EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS

Vol. 2, No. 1, 2009, (58-72)

ISSN 1307-5543 – www.ejpam.com

Annulets in Almost Distributive Lattices

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Abstract. We introduce the concept of annulets in an Almost Distributive lattice(ADL) R with 0. We characterize both generalized stone ADL and normal ADL in terms of their annulets. We characterize $\star$-ADLs by means of their annulets. It is proved that the lattice $\mathscr{A}_0(R)$ of all annulets of a generalized stone ADL R is a relatively complemented sublattice of the lattice $\mathscr{I}(R)$ of all ideals of R. Finally, it is proved that $\mathscr{A}_0(R)$ is relatively complemented iff R is sectionally $\star$-ADL.

AMS subject classifications: 06D99, 06D15.

Key words: Almost Distributive Lattice(ADL), Boolean algebra, dense elements, maximal element, Annihilator ideal, Annulet, normal ADL, $\star$-ADL, generalized stone ADL, Disjunctive ADL.

1. Introduction

The concept of an Almost Distributive Lattice(ADL) was introduced by Swamy. U.M. and Rao.G.C [8] as a common abstraction to most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. Later a more general class called $\star$-ADLs was introduced in the paper [10]. The characterization of $\star$-ADL by means of it's dense elements was studied in [11]. In [5], Mandelker studied the properties of relative annihilators and

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characterized the distributive lattice in terms of relative annihilators. In this paper the concept of Annulet as an ideal of the form $(x]^* = \{ a \in R \mid x \wedge a = 0 \}$ in an ADL R with 0 is introduced, analogous to that in a distributive lattice[4]. It is proved that the set $\mathscr{A}_0(R)$ of all annulets of an ADL R with 0 can be made into a distributive lattice and sublattice of the Boolean algebra $\mathscr{A}(R)$ of all annihilator ideals of R.

We characterize the generalized stone ADL and normal ADL in terms of their annulets. We introduce a more general class of ADLs called disjunctive ADLs with suitable examples and prove that a disjunctive normal ADL is dually isomorphic to the lattice $\mathscr{A}_0(R)$. We characterize $\star$-ADLs by means of their annulets. If R is a generalized stone ADL, then it is proved that the lattice $\mathscr{A}_0(R)$ is a relatively complemented sublattice of the lattice $\mathscr{I}(R)$ of all ideals of R. Finally, it is proved that $\mathscr{A}_0(R)$ is relatively complemented iff R is sectionally $\star$-ADL.

2. Preliminaries

An Almost Distributive Lattice (ADL) is an algebra $(R, \vee, \wedge)$ of type (2,2) satisfying

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$. for any $x, y, z \in R$.

If R has an element 0 and satisfies $0 \wedge x = 0$ and $x \vee 0 = x$ along with the above properties, then R is called an ADL with 0.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define two binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL with x_0 as zero element and is called a discrete ADL.

If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R.

Theorem 2.1. *For any $a, b, c \in R$, we have the following:*

1. $a \vee b = a \Leftrightarrow a \wedge b = b$
2. $a \vee b = b \Leftrightarrow a \wedge b = a$
3. $a \wedge b = b \wedge a$ *whenever* $a \leq b$
4. $\wedge$ *is associative in* R
5. $a \wedge b \wedge c = b \wedge a \wedge c$
6. $(a \vee b) \wedge c = (b \vee a) \wedge c$
7. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
8. $a \vee b = b \vee a$ *whenever* $a \wedge b = 0$
9. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
10. $a \wedge (a \vee b) = a, (a \wedge b) \vee b = b,$ *and* $a \vee (b \wedge a) = a$
11. $a \leq a \vee b$ *and* $a \wedge b \leq b$
12. $a \wedge a = a$ *and* $a \vee a = a$
13. $0 \vee a = a$ *and* $a \wedge 0 = 0$
14. *If* $a \leq c$ *and* $b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
15. $a \vee b = a \vee b \vee a$.

An element $m \in R$ is called maximal if it is maximal in the partial ordered set $(R, \leq)$. That is, for any $x \in R, m \leq x \Rightarrow m = x$.

