EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 8, No. 4, 2015, 462-468
ISSN 1307-5543 – www.ejpam.com

Ore Extensions Over (σ, δ)-Rings

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Abstract. Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers). Let σ be an automorphism of R and δ a σ-derivation of R. A ring R is called a (σ, δ)-ring if $a(\sigma(a)+\delta(a)) \in P(R)$ implies that $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R. We prove that R is 2-primal if $\delta(P(R)) \subseteq P(R)$. We also study the property of minimal prime ideals of R and prove the following in this direction:
Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring. If $P \in Min.Spec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$. Further if $\delta(P(R)) \subseteq P(R)$, then $P[x; \sigma, \delta]$ is a completely prime ideal of $R[x; \sigma, \delta]$.

2010 Mathematics Subject Classifications: 16-XX, 16W20, 16P40, 16S50

Key Words and Phrases: Noetherian ring, Ore extension, endomorphism, automorphism, minimal prime ideals, (σ, δ)-rings and 2-primal.

1. Introduction and Preliminaries

All rings are associative with identity $1 \neq 0$, unless otherwise stated. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The ring of integers is denoted by $\mathbb{Z}$ and the field of rational numbers by $\mathbb{Q}$, unless otherwise stated. The set of minimal prime ideals of R is denoted by $Min.Spec(R)$.

We begin with the following:

Definition 1. *Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R, which is defined as an additive map from R to R such that [12]*

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R.$$

Example 1. *Let $R = \mathbb{Z}[\sqrt{2}]$. Then $\sigma : R \to R$ defined as*

$$\sigma(a + b\sqrt{2}) = a - b\sqrt{2} \text{ for } a + b\sqrt{2} \in R$$

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is an endomorphism of R.

For any $s \in R$, Define $\delta_s : R \to R$ by

$$\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}) \text{ for } a + b\sqrt{2} \in R.$$

Then δ_s is a σ-derivation of R.

Recall that $R[x;\sigma,\delta]$ is the usual polynomial ring with coefficients in R where multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$, for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$. An ideal I of a ring R is called σ-stable if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-stable and δ-invariant, then $I[x;\sigma,\delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$.

Definition 2. *A completely prime ideal in a ring R is any ideal such that R/P is a domain [7].*

Also an ideal P of a ring R is said to be completely prime if $ab \in P$ implies that $a \in P$ or $b \in P$ for $a, b \in R$. In commutative sense completely prime and prime have the same meaning. We also note that a completely prime ideal of a ring R is a prime ideal, but the converse need not be true. The following example shows that a prime ideal need not be a completely prime ideal.

Example 2 (Example 1.1 of [2]). *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R. But is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ we have $ab \in P$, even though $a \notin P$ and $b \notin P$.*

There are examples of rings (non-commutative) in which prime ideals are completely prime.

Example 3 (Example 1.2 of [2]). *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $P_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ are prime ideals of R. Now all these are completely prime also.*

Definition 3. *A minimal prime ideal in a ring R is any prime ideal of R that does not properly contain any other prime ideal [3].*

Example 4. *In example 1.2 of [2] (discussed above), $P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ is minimal prime ideal.*

Further more there are examples of rings in which minimal prime ideals are completely prime. For example a reduced ring. If R is a prime ring, then 0 is a minimal prime ideal of R and it is the only one. In Proposition (3.3) of [6], it has been shown that any prime ideal U in a ring R contains a minimal prime ideal. Further it has been proved that there exists only finitely many minimal prime ideals in a Noetherian ring R and there is a finite product of minimal prime ideals (repetition allowed) that equals zero. An example of a ring which has infinitely many minimal prime ideals is:

Example 5 (Exercise 3C of [7]). *Let X be an infinite set, K a field, and R the ring of all functions from X to K. For $x \in X$, let P_x be the set of those functions in R which vanish at x. Then each P_x is a minimal prime ideal of R.*

It is also known that [6] in a right Noetherian ring which is also an algebra over $\mathbb{Q}$, δ a σ-derivation of R and U a minimal prime ideal of R, $\delta(U) \subseteq U$.

Definition 4. *A ring R is said to be 2-primal if and only if $P(R) = N(R)$ [4].*

Example 6. *Let $R = (\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$. Then R is a commutative ring and hence 2-primal.*

Also a reduced ring is 2-primal and so is a commutative Noetherian ring. Part of the attraction of 2-primal rings in addition to their being a common generalization of commutative rings and rings without nilpotent elements lies in the structure of their prime ideals. We refer to [4, 5, 8, 9, 11, 13, 14] for more details on 2-primal rings.

Definition 5. *Let R be a ring and σ an endomorphism of R. Then R is said to be $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$ for $a \in R$ [3].*

We note that if R is a Noetherian ring and σ an automorphism of R, then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R [Theorem (2.3) of 3].

