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ϕ-Prime and ϕ-Primary Elements in Lattice Modules

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Abstract. In this paper, we introduce ϕ-prime and ϕ-primary elements in an L-module M. Many of its characterizations and properties are obtained. By counter examples, it is shown that a ϕ-prime element of M need not be prime, a ϕ-primary element of M need not be ϕ-prime, a ϕ-primary element of M need not be prime and a ϕ-primary element of M need not be primary. Finally, some results for almost prime and almost primary elements of an L-module M with their characterizations are obtained. Also, we introduce the notions of n-potent prime(respectively n-potent primary) elements in L and M to obtain interrelations among them where $n \geqslant 2$.

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1. Introduction

In multiplicative lattices, the study of ϕ-prime and ϕ-primary elements is done by C. S. Manjarekar and A. V. Bingi in [16]. Our aim is to extend the notion of ϕ-prime and ϕ-primary elements in a multiplicative lattice to the notion of ϕ-prime and ϕ-primary elements in a lattice module and study its properties. According to [1], a proper element N of an L-module M is said to be prime if for all $A \in M,\ a \in L,\ aA \leqslant N$ implies either $A \leqslant N$ or $a \leqslant (N : I_M)$. According to [10], a proper element N of an L-module M is said to be primary if for all $A \in M,\ a \in L,\ aA \leqslant N$ implies either $A \leqslant N$ or $a \leqslant \sqrt{N : I_M}$. By restricting where aA lies, weakly prime and weakly primary elements in lattice modules are studied by C. S. Manjarekar et. al. in [19] and [20], respectively. A proper element N of an L-module M is said to be weakly prime if for all $A \in M,\ a \in L,\ O_M \neq aA \leqslant N$ implies either $A \leqslant N$ or $a \leqslant (N : I_M)$. A proper element N of an L-module M is said to be weakly primary if for all $A \in M,\ a \in L,\ O_M \neq aA \leqslant N$ implies either $A \leqslant N$

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or $a \leqslant \sqrt{N : I_M}$. Keeping this in mind, in this paper we define and study ϕ-prime and ϕ-primary elements of an L-module M.

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $e \in L$ is called meet principal if $a \wedge be = ((a : e) \wedge b)e$ for all $a, b \in L$. An element $e \in L$ is called join principal if $(ae \vee b) : e = (b : e) \vee a$ for all $a, b \in L$. An element $e \in L$ is called principal if e is both meet principal and join principal. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leqslant \vee X$ implies the existence of a finite number of elements $a_1, a_2, \cdots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. The set of compact elements of L will be denoted by L_*. If each element of L is a join of compact elements of L, then L is called a compactly generated lattice or simply a CG-lattice. L is said to be a principally generated lattice or simply a PG-lattice if each element of L is a join of principal elements of L. Throughout this paper, L denotes a compactly generated multiplicative lattice with greatest compact element 1 in which every finite product of compact elements is compact.

An element $a \in L$ is said to be proper if $a < 1$. A proper element $m \in L$ is said to be maximal if for every element $x \in L$ such that $m < x \leqslant 1$ implies $x = 1$. A proper element $p \in L$ is called a prime element if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ where $a, b \in L$ and is called a primary element if $ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$ for some $n \in Z_+$ where $a, b \in L_*$. For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a$, for some $n \in Z_+\}$. A multiplicative lattice is called as a Noether lattice if it is modular, principally generated and satisfies the ascending chain condition. A proper element $a \in L$ is said to be nilpotent if $a^n = 0$ for some $n \in Z_+$. According to [9], a proper element $p \in L$ is said to be almost prime if for all $a,\ b \in L$, $ab \leqslant p$ and $ab \nleqslant p^2$ implies either $a \leqslant p$ or $b \leqslant p$ and according to [15], a proper element $p \in L$ is said to be almost primary if for all $a,\ b \in L$, $ab \leqslant p$ and $ab \nleqslant p^2$ implies either $a \leqslant p$ or $b \leqslant \sqrt{p}$. Further study on almost prime and almost primary elements of a multiplicative lattice L is seen in [16], [5] and [4]. According to [12], a proper element $q \in L$ is said to be 2-absorbing if for all $a, b, c \in L$, $abc \leqslant q$ implies either $ab \leqslant q$ or $bc \leqslant q$ or $ca \leqslant q$. According to [18], a proper element $q \in L$ is said to be 2-absorbing primary if for all $a, b, c \in L$, $abc \leqslant q$ implies either $ab \leqslant q$ or $bc \leqslant \sqrt{q}$ or $ca \leqslant \sqrt{q}$. The reader is referred to [2], [3] and [9] for general background and terminology in multiplicative lattices.

Let M be a complete lattice and L be a multiplicative lattice. Then M is called L-module or module over L if there is a multiplication between elements of L and M written as aB where $a \in L$ and $B \in M$ which satisfies the following properties:
① $(\underset{\alpha}{\vee} a_\alpha)A = \underset{\alpha}{\vee}(a_\alpha\ A)$, ② $a(\underset{\alpha}{\vee} A_\alpha) = \underset{\alpha}{\vee}(a\ A_\alpha)$, ③ $(ab)A = a(bA)$, ④ $1A = A$, ⑤ $0A = O_M$, *for all* $a, a_\alpha, b \in L$ *and* $A, A_\alpha \in M$ where 1 is the supremum of L and 0 is the infimum of L. We denote by O_M and I_M for the least element and the greatest element of M, respectively. Elements of L will generally be denoted by $a, b, c, \cdots$ and elements of M will generally be denoted by $A, B, C, \cdots$

Let M be an L-module. For $N \in M$ and $a \in L$, $(N : a) = \vee\{X \in M \mid aX \leqslant N\}$. For $A, B \in M$, $(A : B) = \vee\{x \in L \mid xB \leqslant A\}$. If $(O_M : I_M) = 0$, then M is called a faithful L-module. M is called a torsion free L-module if for all $c \in L$, $B \in M$, $cB = O_M$ implies either $B = O_M$ or $c = 0$. An L-module M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = aI_M$. By proposition 3 in [10], an L-module M is a multiplication lattice module if and only if $N = (N : I_M)I_M$ $\forall N \in M$. An element $N \in M$ is called meet principal if $(b \wedge (B : N))N = bN \wedge B$ for all $b \in L, B \in M$. An element $N \in M$ is called join principal if $b \vee (B : N) = ((bN \vee B) : N)$ for all $b \in L, B \in M$. An element $N \in M$ is said to be principal if N is both meet principal and join principal. M is said to be a PG-lattice L-module if each element of M is a join of principal elements of M. An element $N \in M$ is called compact if $N \leqslant \underset{\alpha}{\vee} A_\alpha$ implies $N \leqslant A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The set of compact elements of M is denoted by M_*. If each element of M is a join of compact elements of M, then M is called a CG-lattice L-module. An element $N \in M$ is said to be proper if $N < I_M$. A proper element $N \in M$ is said to be maximal if whenever there exists an element $B \in M$ such that $N \leqslant B$ then either $N = B$ or $B = I_M$. If a proper element $N \in M$ is prime, then $(N : I_M) \in L$ is prime. If a proper element $N \in M$ is primary, then $\sqrt{N : I_M} \in L$ is prime. A proper element $N \in M$ is said to be a radical element if $(N : I_M) = \sqrt{N : I_M}$. An L-module M is said to be Noetherian, if M satisfies the ascending chain condition, is modular and is principally generated. According to [17], a proper element Q of an L-module M is said to be 2-absorbing if for all $a, b \in L$, $N \in M$, $abN \leqslant Q$ implies either $ab \leqslant (Q : I_M)$ or $bN \leqslant Q$ or $aN \leqslant Q$. According to [6], a proper element Q of an L-module M is said to be 2-absorbing primary if for all $a, b \in L$, $N \in M$, $abN \leqslant Q$ implies either $ab \leqslant (Q : I_M)$ or $bN \leqslant (\sqrt{Q : I_M})I_M$ or $aN \leqslant (\sqrt{Q : I_M})I_M$. The reader is referred to [1], [10] and [14] for terminology in lattice modules.

This paper is motivated by [24] and [7]. Many of the results obtained in this paper are lattice module version of the results in [16] and principal elements of M are used wherever needed with some more conditions on M. First section of this paper is comprised of ϕ-prime and ϕ-primary elements of an L-module M. Second section is comprised of almost prime and almost primary elements of an L-module M. By counter examples, it is shown that a ϕ-prime element of M need not be prime (see Example 1), a ϕ-primary element of M need not be ϕ-prime (see Example 2), a ϕ-primary element of M need not be prime (see Example 3) and a ϕ-primary element of M need not be primary (see Example 4). We define 2-potent prime and 2-potent primary elements in an L-module M. By counter examples, it is shown that an almost primary element of M need not be 2-potent prime (see Example 5) and a 2-potent prime element of M which is almost primary need not be prime (see Example 6). Also, we introduce the notions of n-potent prime and n-potent primary elements in an L-module M where $n \geqslant 2$. We find condition(s) under which a ϕ-prime element of M is prime (see Theorems 5-10). Also, we find condition(s) under which a ϕ-primary element of M is primary (see Theorems 15-23). Absorbing concepts in an L-module M are related to these notions of ϕ-prime and ϕ-primary in M. In the last section of this paper, many characterizations of almost prime and almost primary elements of M are obtained. By a counter example, it is shown that an almost primary element of M need not be idempotent (see Example 7). By a counter example, it is shown that an almost primary element of M need not be weakly primary (see Example 8). Finally, we show that if an element in M is almost prime (respectively almost primary), then its corresponding element in L is also almost prime (respectively almost primary) and vice versa.

2. ϕ-Prime and ϕ-Primary Elements in M

The study of weakly prime and weakly primary elements of an L-module M is carried out by A. V. Bingi and C. S. Manjarekar in [8]. Also, the notion of an almost prime element of an L-module M is seen in [22]. With weakly prime elements and almost prime elements of an L-module M in mind, we begin with introducing the notion of a ϕ-prime element of an L-module M.

Definition 1. *Let $\phi : M \longrightarrow M$ be a function on an L-module M. A proper element $N \in M$ is said to be ϕ-prime if for all $a \in L$, $A \in M$, $aA \leqslant N$ and $aA \nleqslant \phi(N)$ implies either $A \leqslant N$ or $a \leqslant (N : I_M)$.*

Now if $\phi_\alpha : M \longrightarrow M$ is a function on an L-module M, then ϕ_α-prime elements of M are defined by following settings in the Definition 1 of a ϕ-prime element.