Theorem 2.2. *Let R be an ADL and $m \in R$. Then the following are equivalent:*

1. *m is a maximal element with respect to $\leq$*
2. *$m \vee x = m$, for all $x \in R$*
3. *$m \wedge x = x$, for all $x \in R$*
4. *$x \vee m$ is maximal for all $x \in R$.*

A non-empty subset I of R is called an ideal(filter)of R if $a \vee b \in I (a \wedge b \in I)$ and $a \wedge x \in I (x \vee a \in I)$ whenever $a, b \in I$ and $x \in R$. If I is an ideal of R and $a, b \in R$,then $a \wedge b \in I \Longleftrightarrow b \wedge a \in I$. The set $\mathscr{I}(R)$ of all ideals of R is a complete distributive lattice with least element $\{0\}$ and the greatest element R under set inclusion in which, for any $I, J \in \mathscr{I}(R), I \cap J$ is the infimum of I, J and the supremum is given by $I \vee J = \{ i \vee j \mid i \in I, j \in J \}$. For any $a \in R$, $(a] = \{ a \wedge x \mid x \in R \}$ is the principal ideal generated by a. Similarly, for any $a \in R, [a) = \{ x \vee a \mid x \in R \}$ is the filter generated by a. An ideal I of R is called a direct summand of R if there exists an ideal J in R such that $I \cap J = (0]$ and $I \vee J = R$.

Theorem 2.3. *For any $a, b \in R$, we have the following:*

1. $(a] \vee (b] = (a \vee b] = (b \vee a]$
2. $(a] \cap (b] = (a \wedge b] = (b \wedge a]$
3. $[a) \vee [b) = [a \wedge b) = [b \wedge a)$
4. $[a) \cap [b) = [a \vee b) = [b \vee a)$

Thus the set $\mathscr{P}\mathscr{I}(R)$ of all principal ideals of R is a sublattice of the distributive lattice $\mathscr{I}(R)$ of ideals of R. A proper ideal P of R is said to be prime if for any $x, y \in R$, $x \wedge y \in P \Rightarrow$ either $x \in P$ or $y \in P$. It is clear that a subset P of R is a prime ideal *iff* $R - P$ is a prime filter.

For any $A \subseteq R$, $A^* = \{ x \in R \mid a \wedge x = 0$ for all $a \in A \}$ is an ideal of R. We write $(a]^*$ for $\{a\}^*$. Then clearly $(0]^* = R$ and $R^* = (0]$. An element $a \in R$ is called dense if $(a]^* = (0]$. The set of all dense elements of R is denoted by D. An ideal I of R is called dense if $I^* = (0]$. An ADL R with 0 is called a $\star$-ADL [10], if for each $x \in R$, there exists an element $x' \in R$ such that $(x]^{**} = (x']^*$. R is a $\star$-ADL iff to each $x \in R$, there exists $x' \in R$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense. Every $\star$-ADL possesses a dense element. An ADL R with 0 is called relatively complemented if each interval $[a, b], a \leq b$, in R is a complemented lattice.

An ideal I of R is called an annihilator ideal if $I = I^{**}$, or equivalently, $I = S^* = \{ y \in R \mid y \wedge s = 0 \text{ for all } s \in S \}$ for some non-empty subset S of R. We denote the set of all annihilator ideals of R by $\mathscr{A}(R)$. The set $\mathscr{A}(R)$ forms a complete Boolean algebra with bounds $\{0\}, R$ and the complement of any $I \in \mathscr{A}(R)$ is I^* with respect to the operations $\wedge$ and $\underline{\vee}$ given by $I \wedge J = I \cap J$ and $I \underline{\vee} J = (I^* \cap J^*)^*$.

3. Annulets

In this section, we introduce the concept of annulets in R and study some basic properties of these annulets. We prove charactarization theorems of a few algebraic structures with the help of their annulets. We begin with the following definition.

Definition 3.1. *Let R be an ADL with 0 and $x \in R$. Then define the annulet $(x]^*$ as follows:*

$$(x]^* = \{ y \in R \mid x \wedge y = 0 \}$$

Clearly $(x]^*$ is an ideal in R and hence an annihilator ideal.
Let us denote $\mathscr{A}_0(R) = \{ (x]^* \mid x \in R \}$.

Annulets have many important properties. We give some of them in the following lemma which can be proved directly.

