$$s\,\sigma\,t \Longleftrightarrow ea = eb, \text{ for some } e \in E(S),$$

is the *minimum* group congruence on S (that is, a group congruence which is contained in every other group congruence).[¶] Note that σ is necessarily universal in any semigroup with a zero. Since σ is the *smallest* possible group congruence, the factor semigroup S/σ is the *largest* possible group that can be constructed in this way; for this reason, it is often termed the *maximum group image* of S.

It is worth noting the form that σ takes when we move away from the abstract setting and consider partial bijections. The only *idempotent* partial transformations on a set X are the empty transformation ε and the partial identity transformations I_A, for $A \subseteq X$. Thus, for partial bijections α, β on X, we have

$$\alpha\,\sigma\,\beta \Longleftrightarrow I_A\alpha = I_A\beta, \text{ for some } A \subseteq X.$$

But the effect of composing α (respectively, β) with I_A on the left is to *restrict* α (respectively, β) to A; we can see this by applying (1). Therefore, the minimum group congruence takes the following form in any semigroup of partial bijections:

$$\alpha\,\sigma\,\beta \Longleftrightarrow \alpha|_A = \beta|_A, \text{ for some } A \subseteq X. \tag{4}$$

Note, however, that σ is universal in $\mathscr{I}_X$, indeed in any semigroup of partial bijections that contains ε.

Returning to the matter of proper inverse semigroups, we are now in a position to state that an inverse semigroup S is *proper* if $\sigma \cap \mathscr{R}$ is equality, where $\mathscr{R}$ denotes Green's right-hand relation. In a less compact form, this says that if $s\,\sigma\,t$ and $s\,\mathscr{R}\,t$, then $s = t$. The first appearance of proper inverse semigroups in the literature seems to have been in the work of Saitô [84], who considered them in the ordered case and obtained a characterisation which was subsumed by that of McAlister (see below).

[¶]A different characterisation was given by Howie [43]; this may be found in [44, Theorem 5.3.5]. Although the minimum group congruence was exploited more fully by later authors, such as Munn [62], it was in fact present in a pair of earlier works on inverse semigroups and related topics, namely, those of Gołąb [24] and Rees [81] (on which, see §10.2 and §10.6 of [41], respectively). Both Gołąb and Rees were working with partial bijections, and thus (in essence) defined σ in the form given in (4). We note also that, as with so much else in the theory of inverse semigroups, σ put in a brief appearance in the work of Wagner [98, Theorem 4.39].

Another special property of inverse semigroups that it is useful to define at this point is the property of being *E-unitary*. An inverse semigroup S is said to be *E-unitary* if, for any $s \in S$ and any $e \in E(S)$, $es \in E(S)$ implies that $s \in E(S)$ (that is, $E(S)$ forms a *unitary* subset of S). The reason that this concept is useful is that it does in fact give an alternative characterisation of proper inverse semigroups: an inverse semigroup is proper if and only if it is E-unitary (see [44, §5.9]). We will use both characterisations (and, indeed, both names) throughout this section. Note that, rather than 'proper', O'Carroll [70] used the term *reduced* for these inverse semigroups. Indeed, Preston [78] argued that this is a more natural term, since such a semigroup S occurs when the idempotent of S/σ is smallest: when $E(S)$ is a σ-class. Note that the 'one-sidedness' in the definitions of 'proper' and 'E-unitary' is in fact only apparent: a proper inverse semigroup may equivalently be defined by demanding that $\sigma \cap \mathscr{L}$ be equality, whilst we may replace 'es' by 'se' in the definition of an E-unitary inverse semigroup (again, see [44, §5.9]).