- $\phi_0(N) = O_M$. Then $N \in M$ is called a weakly prime element.
- $\phi_2(N) = (N : I_M)N$. Then $N \in M$ is called a 2-almost prime element or a ϕ_2-prime element or simply an almost prime element.
- $\phi_n(N) = (N : I_M)^{n-1}N$ $(n \geqslant 2)$. Then $N \in M$ is called an n-almost prime element or a ϕ_n-prime element $(n \geqslant 2)$.
- $\phi_\omega(N) = \bigwedge_{i=1}^{\infty}(N : I_M)^i N$. Then $N \in M$ is called a ω-prime element or ϕ_ω-prime element.

Since $N \backslash \phi(N) = N \backslash (N \wedge \phi(N))$, so without loss of generality, throughout this paper, we assume that $\phi(N) \leqslant N$.

Definition 2. *Given two functions $\gamma_1,\ \gamma_2 : M \longrightarrow M$ on an L-module M, we define $\gamma_1 \leqslant \gamma_2$ if $\gamma_1(N) \leqslant \gamma_2(N)$ for all $N \in M$.*

Clearly, we have the following order:

$$\phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2$$

Now before obtaining the characterizations of a ϕ-prime element of an L-module M, we state the following essential lemma which is outcome of Lemma 2.3.13 from [11].

Lemma 1. *Let $a_1,\ a_2 \in L$. Suppose $b \in L$ satisfies the following property:*
(*). If $h \in L_*$ with $h \leqslant b$, then either $h \leqslant a_1$ or $h \leqslant a_2$.
Then either $b \leqslant a_1$ or $b \leqslant a_2$.

Theorem 1. *Let M be a CG-lattice L-module, $N \in M$ be a proper element and $\phi : M \longrightarrow M$ be a function on M. Then the following statements are equivalent:*

① *N is a ϕ-prime element of M.*

② *For every* $A \in M$ *such that* $A \nleqslant N$, *either* $(N : A) = (N : I_M)$ *or* $(N : A) = (\phi(N) : A)$.

③ *For every* $r \in L$ *such that* $r \nleqslant (N : I_M)$, *either* $(N : r) = N$ *or* $(N : r) = (\phi(N) : r)$.

④ *For every* $r \in L_*,\ A \in M_*$, *if* $rA \leqslant N$ *and* $rA \nleqslant \phi(N)$, *then either* $r \leqslant (N : I_M)$ *or* $A \leqslant N$.

Proof. ①⟹②. Suppose ① holds. Let $A \in M$ be such that $A \nleqslant N$. Obviously, $(\phi(N) : A) \leqslant (N : A)$ and $(N : I_M) \leqslant (N : A)$. Let $a \in L_*$ be such that $a \leqslant (N : A)$. Then $aA \leqslant N$. If $aA \leqslant \phi(N)$, then $a \leqslant (\phi(N) : A)$. If $aA \nleqslant \phi(N)$, then since N is ϕ-prime and $A \nleqslant N$, it follows that $a \leqslant (N : I_M)$. Hence by Lemma 1, either $(N : A) \leqslant (\phi(N) : A)$ or $(N : A) \leqslant (N : I_M)$. Thus either $(N : A) = (\phi(N) : A)$ or $(N : A) = (N : I_M)$.

②⟹③. Suppose ② holds. Let $r \nleqslant (N : I_M)$ for $r \in L$. Then $rI_M \nleqslant N$. Using ②, we have, either $(N : rI_M) = (N : I_M)$ or $(N : rI_M) = (\phi(N) : rI_M)$. Now let $K \leqslant (N : r)$ for $K \in M_*$. As $(K : I_M)I_M \leqslant K$, we have, $(K : I_M)I_M \leqslant (N : r)$ and $(K : I_M)I_M \in M_*$. Clearly, $K \leqslant (N : r)$ implies $(K : I_M) \leqslant ((N : r) : I_M) = (N : rI_M)$. So we have either $(K : I_M) \leqslant (N : I_M)$ or $(K : I_M) \leqslant (\phi(N) : rI_M) = (\phi(N) : r : I_M)$. This gives either $(K : I_M)I_M \leqslant N$ or $(K : I_M)I_M \leqslant (\phi(N) : r)$. This implies that either $(N : r) \leqslant N$ or $(N : r) \leqslant (\phi(N) : r)$, by Lemma 3.1 of [22]. Since $rN \leqslant N$ gives $N \leqslant (N : r)$ and $\phi(N) \leqslant N$ gives $(\phi(N) : r) \leqslant (N : r)$, it follows that either $(N : r) = N$ or $(N : r) = (\phi(N) : r)$.

③⟹④. Suppose ③ holds. Let $rA \leqslant N$, $rA \nleqslant \phi(N)$ and $r \nleqslant (N : I_M)$ for $r \in L_*$, $A \in M_*$. Then by ③, we have either $(N : r) = (\phi(N) : r)$ or $(N : r) = N$. If $(N : r) = (\phi(N) : r)$, then as $rA \leqslant N$, it follows that $A \leqslant (\phi(N) : r)$ which contradicts $rA \nleqslant \phi(N)$ and so we must have $(N : r) = N$. Therefore $rA \leqslant N$ gives $A \leqslant N$.

④⟹①. Suppose ④ holds. Let $aQ \leqslant N$, $aQ \nleqslant \phi(N)$ and $Q \nleqslant N$ for $a \in L$, $Q \in M$. As L and M are compactly generated, there exist $x' \in L_*$ and $Y,\ Y' \in M_*$ such that $x' \leqslant a$, $Y \leqslant Q$, $Y' \leqslant Q$, $Y' \nleqslant N$ and $x'Y' \nleqslant \phi(N)$. Let $x \in L_*$ be such that $x \leqslant a$. Then $(x \vee x') \in L_*$, $(Y \vee Y') \in M_*$ such that $(x \vee x')(Y \vee Y') \leqslant aQ \leqslant N$, $(x \vee x')(Y \vee Y') \nleqslant \phi(N)$ and $(Y \vee Y') \nleqslant N$. So by ④, $(x \vee x') \leqslant (N : I_M)$ which implies $a \leqslant (N : I_M)$. Therefore N is ϕ-prime.

The following 2 corollaries are consequences of Theorem 1.

Corollary 1. *Let* M *be a CG-lattice* L*-module and* $N \in M$ *be a proper element. Then the following statements are equivalent:*

① N *is a weakly prime element of* M.

② *For every* $A \in M$ *such that* $A \nleqslant N$, *either* $(N : A) = (N : I_M)$ *or* $(N : A) = (O_M : A)$.

③ *For every* $r \in L$ *such that* $r \nleqslant (N : I_M)$, *either* $(N : r) = N$ *or* $(N : r) = (O_M : r)$.

④ *For every* $r \in L_*,\ A \in M_*$, *if* $O_M \neq rA \leqslant N$, *then either* $r \leqslant (N : I_M)$ *or* $A \leqslant N$.

Corollary 2. *Let M be a CG-lattice L-module and $N \in M$ be a proper element. Then the following statements are equivalent:*

① *N is an almost prime element of M.*

② *For every $A \in M$ such that $A \nleqslant N$, either $(N : A) = ((N : I_M)N : A)$ or $(N : A) = (N : I_M)$.*

③ *For every $r \in L$ such that $r \nleqslant (N : I_M)$, either $(N : r) = ((N : I_M)N : r)$ or $(N : r) = N$.*

④ *For every $r \in L_*$, $A \in M_*$, if $rA \leqslant N$ and $rA \nleqslant (N : I_M)N$, then either $A \leqslant N$ or $r \leqslant (N : I_M)$.*

To obtain the relation among prime, weakly prime, ω-prime, n-almost prime $(n \geqslant 2)$ and almost prime elements of an L-module M, we prove the following result.

Theorem 2. *Let $\gamma_1, \ \gamma_2 : M \longrightarrow M$ be functions on an L-module M such that $\gamma_1 \leqslant \gamma_2$. Then every proper γ_1-prime element of M is γ_2-prime.*

Proof. Let a proper element $N \in M$ be γ_1-prime. Assume that $aA \leqslant N$ and $aA \nleqslant \gamma_2(N)$ for $a \in L, \ A \in M$. Then as $\gamma_1 \leqslant \gamma_2$, we have $aA \nleqslant \gamma_1(N)$. Since N is γ_1-prime, it follows that either $A \leqslant N$ or $a \leqslant (N : I_M)$ and hence N is γ_2-prime.

Theorem 3. *Let N be a proper element of an L-module M. Then N is prime implies N is weakly prime, N is weakly prime implies N is ω-prime, N is ω-prime implies N is n-almost prime $(n \geqslant 2)$ and N is n-almost prime $(n \geqslant 2)$ implies N is almost prime.*

Proof. By definition, every prime element of an L-module M is weakly prime and hence N is prime implies N is weakly prime. The remaining implications follow by using Theorem 2 to the fact that $\phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2$.

From the Theorem 3, we get the following characterization of a ω-prime element of an L-module M.

Corollary 3. *Let N be a proper element of an L-module M. Then N is ω-prime if and only if N is n-almost prime for every $n \geqslant 2$.*

Proof. Assume that $N \in M$ is n-almost prime for every $n \geqslant 2$. Let $aA \leqslant N$ and $aA \nleqslant \bigwedge_{i=1}^{\infty}(N : I_M)^i N$ for $a \in L, \ A \in M$. Then $aA \nleqslant (N : I_M)^{n-1}N$ for some $n \geqslant 2$. Since N is n-almost prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$ and hence N is ω-prime. The converse follows from Theorem 3.

Before going to the characterization of an n-almost prime element of an L-module M, we recall the definition of the Jacobson radical of L. According to [2], in a multiplicative lattice L with 1 compact, the Jacobson radical is the element $\wedge\{m \in L \mid$ m is a maximal element$\}$.

Theorem 4. *Let L be a Noether lattice, M be a torsion free Noetherian L-module and $f \in L$ be the Jacobson radical. Then a proper element $N \in M$ such that $(N : I_M) \leqslant f$ is n-almost prime for every $n \geqslant 2$ if and only if N is prime.*

Proof. Assume that $N \in M$ is n-almost prime where $n \geqslant 2$. Let $aA \leqslant N$ for $a \in L$, $A \in M$. If $aA \nleqslant (N : I_M)^{n-1}N$ for $n \geqslant 2$, then as N is n-almost prime, we have either $A \leqslant N$ or $a \leqslant (N : I_M)$. If $aA \leqslant (N : I_M)^{n-1}N$ for all $n \geqslant 2$, then as $(N : I_M) \leqslant f$, from Corollary 3.3 of [13], it follows that $aA \leqslant \bigwedge_{n=1}^{\infty}(N : I_M)^n N = O_M$ and thus $aA = O_M$. Since M is torsion free, we have either $A = O_M$ or $a = 0$ which implies either $A \leqslant N$ or $a \leqslant (N : I_M)$ and hence N is prime. The converse follows from Theorem 3.