Lemma 3.2. *Let R be an ADL with 0 and $x, y \in R$. Then we have:*

1. $x \leq y \Rightarrow (y]^* \subseteq (x]^*$
2. $(x \wedge y]^* = (y \wedge x]^*$
3. $(x \vee y]^* = (y \vee x]^*$
4. $(x \vee y]^* = (x]^* \cap (y]^*$
5. $(x]^* \vee (y]^* \subseteq (x \wedge y]^*$.

Note: Since each annulet is an annihilator ideal, we can have the following:

$$(x]^* \underline{\vee} (y]^* = \left[(x]^{**} \cap (y]^{**}\right]^* = \left[(x \wedge y]^{**}\right]^* = (x \wedge y]^*$$

$$(x]^* \wedge (y]^* = (x]^* \cap (y]^* = (x \vee y]^*.$$

Now we prove in the following theorem that the set $\mathscr{A}_0(R)$ of all annulets of an ADL R forms a distributive lattice.

Theorem 3.3. *Let R be an ADL with 0. Then $(\mathscr{A}_0(R), \cap, \underline{\vee})$ is a distributive lattice and a sublattice of the Boolean algebra $\langle \mathscr{A}(R), \cap, \underline{\vee}, ^*, (0], R\rangle$ of annihilator ideals of R. $\mathscr{A}_0(R)$ has the same greatest element $R = (0]^*$ as $\mathscr{A}(R)$ while $\mathscr{A}_0(R)$ has the smallest element iff R possesses a dense element.*

Proof: Let $(x]^*, (y]^* \in \mathscr{A}_0(R)$, where $x, y \in R$. Then

1. $(x]^* \wedge (y]^* = (x]^* \cap (y]^* = (x \vee y]^* \in \mathscr{A}_0(R)$ and
2. $(x]^* \underline{\vee} (y]^* = (x \wedge y]^* \in \mathscr{A}_0(R)$.

Hence $\mathscr{A}_0(R)$ is a sublattice of $\mathscr{A}(R)$. Since $\mathscr{A}(R)$ is distributive, we have that $\mathscr{A}_0(R)$ is also distributive. Clearly $(0]^*$ is the greatest element of $\mathscr{A}(R)$. Now for any $(x]^* \in \mathscr{A}_0(R)$, we get $(x]^* \cap (0]^* = (x \vee 0]^* = (x]^*$ and $(x]^* \underline{\vee} (0]^* = (x \wedge 0]^* = (0]^*$. It shows that $(0]^*$ is the greatest element in $\mathscr{A}_0(R)$. Now, it remains to prove the final condition of the theorem. Assume $\mathscr{A}_0(R)$ has the smallest element, say $(d]^*$ where $d \in R$. Suppose $x \in (d]^*$. Then $x \wedge d = 0$. Since $(d]^*$ is the least element, we get $(x]^* = (x]^* \underline{\vee} (d]^* = (x \wedge d]^* = (0]^* = R$. Hence $x = 0$. Thus $(d]^* = (0]$. Therefore d is a dense element in R.
Conversely, suppose that R possesses a dense element, say d. So $(d]^* = (0]$. Clearly $(d]^* \in A_0(R)$. Now for any $x \in R$, consider $(x]^* \cap (d]^* = (x]^* \cap (0] = (0]$. Also $(x]^* \underline{\vee} (d]^* = [(x]^{**} \cap (d]^{**}]^* = [(x]^{**} \cap (0]^*]^* = [(x]^{**} \cap R]^* = (x]^{***} = (x]^*$. Hence $(d]^*$ is the smallest element in $\mathscr{A}_0(R)$. □

The following definition of a normal ADL is taken from [7].

Definition 3.4. *An ADL R with 0 is called normal ADL iff for all $x, y \in R$*

$$(x]^* \vee (y]^* = (x \wedge y]^*.$$

Swamy.U.M., Rao.G.C., Nanaji Rao.G.[9] and [10], have studied the properties of a psuedo-complemented ADL and later introduced the concept of stone ADL [10] as a psuedo-complemented

ADL R with 0, in which $x^* \vee x^{**} = 0^*$, for all $x \in R$. Now we give the definition of a generalized stone ADL in the following.