5.2. The Covering Theorem

As well as obtaining the P-Theorem (to be stated below) in papers of 1974, McAlister also introduced the notion of an *E-unitary* (or *proper*) *cover* for an inverse semigroup (for a general introduction to covers for semigroups, see [19]). Such a notion contrasts with that of the much-studied 'embedding problems' for semigroups: the question of whether a given semigroup can be embedded in another semigroup with 'nice' properties, for example, a group (see [41, Chapter 5] and [42]). The problem of finding covers for semigroups is in some sense the 'opposite' problem: whereas the embedding problem takes a semigroup S and seeks an injective morphism from S into some other semigroup T with particular properties, the 'covering problem' seeks a surjective morphism *from T onto S*. In particular, as noted above, McAlister sought an *E-unitary cover* for an arbitrary inverse semigroup S: an E-unitary semigroup T and a surjective morphism ϕ such that S is the homomorphic image of T under ϕ. In fact, McAlister went further; he showed that such a cover exists for *any* inverse semigroup via an idempotent-separating morphism (which, intuitively, has the effect of making the cover 'tighter') — the so-called *Covering Theorem*:

Theorem 3 ([51, Theorem 2.2.4]). *Every inverse semigroup is the image of a proper inverse semigroup under an idempotent-separating morphism.*

Lawson [50, p. 73] observes that Gérard Joubert, a student of Ehresmann, had earlier proved a result which is equivalent to the Covering Theorem [46].

We will link E-unitary covers with the P-Theorem below, but we must first define the notion of a *P-semigroup*.

5.3. *P*-semigroups

As already commented, McAlister's P-Theorem gives a complete characterisation of the structure of proper inverse semigroups. The central concept in the P-Theorem is that of a *P-semigroup*, a semigroup constructed according to a particular recipe, which we now describe, following the account of Howie [44, §5.9]. Let $\mathscr{X}$ be a partially ordered set (with partial order $\leq$), and let $\mathscr{Y}$ be a subset of $\mathscr{X}$. We assume the following properties of $\mathscr{Y}$:

(1) $\mathscr{Y}$ is a $\wedge$-*semilattice*, that is, every pair of elements $a, b \in \mathscr{Y}$ has a greatest lower bound $a \wedge b \in \mathscr{Y}$ with respect to $\leq$;

(2) $\mathscr{Y}$ is an *order ideal* of $\mathscr{X}$, that is, if $a \in \mathscr{Y}$ and $b \leq a$, for any $b \in \mathscr{X}$, then $b \in \mathscr{Y}$.

Now suppose that G is a group (with identity e) that acts on $\mathscr{X}$ on the left by order automorphisms (with the action denoted by $\cdot$). We may express this as follows:

(3) for all $a \in \mathscr{X}$, $e \cdot a = a$;

(4) for all $g \in G$ and all $a, b \in \mathscr{X}$, $a \leq b$ if and only if $g \cdot a \leq g \cdot b$;

(5) for all $g, h \in G$ and all $a \in \mathscr{X}$, $g \cdot (h \cdot a) = (gh) \cdot a$.

Note that condition (4) gives order-preservation, and (5) gives the 'morphism' property; conditions (3) and (5) imply that the action is by bijections. Going further, the triple $(G, \mathscr{X}, \mathscr{Y})$ is called a *McAlister triple* if it satisfies the following extra conditions:

(6) for each $b \in \mathscr{X}$, there exist $g \in G$ and $a \in \mathscr{Y}$ such that $g \cdot a = b$;

(7) for all $g \in G$, $\mathscr{Y} \cap g \cdot \mathscr{Y} \neq \emptyset$.

Using a McAlister triple we define a semigroup $P(G, \mathscr{X}, \mathscr{Y})$ to have underlying set

$$\{(a, g) \in \mathscr{Y} \times G : g^{-1} \cdot a \in \mathscr{Y}\} \tag{5}$$

and multiplication

$$(a, g)(b, h) = (a \wedge g \cdot b, gh).$$

As McAlister showed, any semigroup constructed in this way is an inverse semigroup with $(A, g)^{-1} = (g^{-1}A, g^{-1})$, and, moreover, it is proper. Such a semigroup is termed a *P-semigroup*. The *P-Theorem* runs as follows (see, for example, [51, Theorem 7.2.15]):