Clearly, every prime element of an L-module M is ϕ-prime. But the converse is not true which is shown in the following example by taking $\phi(N) = (N : I_M)N$ for convenience.

Example 1. *If Z is the ring of integers, then Z_{24} is a $Z-$module. Assume that (k) denotes the cyclic ideal of Z generated by $k \in Z$ and $<\overline{t}>$ denotes the cyclic submodule of $Z-$module Z_{24} where $\overline{t} \in Z_{24}$. Suppose that $L = L(Z)$ is the set of all ideals of Z and $M = L(Z_{24})$ is the set of all submodules of $Z-$module Z_{24}. The multiplication between elements of L and M is given by $(k_i) < \overline{t_j} >=< \overline{k_i t_j} >$ for every $(k_i) \in L$ and $< \overline{t_j} >\in M$ where $k_i, t_j \in Z$. Then M is a lattice module over L [[22], Example 2.5]. Let N be the cyclic submodule of M generated by $\overline{0}$. It is easy to see that $O_M =< \overline{0} >= N$ is weakly prime and hence almost prime (ϕ_2-prime) while N is not prime, since $(2) < \overline{12} >\leqslant N$ but $< \overline{12} >\nleqslant N$ and $(2) \nleqslant (N : I_M) = (0)$ where $I_M =< \overline{1} >$.*

Now we obtain six results that show under which condition(s) a ϕ-prime element of an L-module M is prime. But before that we prove the required cancellation laws of M in the form of following lemmas.

Lemma 2. *Let M be a torsion free L-module and $O_M \neq A \in M$ be a weak join principal element. Then $aA \leqslant bA$ implies $a \leqslant b$ for $a,\ b \in L$ where $b \neq 0$.*

Proof. Let $aA \leqslant bA$ and $O_M \neq A \in M$ be a weak join principal element for $a,\ b \in L$. As M is a torsion free L-module, we have $(O_M : A) = 0$. Then clearly, $a = a \vee 0 = a \vee (O_M : A) = (aA : A) \leqslant (bA : A) = b \vee (O_M : A) = b \vee 0 = b$ which implies $a \leqslant b$.

Lemma 3. *Let M be a torsion free L-module and $O_M \neq A \in M$ be a weak join principal element. Then $aA = bA$ implies $a = b$ for $a,\ b \in L$ where $a \neq 0$, $b \neq 0$.*

Proof. The proof is obvious.

Now we have a characterization of a ϕ-prime element of an L-module M.

Theorem 5. *Let M be a torsion free L-module and $O_M \neq N < I_M$ be a weak join principal element of M. Then N is ϕ-prime for some $\phi \leqslant \phi_2$ if and only if N is prime.*

Proof. Assume that $N \in M$ is a prime element. Then obviously, N is ϕ-prime for every ϕ and hence for some $\phi \leqslant \phi_2$. Conversely, let N be ϕ-prime for some $\phi \leqslant \phi_2$. Then by Theorem 2, N is ϕ_2-prime. Let $aA \leqslant N$ for $a \in L,\ A \in M$. If $aA \nleqslant \phi_2(N)$, then as N is ϕ_2-prime, we have either $A \leqslant N$ or $a \leqslant (N : I_M)$. Next, assume that $aA \leqslant \phi_2(N)$. If $a(A \vee N) \nleqslant \phi_2(N)$, then as $a(A \vee N) \leqslant N$ and N is ϕ_2-prime, we have either $(A \vee N) \leqslant N$ or $a \leqslant (N : I_M)$ and hence either $A \leqslant N$ or $a \leqslant (N : I_M)$. Finally, if $a(A \vee N) \leqslant \phi_2(N)$, then $aN \leqslant (N : I_M)N$ which implies $a \leqslant (N : I_M)$, by Lemma 2 and hence N is prime.

Now we show that the Theorem 5 can also be achieved by changing the conditions on M and L. According to [23], in a Noether lattice L, an element $a \in L$ is said to satisfy the restricted cancellation law (RCL) if for all $b,\ c \in L$, $ab = ac \neq 0$ implies $b = c$.

Theorem 6. *Let L be a Noether PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of M such that $0 \neq (N : I_M) \in L$ satisfies the restricted cancellation law (RCL) and is a non-nilpotent element. Then N is ϕ-prime for some $\phi \leqslant \phi_2$ if and only if N is prime.*

Proof. Assume that $N \in M$ is a prime element. Then obviously, N is ϕ-prime for every ϕ and hence for some $\phi \leqslant \phi_2$. Conversely, let N be ϕ-prime for some $\phi \leqslant \phi_2$. Then by Theorem 2, N is ϕ_2-prime. Let $aA \leqslant N$ for $a \in L,\ A \in M$. If $aA \nleqslant \phi_2(N)$, then as N is ϕ_2-prime, we have either $A \leqslant N$ or $a \leqslant (N : I_M)$. Next, assume that $aA \leqslant \phi_2(N)$. If $a(A \vee N) \nleqslant \phi_2(N)$, then as $a(A \vee N) \leqslant N$ and N is ϕ_2-prime, we have either $(A \vee N) \leqslant N$ or $a \leqslant (N : I_M)$ and hence either $A \leqslant N$ or $a \leqslant (N : I_M)$. Finally, if $a(A \vee N) \leqslant \phi_2(N)$, then $aN \leqslant (N : I_M)N$ which implies $a(N : I_M)I_M \leqslant (N : I_M)^2 I_M$, since M is a multiplication lattice L module. As I_M is compact, this gives $a(N : I_M) \leqslant (N : I_M)^2 \neq 0$, by Theorem 5 of [10]. This implies $a \leqslant (N : I_M)$, by Lemma 1.11 of [23] and hence N is prime.

Now we define a 2-potent prime element in an L-module M.

Definition 3. *A proper element $N \in M$ is said to be 2-potent prime if for all $a \in L,\ A \in M$, $aA \leqslant (N : I_M)N$ implies either $a \leqslant (N : I_M)$ or $A \leqslant N$.*

Theorem 7. *Let a proper element N of an L-module M be 2-potent prime. Then N is ϕ-prime for some $\phi \leqslant \phi_2$ if and only if N is prime.*

Proof. Assume that $N \in M$ is a prime element. Then obviously, N is ϕ-prime for every ϕ and hence for some $\phi \leqslant \phi_2$. Conversely, let N be ϕ-prime for some $\phi \leqslant \phi_2$. Then by Theorem 2, $N \in M$ is ϕ_2-prime. Let $aA \leqslant N$ for $a \in L,\ A \in M$. If $aA \nleqslant (N : I_M)N$, then as N is ϕ_2-prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$. If $aA \leqslant (N : I_M)N$, then as N is 2-potent prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$ and hence N is prime.

Now we define a n-potent prime element in an L-module M where $n \geqslant 2$.

Definition 4. *Let $n \geqslant 2$ and $n \in Z_+$. A proper element $N \in M$ is said to be n-potent prime if for all $a \in L,\ A \in M$, $aA \leqslant (N : I_M)^{n-1}N$ implies either $a \leqslant (N : I_M)$ or $A \leqslant N$.*

Theorem 8. *A proper element N of an L-module M is ϕ-prime for some $\phi \leqslant \phi_n$ where $n \geqslant 2$ if and only if N is prime, provided N is k-potent prime for some $k \leqslant n$.*

Proof. Assume that $N \in M$ is a prime element. Then obviously, N is ϕ-prime for every ϕ and hence for some $\phi \leqslant \phi_n$ where $n \geqslant 2$. Conversely, let N be ϕ-prime for some $\phi \leqslant \phi_n$ where $n \geqslant 2$. Then by Theorem 2, $N \in M$ is ϕ_n-prime. Let $aA \leqslant N$ for $a \in L,\ A \in M$. If $aA \nleqslant \phi_k(N)$, then $aA \nleqslant \phi_n(N)$ as $k \leqslant n$. Since N is ϕ_n-prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$. If $aA \leqslant \phi_k(N)$, then as N is k-potent prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$ and hence N is prime.

The following corollary is outcome of Theorems 5, 6 and 7.

Corollary 4. *An almost prime element N of an L-module M is prime if one the following statements hold true:*

(i) M is torsion free and $O_M \neq N < I_M$ is a weak join principal element.

(ii) N is a 2-potent prime element.

(iii) L is a Noether PG-lattice, M is a faithful multiplication PG-lattice with I_M compact, $0 \neq (N : I_M) \in L$ satisfies the restricted cancellation law (RCL) and is a non-nilpotent element.

Theorem 9. *Let a proper element N of an L-module M be ϕ-prime. If $\phi(N)$ is prime, then N is prime.*

Proof. Let $aA \leqslant N$ for $a \in L,\ A \in M$. If $aA \nleqslant \phi(N)$, then as N is ϕ-prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$ and we are done. If $aA \leqslant \phi(N)$, then as $\phi(N)$ is prime, we have either $aI_M \leqslant \phi(N)$ or $A \leqslant \phi(N)$. This implies that either $aI_M \leqslant N$ or $A \leqslant N$ because $\phi(N) \leqslant N$. Hence N is prime.

Theorem 10. *Let a proper element N of an L-module M be ϕ-prime. If $(N : I_M)N \nleqslant \phi(N)$, then N is prime.*

Proof. Let $aA \leqslant N$ for $a \in L,\ A \in M$. If $aA \nleqslant \phi(N)$, then as N is ϕ-prime, we have either $a \leqslant (N : I_M)$ or $A \leqslant N$. So assume that $aA \leqslant \phi(N)$. First suppose $aN \nleqslant \phi(N)$. Then $aN_0 \nleqslant \phi(N)$ for some $N_0 \leqslant N$ in M. Since N is ϕ-prime, $a(A \vee N_0) = aA \vee aN_0 \leqslant N$ and $a(A \vee N_0) \nleqslant \phi(N)$, we have either $a \leqslant (N : I_M)$ or $(A \vee N_0) \leqslant N$ and hence either $a \leqslant (N : I_M)$ or $A \leqslant N$. Next, assume that $aN \leqslant \phi(N)$. If $(N : I_M)A \nleqslant \phi(N)$, then $k_0 A \nleqslant \phi(N)$ for some $k_0 \leqslant (N : I_M)$ in L. Since N is ϕ-prime, $(a \vee k_0)A \leqslant N$ and $(a \vee k_0)A \nleqslant \phi(N)$, we have either $(a \vee k_0) \leqslant (N : I_M)$ or $A \leqslant N$ and hence either $a \leqslant (N : I_M)$ or $A \leqslant N$. Now let $(N : I_M)A \leqslant \phi(N)$. By hypothesis, as $(N : I_M)N \nleqslant \phi(N)$, there exist $k \leqslant (N : I_M)$ in L and $N_0 \leqslant N$ in M such that $kN_0 \nleqslant \phi(N)$. Since N is ϕ-prime, $(a \vee k)(A \vee N_0) \leqslant N$ and $(a \vee k)(A \vee N_0) \nleqslant \phi(N)$, we have either $(a \vee k) \leqslant (N : I_M)$ or $(A \vee N_0) \leqslant N$ and hence either $a \leqslant (N : I_M)$ or $A \leqslant N$. Therefore N is prime.