Definition 3.5. *An ADL R with 0 is called a generalized Stone ADL iff*

$$(x]^* \vee (x]^{**} = R \text{ for each } x \in R.$$

Example 3.6. *Let* $A = \{0, a\}$ *and* $B = \{0, b_1, b_2\}$ *be two discrete ADLs. Write* $R = A \times B = \{(0,0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. *Then* $(R, \vee, \wedge, 0')$ *is an* ADL *where* $0' = (0, 0)$, *under point-wise operations.*

Now $((a, 0)]^* \vee ((a, 0)]^{**} = \{(0, 0), (0, b_1), (0, b_2)\} \vee \{(0, 0), (a, 0)\} = R$.

$((0, b_1)]^* \vee ((0, b_1)]^{**} = \{(0, 0), (a, 0)\} \vee \{(0, 0), (0, b_1), (0, b_2)\} = R$.

Also $((a, b_1)]^* \vee ((a, b_1)]^{**} = \{(0, 0)\} \vee R = R$.

Hence $(R, \vee, \wedge, 0')$ *is a generalized stone ADL.*

We now characterize normal ADL and the generalized stone ADL in terms of annulets.

Theorem 3.7. *Let R be an ADL with 0. Consider the following conditions:*

(1). *Each annulet is a direct summand of R*

(2). *R is a generalized stone ADL*

(3). *R is normal*

(4). $\mathscr{A}_0(R)$ *is a sublattice of the lattice* $\mathscr{I}(R)$ *of all ideals of R.*

Then (1) *is equivalent to* (2)*,* (3) *is equivalent to* (4)*, and* (2) *implies* (3)*. If R is a* $\star$*-ADL, then* (4) *implies* (1).

Proof: $(1) \Rightarrow (2)$: Let $x \in R$. Then by (1), there exists an ideal J of R such that $(x]^* \cap J = (0]$ and $(x]^* \vee J = R$. Now $(x]^* \cap J = (0]$ implies that $J \subseteq (x]^{**}$. Hence $R = (x]^* \vee J \subseteq (x]^* \vee (x]^{**}$. Thus $R = (x]^* \vee (x]^{**} \ \ \forall\, x \in R$.

$(2) \Rightarrow (1)$: Assume that R is a generalized stone ADL. Let $x \in R$.

We have always $(x]^* \cap (x]^{**} = (0]$. By (2), we get $(x]^* \vee (x]^{**} = R$.

$(2) \Rightarrow (3)$: Assume that R is a generalized stone ADL. Let $x, y \in R$. Always we have $(x]^* \vee (y]^* \subseteq (x \wedge y]^*$. Let $a \in (x \wedge y]^*$. Then $a \wedge x \wedge y = 0$.

$$\Rightarrow (a \wedge x \wedge y] = (0]$$

$\Rightarrow\ (x] \cap (a \wedge y] = (0]$

$\Rightarrow\ (a \wedge y] \subseteq (x]^*$

$\Rightarrow\ (x]^{**} \subseteq (a \wedge y]^*$

$\Rightarrow\ (x]^{**} \cap (a \wedge y] = (0]$

$\Rightarrow\ (x]^{**} \cap \{(a] \cap (y]\} = (0]$

$\Rightarrow\ \{(x]^{**} \cap (a]\} \cap (y] = (0]$

$\Rightarrow\ (x]^{**} \cap (a] \subseteq (y]^*$

It is clear that $(x]^* \cap (a] \subseteq (x]^*$

Thus we get that $\{(x]^* \cap (a]\} \vee \{(x]^{**} \cap (a]\} \subseteq (x]^* \vee (y]^*$

$\Rightarrow\ \{(x]^* \vee (x]^{**}\} \cap (a] \subseteq (x]^* \vee (y]^*$

$\Rightarrow\ R \cap (a] \subseteq (x]^* \vee (y]^*$ (since R is a generalized stone ADL)

$\Rightarrow\ (a] \subseteq (x]^* \vee (y]^*$

$\Rightarrow\ a \in (x]^* \vee (y]^*$

Hence $(x \wedge y]^* \subseteq (x]^* \vee (y]^*$. Thus $(x \wedge y]^* = (x]^* \vee (y]^*$. Therefore R is normal.

Now we prove the equivalency of (3) and (4).