Theorem 4. *Every proper inverse semigroup is isomorphic to a P-semigroup.*

In this way, E-unitary inverse semigroups were described in a similar spirit to Rees' characterisation of completely 0-simple semigroups [80]. McAlister's original proof of this theorem was somewhat involved, but it wasn't long before Schein [88] and Munn [66] provided shorter ones. Indeed, the P-Theorem seems to exert a certain fascination amongst semigroup theorists, as several distinct proofs have been provided over the years: besides those of Schein (to be dealt with shortly) and Munn (described by McAlister [59, p. 138] as a "gem"), there are proofs by Reilly and Munn [83], Petrich and Reilly [73], and Wilkinson [103], for example. A homological proof was given by Loganathan [54], whilst Margolis and Pin [56] proved the theorem using the Grothendieck construction. Perhaps the most recent proof is that of Kellendonk and Lawson [47] in the context of partial group actions; indeed, Petrich and Reilly's previous approach to proper inverse semigroups had also been by partial actions — using these, it is possible to avoid the need for $\mathscr{Y}$ in the construction, which means furthermore that this technique may be adapted to some of the generalisations of inverse semigroups that we deal with in Section 6, for which the approach involving total actions does not work. See [52] for a list of proofs of the P-Theorem, as well as for examples of E-unitary inverse semigroups.

5.4. Background to the Covering and *P*-Theorems

McAlister linked E-unitary covers for semigroups with isomorphism extension theorems for groups. He noted [58, p. 9] that Higman, Neumann and Neumann [35] had shown that any group G may be embedded in a group H in such a way that every isomorphism between subgroups of G is induced by conjugation by an element of H. McAlister observed that a P-semigroup is easily obtained from this set-up. Let L_G denote the $\wedge$-semilattice of subgroups of G, and let L_H denote that of H. Then H acts on L_H, and L_G is an ideal of L_H, which means that we can construct the P-semigroup $P(H, L_H, L_G)$. For any subgroup A of H, let $i_g|A$ denote the isomorphism of A onto $g^{-1}Ag$, induced by conjugation by $g \in H$. Let A_G be the inverse semigroup of isomorphisms between subgroups of G. Then the map which sends $(A, g) \in P(H, L_H, L_G)$ to $i_g|A$ is an idempotent-separating morphism onto A_G, that is, an E-unitary cover for A_G. McAlister [58, p. 9] commented:

> What this example points out, I think, is that the search for E-unitary covers for inverse semigroups has a natural interpretation. It fits into the framework of isomorphism extension theorems like that of Higman and the Neumanns and thus, in turn, is related to questions about amalgamations.

In [59], McAlister described the background to his construction of P-semigroups and his proof of the P-Theorem. He commented that during the 1960s, much work had been done on the structure of inverse semigroups. There existed various decompositions whereby inverse semigroups could be studied in terms of 'simpler' components, such as groups and semilattices (for a survey of these various approaches, see [78]). However, there was a problem with these structure theorems:

> The building blocks ... were simple and natural. The interrelations between the building blocks were not. [59, p. 134]

In many cases, these interrelations were complicated and considerably less than transparent. McAlister commented:

> Frankly, the structure theory of inverse semigroups had reached the point of diminishing returns. One could reasonably say that each new theorem resulted in a gain of information but a loss of insight. [59, p. 134]

A new approach was therefore needed. This was provided by Scheiblich's construction of the free inverse semigroup (which is proper — see [44, §5.10]), in which it is possible to see many similarities with the construction of a P-semigroup ([85]; see also [59, pp. 134–135]). Working on the principle that "if an object is simple [then] so are its homomorphic images" [59, p. 135], McAlister set out to

> construct a family of inverse semigroups from simple, familiar, naturally related objects in such a way that every inverse semigroup is a nice homomorphic image of a member of the family. [59, p. 136]

Scheiblich's construction provided a guide and the result was the notion of a proper inverse semigroup, via that of an E-unitary cover. As we have seen, the "simple, familiar, naturally related objects" are partially ordered sets, semilattices and groups; "nice" in this context means 'idempotent-separating'. McAlister's Covering and P-Theorems were very much simpler than many of the other pre-existing structure theorems for inverse semigroups, so much so that A. H. Clifford's initial reaction to these theorems was to say "That can't possibly be true" [59, p. 137].