The consequences of Theorem 10 are presented in the following corollaries.

Corollary 5. *If a proper element N of a multiplication lattice L-module M is ϕ-prime but not prime, then $(N : I_M)^2 I_M \leqslant \phi(N)$.*

Proof. Since M is a multiplication lattice L-module, by Proposition 3 of [10], we have $N = (N : I_M)I_M$. So $(N : I_M)^2 I_M = (N : I_M)N \leqslant \phi(N)$ by Theorem 10.

Corollary 6. *If a proper element N of an L-module M is weakly prime such that $(N : I_M)N \neq O_M$, then N is prime.*

Proof. The proof is obvious.

Corollary 7. *If a proper element N of an L-module M is ϕ-prime such that $\phi \leqslant \phi_3$, then N is ω-prime.*

Proof. If N is prime, then by Theorem 3, N is ω-prime. So assume that N is not prime. Then by Theorem 10 and hypothesis, we get $(N : I_M)^2 N \leqslant (N : I_M)N \leqslant \phi(N) \leqslant (N : I_M)^2 N$ and so $\phi(N) = (N : I_M)^2 N = (N : I_M)N$. Now consider $(N : I_M)^3 N = ((N : I_M)(N : I_M)^2)N = (N : I_M)((N : I_M)^2 N) = (N : I_M)((N : I_M)N) = ((N : I_M)(N : I_M))N = (N : I_M)^2 N = \phi(N)$ and so on. Hence $\phi(N) = (N : I_M)^{n-1} N$ for every $n \geqslant 2$. Consequently, N is n-almost prime for every $n \geqslant 2$ and thus N is ω-prime by Corollary 3.

Corollary 8. *If a proper element N of a multiplication lattice L-module M is ϕ-prime but not prime, then $\sqrt{N : I_M} = \sqrt{\phi(N) : I_M}$.*

Proof. By Corollary 5, we have $(N : I_M)^2 I_M \leqslant \phi(N)$ which implies $(N : I_M) \leqslant \sqrt{\phi(N) : I_M}$. Hence $\sqrt{N : I_M} \leqslant \sqrt{\sqrt{\phi(N) : I_M}} = \sqrt{\phi(N) : I_M}$, by property (p3) of radicals in [21]. Also, as $\phi(N) \leqslant N$, we have $\sqrt{\phi(N) : I_M} \leqslant \sqrt{N : I_M}$ and thus $\sqrt{N : I_M} = \sqrt{\phi(N) : I_M}$.

Corollary 9. *If a proper element N of a multiplication lattice L-module M is ϕ-prime, then either $\sqrt{\phi(N) : I_M} \leqslant (N : I_M)$ or $(N : I_M) \leqslant \sqrt{\phi(N) : I_M}$.*

Proof. The proof is obvious.

Now we introduce the notion of ϕ-primary element of an L-module M.

Definition 5. *Let $\phi : M \longrightarrow M$ be a function on an L-module M. A proper element $N \in M$ is said to be ϕ-primary if for all $a \in L$, $A \in M$, $aA \leqslant N$ and $aA \nleqslant \phi(N)$ implies either $A \leqslant N$ or $a^n \leqslant (N : I_M)$ for some $n \in Z_+$.*

Now if $\phi_\alpha : M \longrightarrow M$ is a function on an L-module M, then ϕ_α-primary elements of M are defined by following settings in the Definition 5 of a ϕ-primary element.

- $\phi_0(N) = O_M$. Then $N \in M$ is called a weakly primary element.
- $\phi_2(N) = (N : I_M)N$. Then $N \in M$ is called a 2-almost primary element or a ϕ_2-primary element or simply an almost primary element.

- $\phi_n(N) = (N : I_M)^{n-1}N$ $(n \geqslant 2)$. Then $N \in M$ is called an n-almost primary element or a ϕ_n-primary element $(n \geqslant 2)$.
- $\phi_\omega(N) = \bigwedge_{i=1}^{\infty}(N : I_M)^i N$. Then $N \in M$ is called a ω-primary element or ϕ_ω-primary element.

Clearly, every ϕ-prime element of an L-module M is ϕ-primary but the converse is not true as shown in the following example by taking $\phi(N) = (N : I_M)N$ for convenience.

Example 2. *Consider the lattice module as in Example* 1*. Let N be the cyclic submodule of M generated by $\overline{4}$. It is easy to see that the element $N =<\overline{4}>$ is almost primary (ϕ_2-primary) but N is not almost prime (ϕ_2-prime) because $(2)<\overline{6}>\leqslant N$, $(2)<\overline{6}>\nleqslant \phi_2(N) =<\overline{8}>$ but $<\overline{6}>\nleqslant N$ and $(2) \nleqslant (N : I_M) = (4)$ where $I_M =<\overline{1}>$.*

Clearly, every prime element of an L-module M is ϕ-primary. But the converse is not true which is shown in the following example by taking $\phi(N) = (N : I_M)N$ for convenience.

Example 3. *Consider the lattice module as in Example 1. Let N be the cyclic submodule of M generated by $\overline{0}$. It is easy to see that the element $N =<\overline{0}>= O_M$ is almost primary (ϕ_2-primary) but N is not prime.*

The analogous results (from the results of ϕ-prime elements of M) for ϕ-primary elements of M are stated below whose proofs being on similar arguments are omitted. We begin with the characterizations of a ϕ-primary element of an L-module M.

Theorem 11. *Let M be a CG-lattice L-module, $N \in M$ be a proper element and $\phi : M \longrightarrow M$ be a function on M. Then the following statements are equivalent:*

(i) *N is a ϕ-primary element of M.*

(ii) *For every $A \in M$ such that $A \nleqslant N$, either $(N : A) \leqslant \sqrt{N : I_M}$ or $(N : A) = (\phi(N) : A)$.*

(iii) *For every $r \in L$ such that $r \nleqslant \sqrt{N : I_M}$, either $(N : r) = N$ or $(N : r) = (\phi(N) : r)$.*

(iv) *For every $r \in L_*$, $A \in M_*$, if $rA \leqslant N$ and $rA \nleqslant \phi(N)$, then either $r \leqslant \sqrt{N : I_M}$ or $A \leqslant N$.*

The following 2 corollaries are consequences of Theorem 11.

Corollary 10. *Let M be a CG-lattice L-module and $N \in M$ be a proper element. Then the following statements are equivalent:*

① *N is a weakly primary element of M.*

② *For every $A \in M$ such that $A \nleqslant N$, either $(N : A) \leqslant \sqrt{N : I_M}$ or $(N : A) = (O_M : A)$.*

③ *For every $r \in L$ such that $r \nleqslant \sqrt{N : I_M}$, either $(N : r) = N$ or $(N : r) = (O_M : r)$.*

④ *For every $r \in L_*$, $A \in M_*$, if $O_M \neq rA \leqslant N$, then either $r \leqslant \sqrt{N : I_M}$ or $A \leqslant N$.*

Corollary 11. *Let M be a CG-lattice L-module and $N \in M$ be a proper element. Then the following statements are equivalent:*

① *N is an almost primary element of M.*

② *For every $A \in M$ such that $A \nleqslant N$, either $(N : A) = ((N : I_M)N : A)$ or $(N : A) \leqslant \sqrt{N : I_M}$.*

③ *For every $r \in L$ such that $r \nleqslant \sqrt{N : I_M}$, either $(N : r) = ((N : I_M)N : r)$ or $(N : r) = N$.*

④ *For every $r \in L_*$, $A \in M_*$, if $rA \leqslant N$ and $rA \nleqslant (N : I_M)N$, then either $r \leqslant \sqrt{N : I_M}$ or $A \leqslant N$.*

To obtain the relation among primary, weakly primary, ω-primary, n-almost primary ($n \geqslant 2$) and almost primary elements of an L-module M, we have the following result.

Theorem 12. *Let $\gamma_1, \gamma_2 : M \longrightarrow M$ be functions on an L-module M such that $\gamma_1 \leqslant \gamma_2$. Then every proper γ_1-primary element of M is γ_2-primary.*

Theorem 13. *Let N be a proper element of an L-module M . Then N is primary implies N is weakly primary, N is weakly primary implies N is ω-primary, N is ω-primary implies N is n-almost primary ($n \geqslant 2$), N is n-almost primary ($n \geqslant 2$) implies N is almost primary.*

From the Theorem 13, we get the following characterization of a ω-primary element of an L-module M.

Corollary 12. *Let $N \in M$ be a proper element of an L-module M. Then N is ω-primary if and only if N is n-almost primary for every $n \geqslant 2$.*

The following theorem gives the characterization of an n-almost primary element of an L-module M.

Theorem 14. *Let L be a Noether lattice, M be a torsion free Noetherian L-module and $f \in L$ be the Jacobson radical. Then a proper element $N \in M$ such that $(N : I_M) \leqslant f$ is n-almost primary for every $n \geqslant 2$ if and only if N is primary.*

Clearly, every primary element of an L-module M is ϕ-primary. But the converse is not true which is shown in the following example by taking $\phi(N) = (N : I_M)N$ for convenience.

Example 4. *If Z is the ring of integers, then Z_{30} is a $Z-$module. Assume that (k) denotes the cyclic ideal of Z generated by $k \in Z$ and $< \overline{t} >$ denotes the cyclic submodule of $Z-$module Z_{30} where $\overline{t} \in Z_{30}$. Suppose that $L = L(Z)$ is the set of all ideals of Z and $M = L(Z_{30})$ is the set of all submodules of $Z-$module Z_{30}. The multiplication between* *elements of* L *and* M *is given by* $(k_i) < \overline{t_j} >=< \overline{k_i t_j} >$ *for every* $(k_i) \in L$ *and* $< \overline{t_j} >\in M$ *where* $k_i, t_j \in Z$. *Then* M *is a lattice module over* L. *Let* N *be the cyclic submodule of* M *generated by* $\overline{6}$. *It is easy to see that* $N =< \overline{6} >$ *is almost primary* (ϕ_2*-primary) while* N *is not primary, since* $(3) < \overline{2} >\leqslant N$ *but* $< \overline{2} >\nleqslant N$ *and* $(3)^n \nleqslant (N : I_M) = (6)$ *for every* $n \in Z_+$ *where* $I_M =< \overline{1} >$.