Definition 6. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a δ-ring if $a\delta(a) \in P(R)$ implies that $a \in P(R)$ for $a \in P(R)$ [1].*

Note that a ring with identity is not a δ-ring as $1\delta(1) = 0$, but $1 \neq 0$. Also from [1] we know that if R is a δ-Noetherian $\mathbb{Q}$-algebra such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$, for all $P \in Min.Spec(R)$ and $\delta(P(R)) \subseteq P(R)$, then $R[x;\sigma,\delta]$ is 2-primal.

We now generalize these notions as follows:

Definition 7. *Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ,δ)-ring if $a(\sigma(a)+\delta(a)) \in P(R)$ implies that $a \in P(R)$ for $a \in R$.*

Example 7. *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by*

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \text{for all } a,b,c \in \mathbb{Z}.$$

Then it can be seen that σ is an endomorphism of R.

Define $\delta : R \to R$ by

$$\delta(a) = a - \sigma(a), \text{for all} a \in R.$$

Clearly, δ is a σ-derivation of R.

Now let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. $A[\sigma(A) + \delta(A)] \in P(R)$ *implies that*

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big\{\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big)+\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}-\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big)\Big\} \in P(R)$$

or

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big\{\begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}+\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}-\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big)\Big\} \in P(R)$$

which gives on simplification, $\begin{pmatrix} a^2 & ab+bc \\ 0 & c^2 \end{pmatrix} \in P(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ *which implies that* $a^2 = 0, c^2 = 0$, *i.e.* $a = 0$, $c = 0$. *Therefore,* $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P(R)$. *Hence R is a* (σ, δ)*-ring.*

Example 8. *Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring. Define an automorphism $\sigma : R \to R$ by $\sigma((a, b)) = (b, a)$ for $a, b \in \mathbb{Z}_2$. Also $\delta : R \to R$ defined by $\delta((a, b)) = (a - b, 0)$ for $a, b \in \mathbb{Z}_2$ is a σ-derivation of R. Here $P(R) = \{0\}$. But R is not a (σ, δ)-ring, for take $(a, b) = (0, 1)$.*

With this we prove the following:

Theorem 1: Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.

Theorem 2: Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring. If $P \in Min.Spec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.

Theorem 4: Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in Min.Spec(R)$ be such that $\sigma(P) = P$, then $O(P)$ is a completely prime ideal of $O(R)$.

2. Proof of Main Results

We now prove Theorems 1, 2 and 3 as follows:

Theorem 1. *Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\rho : R/P(R) \to R/P(R)$ by

$$\rho(a + P(R)) = \delta(a) + P(R) \text{ for } a \in R$$

Also define $\tau : R/P(R) \to R/P(R)$ by

$$\tau(a + P(R)) = \sigma(a) + P(R) \text{ for } a \in R.$$

Then τ is an automorphism of $R/P(R)$ and ρ is a τ-derivation of $R/P(R)$.

Also $a(\sigma(a) + \delta(a)) \in P(R)$ if and only if

$$(a + P(R))\rho(a + P(R)) + (a + P(R))\tau(a + P(R)) = P(R) in R/P(R).$$

Then as in Proposition (5) of [10], R is a reduced ring. Hence it is 2-primal. □

For the proof of Theorem 2, we need the following:

Proposition 1. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let δ be a derivation of R. Then $\delta(P(R)) \subseteq P(R)$.*

Proof. See Proposition (1.1) of [1]. □

Proposition 2. *Let R be a 2-primal ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. If $P \in Min.Spec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. Let $P \in Min.Spec(R)$. Now P is a completely prime ideal, therefore, for any $a \in P$ there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of Shin [13]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \subseteq P(R)$; i.e., $\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Now $\sigma(P) = P$ implies that $\sigma(b) \notin P$ and since P is completely prime in R, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$. □

Theorem 2. *Let R be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring. If $P \in Min.Spec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. Let $P \in Min.Spec(R)$. Then by Proposition 1, $\delta(P(R)) \subseteq P(R)$ and by Theorem 1, R is 2-primal. Since $\sigma(P) = P$, the result follows by Proposition 2. □

For the proof of Theorem 4, we need the following:

Theorem 3. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

(i) For any completely prime ideal P of R with $\sigma(P) = P$ and $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.

(ii) For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.

Proof. See Theorem (2.4) of [2]. □

Theorem 4. *Let R be a Noetherian, integral domain which is also an algebra over* $\mathbb{Q}$. *Let* σ *be an automorphism of R and* δ *a* σ*-derivation of R such that R is a* (σ, δ)*-ring and* $\delta(P(R)) \subseteq P(R)$. *Let* $P \in Min.Spec(R)$ *be such that* $\sigma(P) = P$, *then* $O(P)$ *is a completely prime ideal of* $O(R)$.

Proof. R is 2-primal by Theorem 1 and so by Proposition 2, $\delta(P) \subseteq P$ and as in proof of Proposition 2 above, P is a completely prime ideal of R. Now use Theorem 3 and the proof is complete. □

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