5.5. A Naive Approach

I have glossed over much of the technical development of proper inverse semigroups. As in previous sections, my aim is to provide an intuitive understanding. To this end, we now turn to an article of 1980, in which McAlister provided

> a naive approach to the structure of inverse semigroups to motivate the introduction of P-semigroups and E-unitary inverse semigroups. [58, p. 1]

Although this is not the way in which the notion of an E-unitary inverse semigroup emerged, it is instructive to consider this "naive approach".

McAlister began by recalling the easy observation that in any inverse semigroup S, there is a group H_e around each idempotent e, consisting of those elements which have e as both a left and a right identity. However, these groups do not necessarily exhaust S and it need not be the case that $H_e H_f \subseteq H_{ef}$. As I have described elsewhere ([37, §7] and [41, §6.6]), Clifford had faced this very problem, but, in a paper of 1941, had shown that if the idempotents of S are *central*, then not only is S the (disjoint) union of the groups H_e, but also $H_e H_f \subseteq H_{ef}$. He had shown further that the multiplication in S is determined by a family of morphisms $\phi_{e,f} : H_e \to H_f$ for $e \geq f$. We see that in this case S is completely determined by the groups H_e and its semilattice of idempotents. As McAlister [58, p. 2] commented:

> In view of this, it is natural to wonder to what extent inverse semigroups can be constructed from groups and semilattices. Indeed much of the structure theory of inverse semigroups has been concerned with this problem.

The most naive way to combine a group and a semilattice is simply to take their direct product. If we do this, the result is certainly an inverse semigroup, but it is an inverse semigroup with central idempotents, so we cannot construct every inverse semigroup in this way. As McAlister [58, p. 2] noted, "[o]ne needs a mechanism to account for the non-centrality of idempotents". He suggested that such a mechanism may be found in the semidirect product construction: suppose that a group G acts on the left of a semilattice E by automorphisms — the *semidirect product of E by G* is the set of all pairs $(e, g) \in E \times G$, under the multiplication

$$(e, g)(f, h) = (e(g \cdot f), gh).$$

Such a semidirect product, $P(G,E,E)$, is an inverse semigroup and, moreover, it is an inverse semigroup with non-central idempotents. However, once again, this construction will not serve to describe all inverse semigroups, this time because it cannot have a zero element. Nevertheless, such a semidirect product can be connected with a geometric example [58, Example 1.3]. Let A_n denote n-dimensional real affine space; a *geometric figure* is a compact connected subset of A_n. The set F of geometric figures forms a semilattice under convex join, and the n-dimensional affine group G acts on F via

$$g \cdot a = ag^{-1},$$

for $a \in F$ and $g \in G$, where ag^{-1} denotes the result of applying the transformation g^{-1} to the figure a. We may of course associate a semidirect product $P(G,F,F)$ with this action. McAlister [58, p. 4] observed that $P(G,F,F)$ has a particularly interesting ideal structure, but that it is "not quite so satisfying" from the geometric viewpoint. The problem is that the set of *all* geometric figures is rather too large to handle; we should 'localise' them in some way. We do this by restricting our attention to those geometric figures which contain the origin; let the set of all such be denoted by E. However, this causes a new problem: G may translate elements of E out of E, and so we cannot construct the semidirect product of E by G. Instead, we limit ourselves to those pairs $(a,g) \in E \times G$ for which $g^{-1} \cdot a = ag$ also lies in E. That is, we confine our attention to the set

$$\{(a,g) \in E \times G : g^{-1} \cdot a \in E\},$$

but we retain the semidirect product multiplication. We denote this new construction by $P(G,F,E)$ and observe that it is an inverse subsemigroup of $P(G,F,F)$. Comparing it also with (5), we see that, as the notation suggests, $P(G,F,E)$ is a special case of a P-semigroup. McAlister noted that, just like semidirect products, P-semigroups in which the group is non-trivial cannot have a zero element, and so not all inverse semigroups can be realised as P-semigroups. Thus, the 'goal' of these considerations has not been reached — we have not managed to give a complete description of inverse semigroups in terms of groups and semilattices — but we have derived an interesting class of inverse semigroups to study. In order to study inverse semigroups *with zero* which satisfy something like the E-unitary property, we must instead consider so-called 0-*E-unitary inverse semigroups*: an inverse semigroup with zero is 0-E-unitary whenever, for any non-zero idempotent e, $e \leq s$ implies that s is idempotent. For more details on these semigroups, see [51, Chapter 9].