In the following successive nine theorems, we show under which condition(s) a ϕ-primary element of an L-module M is primary. Now we have a characterization of a ϕ-primary element of an L-module M.

Theorem 15. *Let* M *be a torsion free* L*-module and* $O_M \neq N < I_M$ *be a weak join principal element of an* L*-module* M. *Then* N *is* ϕ*-primary for some* $\phi \leqslant \phi_2$ *if and only if* N *is primary.*

The following result shows that the Theorem 15 can also be achieved by changing the conditions on M and L.

Theorem 16. *Let* L *be a Noether PG-lattice and* M *be a faithful multiplication PG-lattice* L*-module with* I_M *compact. Let* N *be a proper element of* M *such that* $0 \neq (N : I_M) \in L$ *satisfies the restricted cancellation law (RCL) and is a non-nilpotent element. Then* N *is* ϕ*-primary for some* $\phi \leqslant \phi_2$ *if and only if* N *is primary.*

Now we define a 2-potent primary element in an L-module M.

Definition 6. *A proper element* $N \in M$ *is said to be* 2*-potent primary if for all* $a \in L$, $A \in M$, $aA \leqslant (N : I_M)N$ *implies either* $A \leqslant N$ *or* $a^m \leqslant (N : I_M)$ *for some* $m \in Z_+$.

Theorem 17. *Let a proper element* N *of an* L*-module* M *be* 2*-potent primary. Then* N *is* ϕ*-primary for some* $\phi \leqslant \phi_2$ *if and only if* N *is primary.*

Clearly, every 2-potent prime element of an L-module M is 2-potent primary.

Theorem 18. *Let a proper element* N *of an* L*-module* M *be* 2*-potent prime. Then* N *is* ϕ*-primary for some* $\phi \leqslant \phi_2$ *if and only if* N *is primary.*

Now we define a n-potent primary element in an L-module M where $n \geqslant 2$.

Definition 7. *Let* $n \geqslant 2$ *and* $n \in Z_+$. *A proper element* $N \in M$ *is said to be* n*-potent primary if for all* $a \in L$, $A \in M$, $aA \leqslant (N : I_M)^{n-1}N$ *implies either* $A \leqslant N$ *or* $a^m \leqslant (N : I_M)$ *for some* $m \in Z_+$.

Theorem 19. *A proper element* N *of an* L*-module* M *is* ϕ*-primary for some* $\phi \leqslant \phi_n$ *where* $n \geqslant 2$ *if and only if* N *is primary, provided* N *is* k*-potent primary for some* $k \leqslant n$.

Clearly, every n-potent prime element of an L-module M is n-potent primary.

Theorem 20. *A proper element* N *of an* L*-module* M *is* ϕ*-primary for some* $\phi \leqslant \phi_n$ *where* $n \geqslant 2$ *if and only if* N *is primary, provided* N *is* k*-potent prime for some* $k \leqslant n$.

The following corollary is outcome of Theorems 15, 16, 17 and 18.

Corollary 13. *An almost primary element N of an L-module M is primary if one the following statements hold true:*

(i) M *is torsion free and* $O_M \neq N < I_M$ *is a weak join principal element.*

(ii) N *is a* 2*-potent primary element.*

(iii) N *is a* 2*-potent prime element.*

(iv) L *is a Noether PG-lattice,* M *is a faithful multiplication PG-lattice with* I_M *compact,* $0 \neq (N : I_M) \in L$ *satisfies the restricted cancellation law (RCL) and is a non-nilpotent element.*

From the following examples, it is clear that, an almost primary element of an L module M need not be 2-potent prime and a 2-potent prime element of an L module M which is almost primary need not be prime.

Example 5. *Consider the lattice module as in Example 4. Let N be the cyclic submodule of M generated by $\overline{6}$. It is easy to see that the element $N =< \overline{6} >$ is almost primary but not* 2*-potent prime.*

Example 6. *If Z is the ring of integers, then Z_8 is a $Z-$module. Assume that (k) denotes the cyclic ideal of Z generated by $k \in Z$ and $< \overline{t} >$ denotes the cyclic submodule of $Z-$module Z_8 where $\overline{t} \in Z_8$. Suppose that $L = L(Z)$ is the set of all ideals of Z and $M = L(Z_8)$ is the set of all submodules of $Z-$module Z_8. The multiplication between elements of L and M is given by $(k_i) < \overline{t_j} >=< \overline{k_i t_j} >$ for every $(k_i) \in L$ and $< \overline{t_j} >\in M$ where $k_i, t_j \in Z$. Then M is a lattice module over L. Let N be the cyclic submodule of M generated by $\overline{4}$. It is easy to see that $N =< \overline{4} >$ is almost primary (ϕ_2-primary) and* 2*-potent prime but not prime.*

Theorem 21. *Let a proper element N of an L-module M be ϕ-primary. If $\phi(N)$ is primary, then N is primary.*

Theorem 22. *Let a proper element N of an L-module M be ϕ-primary. If $(N : I_M)N \nleqslant \phi(N)$, then N is primary.*

The consequences of Theorem 22 are presented in the form of following corollaries.

Corollary 14. *If a proper element N of a multiplication lattice L-module M is ϕ-primary but not primary, then $(N : I_M)^2 I_M \leqslant \phi(N)$.*

Corollary 15. *If a proper element N of an L-module M is weakly primary such that $(N : I_M)N \neq O_M$, then N is primary.*

Corollary 16. *If a proper element N of an L-module M is ϕ-primary such that $\phi \leqslant \phi_3$, then N is ω-primary.*

Corollary 17. *If a proper element N of a multiplication lattice L-module M is ϕ-primary but not primary, then* $\sqrt{N : I_M} = \sqrt{\phi(N) : I_M}$.

Corollary 18. *If a proper element N of a multiplication lattice L-module M is ϕ-primary, then either* $\sqrt{\phi(N) : I_M} \leqslant (N : I_M)$ *or* $(N : I_M) \leqslant \sqrt{\phi(N) : I_M}$.

Theorem 23. *Let a proper element N of an L-module M be ϕ-primary. If* $(\sqrt{N : I_M})N \nleqslant \phi(N)$*, then N is primary.*

Proof. Just mimic the proof of Theorem 10.

Now, the interrelations among prime, primary, 2-absorbing and 2-absorbing primary elements of an L-module M are given in following theorems whose proofs being obvious are omitted.

Theorem 24. *Every prime element of an L-module M is primary and 2-absorbing.*

Theorem 25. *If Q is a primary element of an L-module M, then $\sqrt{Q : I_M}$ is a prime element and hence a 2-absorbing element of L. Also, it is a 2-absorbing primary element of L.*

Theorem 26. *If Q is a 2-absorbing element of an L-module M, then both $\sqrt{Q : I_M}$ and $(Q : I_M)$ are 2-absorbing elements of L. Also, they are 2-absorbing primary elements of L.*

Theorem 27. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. If Q is a 2-absorbing primary element of M, then $(Q : I_M)$ is a 2-absorbing primary element of L and $\sqrt{Q : I_M}$ is a 2-absorbing element of L.*

Proof. Let $abc \leqslant (Q : I_M)$ for $a, b, c \in L$. Then as $ab(cI_M) \leqslant Q$ and Q is a 2-absorbing primary element of M, we have, either $ab \leqslant (Q : I_M)$ or $a(cI_M) \leqslant (\sqrt{Q : I_M})I_M$ or $b(cI_M) \leqslant (\sqrt{Q : I_M})I_M$. Since I_M is compact, by Theorem 5 of [10], it follows that, either $ab \leqslant (Q : I_M)$ or $ac \leqslant \sqrt{Q : I_M}$ or $bc \leqslant \sqrt{Q : I_M}$ and hence $(Q : I_M)$ is a 2-absorbing primary element of L. By Theorem 2.4 in [18], it follows that $\sqrt{Q : I_M}$ is a 2-absorbing element of L.

By relating the absorbing concepts with ϕ-prime and ϕ-primary elements of an L-module M, we obtain the following results.

Theorem 28. *Let a proper element N of an L-module M be ϕ-prime. If $(N : I_M)N \nleqslant \phi(N)$, then N is primary and 2-absorbing. Also, then both $\sqrt{N : I_M}$ and $(N : I_M)$ are 2-absorbing and hence 2-absorbing primary elements of L.*

Proof. The proof follows from Theorems 10, 24 and 26.

Clearly, every primary element of a multiplication L-module M is 2-absorbing primary.

Theorem 29. *Let a proper element N of a multiplication L-module M be ϕ-prime. If $(N : I_M)N \nleqslant \phi(N)$, then N is 2-absorbing primary. Also, then $(N : I_M)$ is a 2-absorbing primary element of L provided M is a faithful PG-lattice with I_M compact and L as a PG-lattice. Further, $\sqrt{N : I_M}$ is a 2-absorbing element of L.*

Proof. The proof follows from Theorems 10, 24 and 27.

Theorem 30. *Let a proper element N of a multiplication L-module M be ϕ-primary. If $(N : I_M)N \nleq \phi(N)$, then N is 2-absorbing primary.*

Proof. The proof follows from Theorem 22.

Theorem 31. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let a proper element N of an L-module M be ϕ-primary. If $(N : I_M)N \nleq \phi(N)$, then $(N : I_M)$ is a 2-absorbing primary element of L and $\sqrt{N : I_M}$ is a 2-absorbing element of L.*

Proof. The proof follows from Theorems 30 and 27.

The following results are obtained by relating the absorbing concepts with almost prime and almost primary elements of an L-module M.

Theorem 32. *Let M be a torsion free L-module and $O_M \neq N < I_M$ be a weak join principal element of M. If N is almost prime, then N is primary and 2-absorbing. Also, then both $\sqrt{N : I_M}$ and $(N : I_M)$ are 2-absorbing and hence 2-absorbing primary elements of L.*

Proof. The proof follows from Theorems 5, 24 and 26.

Theorem 33. *Let M be a torsion free, multiplication L-module and $O_M \neq N < I_M$ be a weak join principal element of M. If N is almost prime, then N is 2-absorbing primary. Also, then $(N : I_M)$ is a 2-absorbing primary element of L provided M is a faithful PG-lattice with I_M compact and L as a PG-lattice. Further, $\sqrt{N : I_M}$ is a 2-absorbing element of L.*

Proof. The proof follows from Theorems 5, 24 and 27.

Theorem 34. *Let M be a torsion free, multiplication L-module and $O_M \neq N < I_M$ be a weak join principal element of M. If N is almost primary, then N is 2-absorbing primary.*

Proof. The proof follows from Theorem 15.