(3) $\Rightarrow$ (4): Assume that R is normal. Let $x, y \in R$. We have always $(x]^* \cap (y]^* = (x \vee y]^* \in \mathscr{A}_0(R)$. Since R is normal, we get $(x]^* \vee (y]^* = (x \wedge y]^* \in \mathscr{A}_0(R)$. Therefore $\mathscr{A}_0(R)$ is a sublattice of $\mathscr{I}(R)$.

(4) $\Rightarrow$ (3): Assume the condition (4). Let $x, y \in R$. Then by (4), $(x]^* \vee (y]^* = (z]^*$, for some $z \in R$. Now $(z]^{**} = \{(x]^* \vee (y]^*\}^* = (x]^{**} \cap (y]^{**} = (x \wedge y]^{**}$. Hence $(x]^* \vee (y]^* = (x \wedge y]^*$. Therefore R is normal.

(4) $\Rightarrow$ (1): Suppose R is a $\star - ADL$. Assume that $\mathscr{A}_0(R)$ is a sublattice of $\mathscr{I}(R)$. Let $x \in R$. Then there exists $x' \in R$ such that $(x]^{**} = (x']^*$. We have always $(x]^* \cap (x]^{**} = (0]$. Now $(x]^* \vee (x]^{**} = (x]^* \vee (x']^* = (z]^*$, for some $z \in R$(by condition (4)). Hence $(z]^{**} = \{(x]^* \vee (x']^*\}^* = (x]^{**} \cap (x']^{**} = (x]^{**} \cap (x]^{***} = (0]$.

Thus $(x]^* \vee (x]^{**} = (z]^* = (0]^* = R$. Thus $(x]^*$ is a direct summand of R. $\square$

Definition 3.8. *An ADL R with 0, is called disjunctive iff for all $a, b \in R$,*

$$(a]^* = (b]^* \text{ implies } a = b.$$

Example 3.9. *Let $R = \{0, a, b, c\}$ be a set. Define $\vee$ and $\wedge$ on R as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then clearly $(R, \vee, \wedge, 0)$ is an ADL with 0.
Now, $(a]^* = (0]$, $(b]^* = \{0, c\}$ and $(c]^* = \{0, b\}$.
Thus $x \neq y$ implies that $(x]^* \neq (y]^*$ for all $x, y \in R$. Hence R is disjunctive.

Theorem 3.10. *A disjunctive ADL R is dually isomorphic to $\mathscr{A}_0(R)$.*

Proof: Let R be a disjunctive ADL. Define a mapping $\Phi : R \longrightarrow \mathscr{A}_0(R)$ by $\Phi(x) = (x]^*$, for all $x \in R$. Clearly Φ is well-defined.

(i). Let $x, y \in R$ be such that $\Phi(x) = \Phi(y)$. Then $(x]^* = (y]^*$. Since R is disjunctive, we get that $x = y$. Therefore Φ is One-one.

(ii). Let $y \in \mathscr{A}_0(R)$. Then $y = (x]^*$, for some $x \in R$. Now for this x, $\Phi(x) = (x]^* = y$. Therefore Φ is onto.

(iii). Let$(x]^*, (y]^* \in \mathscr{A}_0(R)$, where $x, y \in R$.
Then $\Phi(x \wedge y) = (x \wedge y]^* = (x]^* \underline{\vee} (y]^* = \Phi(x) \underline{\vee} \Phi(y)$.
Again $\Phi(x \vee y) = (x \vee y]^* = (x]^* \cap (y]^* = \Phi(x) \wedge \Phi(y)$.
Hence Φ is a dual isomorphism. □

In an ADL R with 0, we know that a maximal element is always a dense element. Now we prove the converse in disjunctive ADL.

Theorem 3.11. *If R is a disjunctive ADL, then every dense element of R is a maximal element.*

Proof: Assume that R is disjunctive. Let m be a dense element of R. That is $(m]^* = (0]$. For any $x \in R, (m \vee x]^* = (m]^* \cap (x]^* = (0] \cap (x]^* = (0] = (m]^*$. Since R is disjunctive, we get that $m \vee$

$x = m$. Therefore m is a maximal element of R. □

We now characterize a $\star$-ADl in terms of it's lattice of annulets in the following theorem.