5.6. Gołąb's Approach

Earlier on, I indicated that the 1939 work of Gołąb contained the ingredients for the proof of the P-Theorem (to quote [92, p. 152], it contained the "key idea"). In order to explain this, we need to recall the *compatibility relation* from Section 3. This was a relation $\sim$ that we defined on partial bijections f, g by the rule that $f \sim g$ if and only if $f \cup g$ is also a partial bijection. Lawson [51, Proposition 1.2.1] demonstrates that $f \sim g$ precisely when fg^{-1} and $f^{-1}g$ are partial identity transformations. Thus, when we pass from partial bijections to the abstract setting, the compatibility relation on an inverse semigroup S is defined as follows:

$$s \sim t \Longleftrightarrow st^{-1}, s^{-1}t \in E(S).$$

The relation $\sim$ is *not* an equivalence relation, since it fails to be transitive in general. This begs the question: for which inverse semigroups is $\sim$ transitive? The answer: $\sim$ is transitive on an inverse semigroup S if and only if S is E-unitary. Moreover, in an E-unitary inverse semigroup, $\sim$ coincides with the minimum group congruence σ [51, Theorem 2.4.6]. The transitivity of the compatibility relation means that an E-unitary inverse semigroup S of partial bijections has the 'unique extension' property (as used by Gołąb — see [41, p. 257]): any $\alpha \in S$ may be extended to at most one partial bijection on a set $A \supseteq \operatorname{dom}\alpha$.

In explaining Gołąb's approach to the P-Theorem, we follow Schein [91]. Let Σ be any inverse semigroup of partial bijections in which the compatibility relation is transitive. Thus, the union (as partial mappings, in the sense of p. 298) of any collection of elements from Σ is also a partial bijection. Since $\sim$ and σ coincide in such a semigroup, any σ-class in Σ is a compatible subset (p. 298). For any $\alpha \in \Sigma$, let $\underline{\alpha}$ denote the partial bijection formed as the union of all elements of the σ-class of α. It is reasonably clear that $\alpha = \underline{\alpha}|_{\operatorname{dom}\alpha}$ and also that such partial bijections $\underline{\alpha}$ are in a one-one correspondence with the σ-classes of Σ. Let $G = \{\underline{\alpha} : \alpha \in \Sigma\}$. Defining an operation $\underline{\alpha} \circ \underline{\beta} = \underline{\alpha\beta}$ in G, we obtain a group $(G, \circ)$ which is isomorphic to Σ/σ. Next, we observe that the elements $\alpha \in \Sigma$ are in a one-one correspondence with pairs of the form $(\operatorname{dom}\alpha, \underline{\alpha})$. This becomes an isomorphism if we define the following operation on the pairs $(\operatorname{dom}\alpha, \underline{\alpha})$:

$$(\operatorname{dom}\alpha, \underline{\alpha})(\operatorname{dom}\beta, \underline{\beta}) = ((\operatorname{dom}\beta)\underline{\alpha}^{-1}, \underline{\alpha} \circ \underline{\beta}).$$

We note that $Y = \{\operatorname{dom}\alpha : \alpha \in \Sigma\}$ is a semilattice isomorphic to $E(\Sigma)$, and, moreover, that Y is contained in X, the inclusion-ordered set of sets of the form $(\operatorname{dom}\beta)\underline{\alpha}^{-1}$, for $\alpha, \beta \in \Sigma$. Finally, we observe that G acts on X:

$$\underline{\alpha} \cdot \operatorname{dom}\beta = (\operatorname{dom}\beta)\underline{\alpha}^{-1}.$$