Theorem 35. *Let M be a torsion free, faithful, multiplication PG-lattice L-module with I_M compact and L be a PG-lattice. Let $O_M \neq N < I_M$ be a weak join principal element of M. If N is almost primary, then $(N : I_M)$ is a 2-absorbing primary element of L and $\sqrt{N : I_M}$ is a 2-absorbing element of L.*

Proof. The proof follows from Theorems 34 and 27.

3. Almost Prime and Almost Primary Elements in M

In this section, we will obtain some more results on an almost prime (respectively almost primary) element of an L-module M by relating it with an idempotent element and a weakly prime (respectively weakly primary) element of an L-module M. Also, many characterizations of an almost prime and almost primary element of an L-module M are obtained. Finally, we define n-potent prime(respectively n-potent primary) elements in L and these notions are related with n-potent prime(respectively n-potent primary) elements in M where $n \geqslant 2$.

Clearly, every almost prime element of an L-module M is almost primary but the converse need not be true as seen in Example 2. It is easy to see that converse holds for radical elements of an L-module M. Every prime element of an L-module M is almost prime and every primary element of an L-module M is almost primary but their converses are not true as seen in Example 1 and Example 4, respectively. Also, every prime element of an L-module M is almost primary.

According to Definition 2.6 of [22], an idempotent element of an L-module M is defined in the following way.

Definition 8. *A proper element N of an L-module M is said to be idempotent if* $(N : I_M)N = N$.

Clearly, every idempotent element of an L-module M is almost prime and hence almost primary. But an almost primary element of an L-module M need not be idempotent as shown in the following example.

Example 7. *Consider the lattice module as in Example 6. Let N be the cyclic submodule of M generated by $\overline{4}$. It is easy to see that the element $N =< \overline{4} >$ is almost primary but not idempotent.*

Theorem 36. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. For an idempotent element $N \in M$, $(\sqrt{(N : I_M)N : I_M})N = (N : I_M)N$.*

Proof. As $N < I_M$ is idempotent, N is almost prime ($\phi_2 - prime$). Since M is a multiplication lattice L-module, we have $(N : I_M)^2 I_M = (N : I_M)N$ which implies $(N : I_M) \leqslant \sqrt{(N : I_M)N : I_M}$. Thus $(N : I_M)N \leqslant (\sqrt{(N : I_M)N : I_M})N$. Now to prove that $(\sqrt{(N : I_M)N : I_M})N \leqslant (N : I_M)N$, let $a \leqslant \sqrt{(N : I_M)N : I_M}$ for $a \in L$. If $a \leqslant (N : I_M)$, then we are done. So let $a \nleqslant (N : I_M)$. Then as N is $\phi_2 - prime$, by Theorem 1, we have either $(N : a) = N$ or $(N : a) = ((N : I_M)N : a)$. Let $(N : a) = N$ and n be the least positive integer such that $a^n \leqslant ((N : I_M)N : I_M)$. If $n = 1$, then $aI_M \leqslant (N : I_M)N = (N : I_M)^2 I_M$. As I_M is compact, by Theorem 5 of [10], we have $a \leqslant (N : I_M)^2 \leqslant (N : I_M)$ which contradicts $a \nleqslant (N : I_M)$. So assume that $n \geqslant 2$. Then $a^n I_M \leqslant (N : I_M)N \leqslant N$ with $a^k I_M \nleqslant (N : I_M)N$ for every $k \leqslant (n - 1)$. Since

$a(a^{n-1}I_M) \leqslant N$, we have $a^{n-1}I_M \leqslant (N : a) = N$ with $a^{n-1}I_M \nleqslant (N : I_M)N$. If $n = 2$, then $aI_M \leqslant N$ which contradicts $a \nleqslant (N : I_M)$. If $n \geqslant 3$, then $a(a^{n-2}I_M) \leqslant N$ but $a(a^{n-2}I_M) \nleqslant (N : I_M)N$. As N is almost prime, we have either $a \leqslant (N : I_M)$ or $a^{n-2}I_M \leqslant N$. As $a \leqslant (N : I_M)$ is a contradiction, let $a^{n-2}I_M \leqslant N$. Then $a(a^{n-3}I_M) \leqslant N$ but $a(a^{n-3}I_M) \nleqslant (N : I_M)N$. As N is almost prime, we have either $a \leqslant (N : I_M)$ or $a^{n-3}I_M \leqslant N$. Continuing this process we conclude that $a \leqslant (N : I_M)$ which contradicts $a \nleqslant (N : I_M)$. Hence we must have $(N : a) = ((N : I_M)N : a)$. Then $aN \leqslant a(N : a) = a((N : I_M)N : a) \leqslant (N : I_M)N$ which implies $a \leqslant ((N : I_M)N : N)$ and so $\sqrt{(N : I_M)N : I_M} \leqslant ((N : I_M)N : N)$. It follows that $(\sqrt{(N : I_M)N : I_M})N \leqslant (N : I_M)N$ and hence $(\sqrt{(N : I_M)N : I_M})N = (N : I_M)N$.

From following example, it is clear that an almost primary element of an L-module M need not be weakly primary.

Example 8. *Consider the lattice module as in Example* 4. *Let N be the cyclic submodule of M generated by $\overline{6}$. It is easy to see that the element $N =< \overline{6} >$ is almost primary (ϕ_2-primary) but not weakly primary.*

Before obtaining the characterization of an almost primary element of an L-module M in terms of a weakly primary element of M, we recall the definition of a local module M. According to [1], an L-module M is said to be a local module if it has a unique maximal element.

Theorem 37. *Let M be a local L-module with a unique maximal element $Q \in M$ such that $(Q : I_M)Q = O_M$. Then a proper element $N \in M$ is almost primary if and only if N is weakly primary.*

Proof. Assume that a proper element $N \in M$ is almost primary. Then $N \leqslant Q$. It follows that $(N : I_M)N \leqslant (Q : I_M)Q = O_M$ and hence $(N : I_M)N = O_M$. Let $O_M \neq aA \leqslant N$ for $a \in L$, $A \in M$. As $aA \leqslant N$, $aA \nleqslant (N : I_M)N = O_M$ and N is almost primary, we have either $A \leqslant N$ or $a \leqslant \sqrt{N : I_M}$ and hence N is weakly primary. The converse is obvious from Theorem 13.

Now we prove the result required to show that if an element in M (or L) is almost primary, then its corresponding element in L (or M) is also almost primary.

Lemma 4. *Let M be a torsion free multiplication lattice L-module and I_M be a weak join principal element of M. Let N be a proper element of M. Then $a(N : I_M) = (aN : I_M)$ for $a \in L$.*

Proof. Since M is a multiplication lattice L-module, $N = (N : I_M)I_M$. Then $a(N : I_M)I_M = aN = (aN : I_M)I_M$ and so the result follows by Lemma 3.

Theorem 38. *Let L be a PG-lattice and M be a faithful multiplication torsion free PG-lattice L-module with I_M compact. Let I_M be a weak join principal element and N be a proper element of M. Then the following statements are equivalent:*

① *N is an almost primary element of M.*

② *$(N : I_M)$ is an almost primary element of L.*

③ *$N = qI_M$ for some almost primary element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$ where $a \in L$.*

Proof. ①$\Longrightarrow$②. Assume that N is an almost primary element of M. Let $ab \leqslant (N : I_M)$ and $ab \nleqslant (N : I_M)^2$ for $a,\ b \in L$. Then $abI_M \leqslant N$. If $abI_M \leqslant (N : I_M)N$, then by Lemma 4, we have $ab \leqslant ((N : I_M)N : I_M) = (N : I_M)(N : I_M)$ which contradicts $ab \nleqslant (N : I_M)^2$. So let $a(bI_M) \nleqslant (N : I_M)N$. Then as N is almost primary, we have either $a \leqslant \sqrt{N : I_M}$ or $bI_M \leqslant N$ and thus $(N : I_M)$ is an almost primary element of L.

②$\Longrightarrow$③. Assume that $(N : I_M) = q$ is an almost primary element of L. Then $qI_M \leqslant N$. Since M is a multiplication lattice module, $N = aI_M$ for some $a \in L$. So $a \leqslant (N : I_M) = q$ and thus $N = aI_M \leqslant qI_M$. Hence $N = qI_M$ for some almost primary element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$.

③$\Longrightarrow$①. Suppose $N = qI_M$ for some almost primary element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$ where $a \in L$. Then $q \leqslant (N : I_M)$. Now, let $rX \leqslant N$, $rX \nleqslant (N : I_M)N$ and $X \nleqslant N$ for $r \in L$, $X \in M$. Since M is a multiplication lattice module, $X = cI_M$ for some $c \in L$. Then $rc \leqslant (N : I_M) \leqslant q$, using maximality of q to $N = (N : I_M)I_M$(by Proposition 3 of [10]). If $rc \leqslant q^2$, then $rX \leqslant qN \leqslant (N : I_M)N$, a contradiction. So $rc \nleqslant q^2$. Also, $c \nleqslant q$ because if $c \leqslant q$, then $X \leqslant N$, a contradiction. Now, as $rc \leqslant q$, $rc \nleqslant q^2$, $c \nleqslant q$ and q is almost primary, we have, $r \leqslant \sqrt{q}$ which implies $r \leqslant \sqrt{N : I_M}$ and hence N is almost primary

Theorem 39. *Let L be a PG-lattice and M be a faithful multiplication torsion free PG-lattice L-module with I_M compact. Let I_M be a weak join principal element and N be a proper element in M. Then the following statements are equivalent:*

① *N is an almost primary element of M.*

② *$(N : I_M)$ is an almost primary element of L.*

③ *$N = qI_M$ for some almost primary element $q \in L$.*

Proof. ①$\Longrightarrow$② follows from ①$\Longrightarrow$② in the proof of Theorem 38.

②$\Longrightarrow$①. Assume that $(N : I_M)$ is an almost primary element of L. Let $rQ \leqslant N$ and $rQ \nleqslant (N : I_M)N$ for $r \in L$, $Q \in M$. Then $(rQ : I_M) \leqslant (N : I_M)$ and so by Lemma 4, we have $r(Q : I_M) = (rQ : I_M) \leqslant (N : I_M)$. If $r(Q : I_M) \leqslant (N : I_M)^2 = ((N : I_M)N : I_M)$, then $r(Q : I_M)I_M \leqslant (N : I_M)N$ which implies $rQ \leqslant (N : I_M)N$, a contradiction. If $r(Q : I_M) \nleqslant (N : I_M)^2$, then as $r(Q : I_M) \leqslant (N : I_M)$ and $(N : I_M)$ is almost primary, we have either $r \leqslant \sqrt{N : I_M}$ or $(Q : I_M) \leqslant (N : I_M)$ which implies either $r \leqslant \sqrt{N : I_M}$ or $Q \leqslant N$ and thus N is an almost primary element of M.