Theorem 3.12. *Let R be an ADL with 0. Then R is a $\star$-ADL iff $\mathscr{A}_0(R)$ is a Boolean subalgebra of $\mathscr{A}(R)$.*

Proof: Assume that R is a $\star$-ADL.

Then R has a dense element, say d. Then $(d]^* = (0]$ is the least element and$(0]^*$ is the greatest element of the sublattice $\mathscr{A}_0(R)$ of $\mathscr{A}(R)$. Let $x \in R$. Since R is a $\star$-ADL, there exists $x' \in R$ such that $(x]^{**} = (x']^*$.

We now show that $(x']^*$ is the complement of $(x]^*$ in $\mathscr{A}_0(R)$, for each $x \in R$.

Now $(x]^* \cap (x']^* = (x]^* \cap (x]^{**} = (0]$ and $(x]^* \underline{\vee} (x']^* = \left[(x]^{**} \cap (x']^{**}\right]^* = [(x]^{**} \cap (x]^{***}]^* = [(x]^{**} \cap (x]^*]^* = (0]^*$. Thus $\mathscr{A}_0(R)$ is a Boolean subalgebra of $\mathscr{A}(R)$. Conversely assume that $\mathscr{A}_0(R)$ is a Boolean subalgebra of $\mathscr{A}(R)$.

Let $x \in R$. Then $(x]^* \in \mathscr{A}_0(R)$. Since $\mathscr{A}_0(R)$ is a subalgebra of $\mathscr{A}(R)$, there exists $(y]^* \in \mathscr{A}_0(R)$, with $y \in R$ such that $(x]^* \cap (y]^* = (0]$ and $(x]^* \underline{\vee} (y]^* = (0]^*$.

Now $(x]^* \underline{\vee} (y]^* = (0]^* \Rightarrow (x \wedge y]^* = (0]^* = R \Rightarrow x \wedge y = 0$. Again, $(x]^* \cap (y]^* = (0] \Rightarrow (x \vee y]^* = (0] \Rightarrow x \vee y$ is a dense element. Thus we proved that for each $x \in R$, there exists $y \in R$ such that $x \wedge y = 0$ and $x \vee y$ is a dense element. Therefore R is a $\star$-ADL. □

Definition 3.13. *An ADL R with 0 is called sectionally $\star$-ADL iff for any $x(\neq 0) \in R$, the interval $[0, x]$ is a $\star$-ADL.*

Before proving the next theorem, we need the following lemma.

Lemma 3.14. *Let I, J be two ideals in an ADL R. If $I \cap J$ and $I \vee J$ (i.e. The infimum and the supremum of I, J in the distributive lattice $\mathscr{I}(R)$) are both principal ideals, then I, J are also principal ideals.*

Proof: Suppose $I \vee J = (a]$ and $I \cap J = (b]$, for some $a, b \in R$.

Now $a \in I \vee J \Rightarrow a = c \vee d$ for some $c \in I$ and $d \in J$. Then $c \vee (b \wedge d) \in I$. So that

$(c \vee (b \wedge d)] \subseteq I$. We now prove that $I = (c \vee (b \wedge d)]$.

Let $x \in I$. Then $x \in I \vee J = (a]$.

So $x = a \wedge x = (c \vee d) \wedge x = (c \wedge x) \vee (d \wedge x) \longrightarrow (1)$.

Now $x \in I$ and $d \in J \Rightarrow x \wedge d \in I \cap J = (b] \Rightarrow d \wedge x \in (b]$.

Hence $d \wedge x = b \wedge d \wedge x \longrightarrow (2)$.

From (1) and (2), we can obtain $x = (c \wedge x) \vee (b \wedge d \wedge x) = [c \vee (b \wedge d)] \wedge x$. Hence $x \in (c \vee (b \wedge d)]$. Therefore $I \subseteq (c \vee (b \wedge d)]$.