In this way, the G, X and Y that we have just constructed serve as the ingredients for the representation of Σ as a P-semigroup $P(G, X, Y)$. An abstract version of this proof was given by Schein in 1975 as his new proof of the P-Theorem. He commented, however, that what is

> remarkable is that after [this] was published, I discovered that something like the argument leading to this result was made as early as in 1939 by Gołąb ... [91]

Indeed, analogues of $\underline{\alpha}$, and also the composition $\circ$, appear in Gołąb's work. The fact that he was not working with the full composition of (1) (see [41, §10.2]) does not seem to have hampered him. In fact, Schein noted:

> If Σ does contain the empty transformation [which Schein denoted here by $\emptyset$], then any two elements of Σ are compatible (because each one is compatible with $\emptyset$ and the compatibility relation was supposed to be transitive), so Σ is a semilattice, and the whole theorem degenerates. Thus we may as well consider the case when $\emptyset \notin \Sigma$.

Allowing for the lack of $\emptyset$, the objects studied by Gołąb (specifically, his 'pseudogroups in the narrower sense' — see [41, p. 257]) were in fact E-unitary inverse semigroups. The avoidance of the empty transformation may not have caused such a great difficulty after all.

By way of concluding this section, we comment upon the legacy of McAlister's work. Just like the constructions described in Section 4, as well as those of Rees and Clifford (see the comments in [37] and [41, Chapter 6]), McAlister's Covering and P-Theorems have provided models for the research of subsequent semigroup theorists. We mention, in particular, O'Carroll's Embedding Theorem ([71]; see also [51, Theorem 7.1.5]), which was proved on the basis of the P-Theorem, and provides a useful characterisation of proper inverse semigroups:

Theorem 5. *An inverse semigroup is proper if and only if it can be embedded in the semidirect product of a semilattice by a group.*

Moreover, there are several analogues of the P-Theorem in the literature for various generalisations of inverse semigroups: see the next section. McAlister's goal of constructing a family of inverse semigroups from "simple, familiar, naturally related objects" (see p. 310) was certainly achieved.

6. Some generalisations

I give here a brief indication (with just one or two sample results) of some of the ways in which the constructions and notions of the foregoing sections have been extended to more general classes of semigroups. In the interests of saving space, I confine my attention mostly to non-regular generalisations of inverse semigroups (specifically, those discussed in [36]). Moreover, I do not define any of these semigroups here — I instead refer the reader to the sources cited. A more detailed survey of the use of these methods in the non-regular setting may be found in [29].

6.1. Inductive Categories

The connection between inverse semigroups and inductive groupoids was first generalised to the *ample* semigroups of Fountain [17, 18] by Armstrong [1], where the object to which an ample semigroup corresponds is an *inductive cancellative category*. The latter is obtained from an inductive groupoid by dropping the requirement that all arrows be invertible (modifying the statements of certain of the defining axioms in order to take account of the lack of inverses), but nevertheless insisting that their composition be 'cancellative' in an appropriate sense. Successive generalisations of Armstrong's result were given by Lawson, first for what are now termed *full restriction semigroups* and inductive unipotent categories, and then for arbitrary restriction semigroups and arbitrary inductive categories. In addition, Lawson provided a category-theoretic formulation of these results, in which terms the original Ehresmann–Schein–Nambooripad Theorem may also be expressed (see [51, Theorem 4.1.8]), although this language was not employed by Armstrong. Further technical details were added to the general case in [38]. The historical development of these generalisations is dealt with in more detail in [40], where their precise formulations may also be found.

We note that inverse, ample and restriction semigroups are all inherently 'two-sided': elements of inverse semigroups have two-sided inverses, whilst members of the latter two classes of semigroups come equipped with two unary operations. The connection between ample and restriction semigroups and the appropriate classes of inductive categories therefore emerges as being quite natural: the unary operations become the domain and range operations. Nevertheless, it is also possible to extend the 'Ehresmann–Schein–Nambooripad' approach to *one-sided* restriction semigroups (in which only one unary operation is present). In this case, however, we do not employ inductive categories, but objects that have been dubbed *inductive constellations*. These are 'one-sided' analogues of inductive categories, in which we have a notion corresponding to 'domain', but no concept of 'range'. The definition of these objects, together with the formulation of an Ehresmann–Schein–Nambooripad-type Theorem for these and left restriction semigroups, may be found in [30, 39].