②$\Longrightarrow$③. Suppose $(N : I_M)$ is an almost primary element of L. Since M is a multiplication lattice L-module, $N = (N : I_M)I_M$ and hence ③ holds.

③$\Longrightarrow$②. Suppose $N = qI_M$ for some almost primary element $q \in L$. As M is a multiplication lattice L-module, $N = (N : I_M)I_M$. Since I_M is compact, ② holds by Theorem 5 of [10].

Now we relate the almost primary element $N \in M$ with $rad(N) \in M$, the radical of N. According to definition 3.1 in [17], the radical of a proper element N in an L module M is defined as $\wedge\{P \in M \mid P$ is a prime element and $N \leqslant P\}$ and is denoted as $rad(N)$. Using Theorem 3.6 of [17], we have the following interesting characterization of an almost primary element of M.

Theorem 40. *Let L be a PG-lattice and M be a faithful multiplication torsion free PG-lattice L-module with I_M compact. Let $I_M \in M$ be a weak join principal element. Then a proper element $P \in M$ is almost primary (ϕ_2-primary) if and only if whenever $N = aI_M$ and $K = bI_M$ in M are such that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ then either $N \leqslant P$ or $K \leqslant rad(P)$ for $a,\ b \in L$.*

Proof. Assume that $P \in M$ is almost primary. Let $N = aI_M$ and $K = bI_M$ in M be such that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ for $a,\ b \in L$. Since M is a multiplication lattice L-module, we have $a = (N : I_M)$ and $b = (K : I_M)$ and so $(K : I_M)(N : I_M)I_M = abI_M \leqslant P$ and $(K : I_M)(N : I_M)I_M \nleqslant (P : I_M)P$. As $P \in M$ is almost primary, we have either $(N : I_M)I_M \leqslant P$ or $(K : I_M) \leqslant \sqrt{P : I_M}$ which implies either $N = (N : I_M)I_M \leqslant P$ or $K = (K : I_M)I_M \leqslant (\sqrt{P : I_M})I_M = rad(P)$ by Theorem 3.6 of [17]. Conversely, assume that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ implies either $N \leqslant P$ or $K \leqslant rad(P)$ where $N = aI_M$ and $K = bI_M$ are in M for $a,\ b \in L$. Let $rs \leqslant (P : I_M)$ and $rs \nleqslant (P : I_M)^2$ where $S = rI_M$ and $Q = sI_M$ are in M for $r,\ s \in L$. If $rsI_M \leqslant (P : I_M)P$, then since M is a multiplication lattice L-module, we have $rsI_M \leqslant (P : I_M)^2 I_M$. So by Theorem 5 of [10], we have $rs \leqslant (P : I_M)^2$, a contradiction. So let $rsI_M \nleqslant (P : I_M)P$. Since $rsI_M \leqslant P$, by hypothesis, we have either $S \leqslant P$ or $Q \leqslant rad(P)$ which implies either $rI_M \leqslant P$ or $sI_M \leqslant rad(P) = (\sqrt{P : I_M})I_M$, by Theorem 3.6 of [17]. So either $r \leqslant (P : I_M)$ or $s \leqslant \sqrt{P : I_M}$, by Theorem 5 of [10]. Thus $(P : I_M)$ is an almost primary element of L and hence by Theorem 39, P is an almost primary element of M.

Now we show that Lemma 4 can also be achieved by changing the conditions on M and I_M.

Lemma 5. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of M. Then $a(N : I_M) = (aN : I_M)$ for $a \in L$.*

Proof. Since M is a multiplication lattice L-module, $N = (N : I_M)I_M$. Then $a(N : I_M)I_M = aN = (aN : I_M)I_M$ and we are done, by Theorem 5 of [10].

Lemma 5 is Lemma 3.5 of [22].

In view of Lemma 5, the Theorems 38, 39 and 40 can be restated in the following way.

Theorem 41. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of an L-module M. Then the following statements are equivalent:*

① *N is an almost primary element of M.*

② *$(N : I_M)$ is an almost primary element of L.*

③ *$N = qI_M$ for some almost primary element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$ where $a \in L$.*

Theorem 42. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of an L-module M. Then the following statements are equivalent:*

① *N is an almost primary element of M.*

② *$(N : I_M)$ is an almost primary element of L.*

③ *$N = qI_M$ for some almost primary element $q \in L$.*

Theorem 43. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then a proper element $P \in M$ is almost primary (ϕ_2-primary) if and only if whenever $N = aI_M$ and $K = bI_M$ in M are such that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ then either $N \leqslant P$ or $K \leqslant rad(P)$ for $a,\ b \in L$.*

The following result is a consequence of the Theorem 42.

Corollary 19. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then a proper element N of an L-module M is almost primary if and only if $(N : I_M)$ is an almost primary element of L.*

The analogous results (from the results of almost primary elements of M) for almost prime elements of M are as follows.

In Example 2.5 of [22], it is shown that an almost prime element of an L-module M need not be weakly prime. The following characterization of an almost prime element of an L-module M shows that under a certain condition, an almost prime element of an L-module M is weakly prime.

Theorem 44. *Let M be a local L-module with a unique maximal element $Q \in M$ such that $(Q : I_M)Q = O_M$. Then a proper element $N \in M$ is almost prime if and only if N is weakly prime.*

Proof. Assume that a proper element $N \in M$ is almost prime. Then $N \leqslant Q$. It follows that $(N : I_M)N \leqslant (Q : I_M)Q = O_M$ and hence $(N : I_M)N = O_M$. Let $O_M \neq aA \leqslant N$ for $a \in L$, $A \in M$. As $aA \leqslant N$, $aA \nleqslant (N : I_M)N = O_M$ and N is almost prime, we have either $A \leqslant N$ or $a \leqslant (N : I_M)$ and hence N is weakly prime. The converse is obvious from Theorem 3.

The following result shows that if an element in M (or L) is almost prime, then its corresponding element in L (or M) is also almost prime.

Theorem 45. *Let L be a PG-lattice and M be a faithful multiplication torsion free PG-lattice L-module with I_M compact. Let I_M be a weak join principal element and N be a proper element of M. Then the following statements are equivalent:*

① *N is an almost prime element of M.*

② *$(N : I_M)$ is an almost prime element of L.*

③ *$N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$ where $a \in L$.*

Proof. ①$\Longrightarrow$②. Assume that N is an almost prime element of M. Let $ab \leqslant (N : I_M)$ and $ab \nleqslant (N : I_M)^2$ for $a,\ b \in L$. Then $abI_M \leqslant N$. If $abI_M \leqslant (N : I_M)N$, then by Lemma 4, we have $ab \leqslant ((N : I_M)N : I_M) = (N : I_M)(N : I_M)$ which contradicts $ab \nleqslant (N : I_M)^2$. So let $a(bI_M) \nleqslant (N : I_M)N$. Then as N is almost prime, we have either $a \leqslant (N : I_M)$ or $bI_M \leqslant N$ and thus $(N : I_M)$ is an almost prime element of L.

②$\Longrightarrow$③. Assume that $(N : I_M) = q$ is an almost prime element of L. Then $qI_M \leqslant N$. Since M is a multiplication lattice module, $N = aI_M$ for some $a \in L$. So $a \leqslant (N : I_M) = q$ and thus $N = aI_M \leqslant qI_M$. Hence $N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$.

③$\Longrightarrow$①. Suppose $N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$ where $a \in L$. Then $q \leqslant (N : I_M)$. Now, let $rX \leqslant N$, $rX \nleqslant (N : I_M)N$ and $X \nleqslant N$ for $r \in L$, $X \in M$. Since M is a multiplication lattice module, $X = cI_M$ for some $c \in L$. Then $rc \leqslant (N : I_M) \leqslant q$, using maximality of q to $N = (N : I_M)I_M$(by Proposition 3 of [10]). If $rc \leqslant q^2$, then $rX \leqslant qN \leqslant (N : I_M)N$, a contradiction. So $rc \nleqslant q^2$. Also, $c \nleqslant q$ because if $c \leqslant q$, then $X \leqslant N$, a contradiction. Now, as $rc \leqslant q$, $rc \nleqslant q^2$, $c \nleqslant q$ and q is almost prime, we have, $r \leqslant q$ which implies $r \leqslant (N : I_M)$ and hence N is almost prime

Theorem 46. *Let L be a PG-lattice and M be a faithful multiplication torsion free PG-lattice L-module with I_M compact. Let I_M be a weak join principal element and N be a proper element of M. Then the following statements are equivalent:*

① *N is an almost prime element of M.*

② *$(N : I_M)$ is an almost prime element of L.*

③ *$N = qI_M$ for some almost prime element $q \in L$.*

Proof. ①$\Longrightarrow$② follows from ①$\Longrightarrow$② in the proof of Theorem 45.

②$\Longrightarrow$①. Assume that $(N : I_M)$ is an almost prime element of L. Let $rQ \leqslant N$ and $rQ \nleqslant (N : I_M)N$ for $r \in L$, $Q \in M$. Then $(rQ : I_M) \leqslant (N : I_M)$ and so by Lemma 4, we have $r(Q : I_M) = (rQ : I_M) \leqslant (N : I_M)$. If $r(Q : I_M) \leqslant (N : I_M)^2 = ((N : I_M)N : I_M)$, then $r(Q : I_M)I_M \leqslant (N : I_M)N$ which implies $rQ \leqslant (N : I_M)N$, a contradiction. If $r(Q : I_M) \nleqslant (N : I_M)^2$, then as $r(Q : I_M) \leqslant (N : I_M)$ and $(N : I_M)$ is almost prime, we have either $r \leqslant (N : I_M)$ or $(Q : I_M) \leqslant (N : I_M)$ which implies either $r \leqslant (N : I_M)$ or $Q \leqslant N$ and thus N is an almost prime element of M.

②⟹③. Suppose $(N : I_M)$ is an almost prime element of L. Since M is a multiplication lattice L-module, $N = (N : I_M)I_M$ and hence ③ holds.

③⟹②. Suppose $N = qI_M$ for some almost prime element $q \in L$. As M is a multiplication lattice L-module, $N = (N : I_M)I_M$. Since I_M is compact, ② holds by Theorem 5 of [10].

The following result is another characterization of an almost prime element of an L-module M.

Theorem 47. *Let L be a PG-lattice and M be a faithful multiplication torsion free PG-lattice L-module with I_M compact. Let I_M be a weak join principal element. Then a proper element $P \in M$ is almost prime ($\phi_2 -$ prime) if and only if whenever $N = aI_M$ and $K = bI_M$ in M are such that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ then either $N \leqslant P$ or $K \leqslant P$ for $a,\ b \in L$.*

Proof. Assume that $P \in M$ is almost prime. Let $N = aI_M$ and $K = bI_M$ in M be such that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ for $a,\ b \in L$. Since M is a multiplication lattice L-module, we have $a = (N : I_M)$ and $b = (K : I_M)$ and so $(K : I_M)(N : I_M)I_M = abI_M \leqslant P$ and $(K : I_M)(N : I_M)I_M \nleqslant (P : I_M)P$. As $P \in M$ is almost prime, we have either $(N : I_M)I_M \leqslant P$ or $(K : I_M) \leqslant (P : I_M)$ which implies either $N = (N : I_M)I_M \leqslant P$ or $K = (K : I_M)I_M \leqslant P$. Conversely, assume that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ implies either $N \leqslant P$ or $K \leqslant P$ where $N = aI_M$ and $K = bI_M$ are in M for $a,\ b \in L$. Let $rs \leqslant (P : I_M)$ and $rs \nleqslant (P : I_M)^2$ where $S = rI_M$ and $Q = sI_M$ are in M for $r,\ s \in L$. If $rsI_M \leqslant (P : I_M)P$, then since M is a multiplication lattice L-module, we have $rsI_M \leqslant (P : I_M)^2 I_M$. So by Theorem 5 of [10], we have $rs \leqslant (P : I_M)^2$, a contradiction. So let $rsI_M \nleqslant (P : I_M)P$. Since $rsI_M \leqslant P$, by hypothesis, we have either $S \leqslant P$ or $Q \leqslant P$ which implies either $rI_M \leqslant P$ or $sI_M \leqslant P$ and so either $r \leqslant (P : I_M)$ or $s \leqslant (P : I_M)$. Thus $(P : I_M)$ is an almost prime element of L and hence by Theorem 46, P is an almost prime element of M.

In view of Lemma 5, the Theorems 45, 46 and 47 can be restated in the following way.

Theorem 48. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of an L-module M. Then the following statements are equivalent:*

① *N is an almost prime element of M.*

② *$(N : I_M)$ is an almost prime element of L.*

③ *$N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leqslant q$ where $a \in L$.*

Theorem 49. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of an L-module M. Then the following statements are equivalent:*

① *N is an almost prime element of M.*

② *$(N : I_M)$ is an almost prime element of L.*

③ *$N = qI_M$ for some almost prime element $q \in L$.*

Theorem 49 is Theorem 3.8 of [22].

Theorem 50. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then a proper element $P \in M$ is almost prime ($\phi_2 - prime$) if and only if whenever $N = aI_M$ and $K = bI_M$ in M are such that $abI_M \leqslant P$ and $abI_M \nleqslant (P : I_M)P$ then either $N \leqslant P$ or $K \leqslant P$ for $a,\ b \in L$.*

Theorem 50 is Theorem 3.14 of [22].

The following result is a consequence of the Theorem 49.

Corollary 20. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then a proper element N of an L-module M is almost prime if and only if $(N : I_M)$ is an almost prime element of L.*

According to [16], a proper element $q \in L$ is said to be 2-potent prime if for all $a, b \in L$, $ab \leqslant q^2$ implies either $a \leqslant q$ or $b \leqslant q$ and a proper element $q \in L$ is said to be 2-potent primary if for all $a, b \in L$, $ab \leqslant q^2$ implies either $a \leqslant q$ or $b \leqslant \sqrt{q}$.

In view of these definitions, we define n-potent prime and n-potent primary elements (where $n \geqslant 2$) in a multiplicative lattice L in following way.

Definition 9. *Let $n \geqslant 2$ and $n \in Z_+$. A proper element $q \in L$ is said to be n-potent prime if for all $a, b \in L$, $ab \leqslant q^n$ implies either $a \leqslant q$ or $b \leqslant q$.*

Definition 10. *Let $n \geqslant 2$ and $n \in Z_+$. A proper element $q \in L$ is said to be n-potent primary if for all $a, b \in L$, $ab \leqslant q^n$ implies either $a \leqslant q$ or $b \leqslant \sqrt{q}$.*

Now we show that if an element in M is n-potent prime (respectively n-potent primary), then its corresponding element in L is also n-potent prime (respectively n-potent primary) and vice-versa where $n \geqslant 2$.

Theorem 51. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of an L-module M and $n \geqslant 2$. Then the following statements are equivalent:*

① *N is a n-potent prime element of M.*

② *$(N : I_M)$ is a n-potent prime element of L.*

③ *$N = qI_M$ for some n-potent prime element $q \in L$.*

Proof. Since M is a multiplication lattice L-module, by Proposition 3 of [10], we have $N = (N : I_M)I_M$.

①⟹②. Assume that N is a n-potent prime element of M. Let $ab \leqslant (N : I_M)^n$ for $a,\ b \in L$. Then $a(bI_M) \leqslant (N : I_M)^{n-1}N$. As N is n-potent prime, we have either $a \leqslant (N : I_M)$ or $bI_M \leqslant N$ and thus $(N : I_M)$ is a n-potent prime element of L.

②⟹①. Assume that $(N : I_M)$ is a n-potent prime element of L. Let $aX \leqslant (N : I_M)^{n-1}N$ for $a \in L$ and $X \in M$. M being a multiplication lattice L-module, we have $X = cI_M$ for some $c \in L$. Clearly, $a(cI_M) \leqslant (N : I_M)^n I_M$. This implies that $ac \leqslant (N : I_M)^n$ by Theorem 5 of [10]. As $(N : I_M)$ is a n-potent prime, we have either $a \leqslant (N : I_M)$ or $c \leqslant (N : I_M)$ which implies either $a \leqslant (N : I_M)$ or $X = cI_M \leqslant (N : I_M)I_M = N$ and thus N is a n-potent prime element of M.

②⟹③. Suppose $q = (N : I_M)$ is a n-potent prime element of L. Since M is a multiplication lattice L-module, $N = (N : I_M)I_M = qI_M$ and hence ③ holds.

③⟹②. Suppose $N = qI_M$ for some n-potent prime element $q \in L$. As M is a multiplication lattice L-module, $N = (N : I_M)I_M$. Since I_M is compact, ② holds by Theorem 5 of [10].

Theorem 52. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Let N be a proper element of an L-module M and $n \geqslant 2$. Then the following statements are equivalent:*

① *N is a n-potent primary element of M.*

② *$(N : I_M)$ is a n-potent primary element of L.*

③ *$N = qI_M$ for some n-potent primary element $q \in L$.*

Proof. Just mimic the proof of Theorem 51.

We conclude this paper with following 2 results which are outcomes of Theorems 51 and 52, respectively.

Corollary 21. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then a proper element N of an L-module M is 2-potent prime if and only if $(N : I_M)$ is a 2-potent prime element of L.*

Corollary 22. *Let L be a PG-lattice and M be a faithful multiplication PG-lattice L-module with I_M compact. Then a proper element N of an L-module M is 2-potent primary if and only if $(N : I_M)$ is a 2-potent primary element of L.*

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References

[1] Eaman A Al-Khouja. Maximal elements and prime elements in lattice modules. *Damascus University for Basic Sciences*, 19(2):9–20, 2003.

[2] Francisco Alarcon, DD Anderson, and C Jayaram. Some results on abstract commutative ideal theory. *Periodica Mathematica Hungarica*, 30(1):1–26, 1995.

[3] DD Anderson. Abstract commutative ideal theory without chain condition. *Algebra Universalis*, 6(2):131–145, 1976.

[4] Sachin Ballal, Machchhindra Gophane, and Vilas Kharat. On weakly primary elements in multiplicative lattices. *Southeast Asian Bulletin of Mathematics*, 40(1):439–449, 2016.

[5] Sachin Ballal and Vilas Kharat. On generalization of prime, weakly prime and almost prime elements in multiplicative lattices. *Int. J. Algebra*, 8(9):439–449, 2014.

[6] Sachin Ballal and Vilas Kharat. On ϕ-absorbing primary elements in lattice modules. *Algebra*, 2015:1–6, 2015.

[7] Malik Bataineh and S Kuhail. Generalizations of primary ideals and submodules. *International Journal of Contemporary Mathematical Sciences*, 6(17):811–824, 2011.

[8] Ashok V Bingi and CS Manjarekar. Weakly prime and weakly primary elements in multiplication lattice modules. (to appear).

[9] Fethi Çallıalp, C Jayaram, and Ünsal Tekir. Weakly prime elements in multiplicative lattices. *Communications in Algebra*, 40(8):2825–2840, 2012.

[10] Fethi Çallıalp and Ünsal Tekir. Multiplication lattice modules. *Iranian Journal of Science and Technology*, 35(4):309–313, 2011.

[11] Dustin Scott Culhan. *Associated Primes and Primal Decomposition in modules and Lattice modules, and their duals.* University of Michigan Press, University of California, Riverside, 2005.

[12] C Jayaram, Ünsal Tekir, and Ece Yetkin. 2-absorbing and weakly 2-absorbing elements in multiplicative lattices. *Communications in Algebra*, 42(6):2338–2353, 2014.

[13] EW Johnson and JA Johnson. Lattice modules over semi-local noether lattices. *Fundamenta Mathematicae*, 68(2):187–201, 1970.

[14] J Johnson. a-adic completions of noetherian lattice modules. *Fundamenta Mathematicae*, 66:347–373, 1970.

[15] Zeliha Kılıç. Almost primary elements in multiplicative lattices. *International Journal of Algebra*, 7(18):881–888, 2013.

[16] CS Manjarekar and AV Bingi. ϕ-prime and ϕ-primary elements in multiplicative lattices. *Algebra*, 2014:1–7, 2014.

[17] CS Manjarekar and AV Bingi. Absorbing elements in lattice modules. *International Electronic Journal of Algebra*, 19(19):58–76, 2016.

[18] CS Manjarekar and AV Bingi. On 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices. *Trans. Algebra Appl.*, 2:1–13, 2016.

[19] CS Manjarekar and UN Kandale. Weakly prime elements in lattice modules. *International Journal of Scientific and Research Publications*, 3(8):1–6, 2013.

[20] CS Manjarekar and UN Kandale. Residuation properties and weakly primary elements in lattice modules. *Algebra*, 2014:1–4, 2014.

[21] NK Thakare and CS Manjarekar. Radicals and uniqueness theorem in multiplicative lattices with chain conditions. *Studia Scientifica Mathematicarum Hungarica*, 18:13–19, 1983.

[22] Emel Aslankarayigit Ugurlu, Fethi Callialp, and Unsal Tekir. Prime, weakly prime and almost prime elements in multiplication lattice modules. *Open Mathematics*, 14(1):673–680, 2016.

[23] Jane Wells. The restricted cancellation law in a noether lattice. *Fundamenta Mathematicae*, 3(75):235–247, 1972.

[24] Naser Zamani. φ-prime submodules. *Glasgow Mathematical Journal*, 52(2):253–259, 2010.