6.2. Munn-type Representations

The first generalisation of Munn's major result (Theorem 1) was derived by Fountain [18] for a particular class of so-called adequate semigroups, which, in common with several of the other classes of non-regular semigroups dealt with here, may be defined in terms of the 'starred' generalisations of Green's relations. Fountain observed, however, that, unlike an inverse semigroup, an adequate semigroup need not have a maximum idempotent-separating congruence. Recall, however, that Munn [63] had observed that the maximum idempotent-separating congruence on an inverse semigroup is the largest congruence contained in Green's relation $\mathscr{H}$. Taking inspiration from this, Fountain therefore pursued a different line of enquiry by investigating the largest congruence contained in the starred Green's relation $\mathscr{H}^*$; any congruence contained in $\mathscr{H}^*$ is idempotent-separating, but the converse does not necessarily hold. Furthermore, as in other situations in the study of adequate semigroups, it transpired that in order to develop a suitable analogue of Theorem 1 (using the same notion of Munn semigroup as in Section 4), it was necessary to deal with a special class of adequate semigroups, namely ample semigroups:

Theorem 6 ([18, Proposition 4.5]). *For any ample semigroup S, there is a morphism $S \to T_{E(S)}$ which maps $E(S)$ isomorphically onto $E(T_{E(S)})$ and which induces the largest congruence contained in $\mathscr{H}^*$.*

Further generalisations of the notions of a Munn semigroup and of a fundamental inverse semigroup to other non-regular cases were later obtained in [15, 16, 21, 27, 48].

Generalisations of Munn's methods to the regular case were considered in [32, 33, 67]. It is interesting to note also that Zhitomirskii [104] even extended the Munn representation to Wagner's generalised heaps (on which, see [41, §10.4]).

6.3. Proper Semigroups of Other Types

Recall the observation in Section 5 that, although defined separately, the notions of 'proper' and 'E-unitary' coincide for inverse semigroups. This, however, is not the case for the non-regular generalisations of inverse semigroups that we deal with here, an observation first made by Fountain [17, Example 3]. Efforts to generalise McAlister's notions to the non-regular case appear therefore to have focused on the semigroups which are 'proper' in a suitable sense (obtained from the original definition by replacement of Green's relations with their appropriate generalisations). We have, for example, the following right ample (monoid) version of Theorem 3 (McAlister's Covering Theorem):

Theorem 7 ([17, Theorem 3.3]). *Every right ample monoid is the image of a proper right ample monoid under an* $\mathscr{L}^*$*-morphism, where an* $\mathscr{L}^*$*-morphism is a morphism* θ *for which* $s\,\mathscr{L}^*\,t$ *whenever* $s\theta = t\theta$.

Note that such an $\mathscr{L}^*$-morphism is idempotent-separating. Fountain subsequently described a generalisation of McAlister's P-semigroups, which he termed *McAlister monoids*, and used these to derive the following generalised P-Theorem:

Theorem 8 ([17, Theorem 4.3]). *Every proper right ample monoid is isomorphic to a McAlister monoid.*

These theorems are easily adapted to the semigroup case. Versions for two-sided ample semigroups appear in [49] as Theorems 3.8 and 2.11, respectively. Further generalisations of McAlister's notions in certain one-sided non-regular cases appeared in [25, 26, 28]. Results for two-sided restriction semigroups took a little longer to develop, but may be found in [8, 9]. We note that the generalisation to the two-sided cases here are rather harder than to the one-sided, and, to pick up on the comments at the end of Section 5.3, are achieved through the use of partial actions.

Regular generalisations of the P-Theorem are discussed in [94], whilst versions for inductive groupoids feature in [23, 60]; the inductive groupoids to which E-unitary inverse semigroups correspond under the Ehresmann–Schein–Nambooripad Theorem are termed *incompressible* inductive groupoids.

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