By symmetry, we get that J is also a principal ideal. □

Theorem 3.15. *Let R be a generalized stone ADL. Then $\mathscr{A}_0(R)$ is a relatively complemented sublattice of the lattice $\mathscr{I}(R)$ of all ideals of R.*

Proof: Let R be a generalized stone ADL. By theorem 3.7, $\mathscr{A}_0(R)$ is a sublattice of $\mathscr{I}(R)$. So we can treate $\underline{\vee}$ as $\vee$. Since $\mathscr{A}_0(R)$ is a distributive lattice with the greatest element $(0]^* = R$, it is enough to prove that each interval of the form $[I,R]$, where $I \in \mathscr{A}_0(R)$, is complemented. Let $J = [(x]^*,R]$ be an interval in $\mathscr{A}_0(R)$ and $(y]^* \in J$. We have clearly $(y]^* \cap (y]^{**} = (0]$.

Since R is generalized stone ADL, we have $(y]^* \vee (y]^{**} = R$ for all $y \in R$.

Now $\{(x] \cap (y]^*\} \vee \{(x] \cap (y]^{**}\} = (x] \cap \{(y]^* \vee (y]^{**}\} = (x] \cap R = (x]$.

Also $\{(x] \cap (y]^*\} \cap \{(x] \cap (y]^{**}\} = (x] \cap \{(y]^* \cap (y]^{**}\} = (x] \cap (0] = (0]$.

Thus we have that the infimum and the supremum of the ideals $(x] \cap (y]^*$ and $(x] \cap (y]^{**}$ are the principal ideals $(0]$ and $(x]$.

Therefore, by the above lemma, $(x] \cap (y]^*$ and $(x] \cap (y]^{**}$ must be the principal ideals. Suppose $(x] \cap (y]^* = (a]$ and $(x] \cap (y]^{**} = (b]$ for some $a, b \in R$.

Now $a \in (x] \cap (y]^* \Rightarrow (a] \subseteq (x] \Rightarrow (x]^* \subseteq (a]^*$.Therefore $(a]^* \in J$.

Also $(a] = (x] \cap (y]^* \subseteq (y]^* \Rightarrow (y]^{**} \subseteq (a]^*$. Hence $(y]^* \vee (y]^{**} \subseteq (y]^* \vee (a]^* \Rightarrow R \subseteq (a]^* \vee (y]^*$. Thus $R = (a]^* \vee (y]^* \longrightarrow (1)$

Again $(a]^* \cap (y]^* \cap (x] = (a]^* \cap (a] = (0]$. Hence $(a]^* \cap (y]^* \subseteq (x]^*$.

But $(x]^* \subseteq (y]^*$ and $(x]^* \subseteq (a]^*$ imply that $(x]^* \subseteq (a]^* \cap (y]^*$.

Hence $(a]^* \cap (y]^* = (x]^* \longrightarrow (2)$

From (1) and (2), $(a]^*$ is the required complement of $(y]^*$ in J.

Hence $\mathscr{A}_0(R)$ is a relatively complemented sublattice of $\mathscr{I}(R)$. □

Definition 3.16. *Let $I=[0,x],0<x$, be an interval in an ADL R with 0. For $a\in I$, define the annihilator $(a]^+$ of a with respect to I as follows:*

$$(a]^+ = \{\ y \in I \mid y\wedge a = 0\ \}.$$

Observe that $(a]^* \cap I = (a]^+$.

Lemma 3.17. *For $a\in I$, the annihilator $(a]^+$ is an ideal in I.*

Proof: Since $0\in I$ and $0\wedge a=0$, we get that $0\in(a]^+$. Let $r,s\in(a]^+$. Then $r,s\in I$ and $r\wedge a=s\wedge a=0$.

Since $r,s\in I$, we get $r\vee s\in I$, and $(r\vee s)\wedge a=(r\wedge a)\vee(s\wedge a)=0\vee 0=0$.

Hence $r\vee s\in(a]^+$. Let $y\in(a]^+$ and $t\in I$. Then $y\in I$ and $y\wedge a=0$. Hence $y\wedge t\in I$. Now $(y\wedge t)\wedge a=t\wedge y\wedge a=t\wedge 0=0$, which implies that $y\wedge t\in(a]^+$. Thus $(a]^+$ is an ideal of I.

□

Lemma 3.18. *Let $I=[0,x],0<x$, be an interval in an ADL R with 0. Then we have the following:*

(i). For $a,b\in I,(a]^+\subseteq(b]^+$ implies $(a]^*\subseteq(b]^*$.

(ii). If $z\in R$, then $(z]^*\cap I=(z\wedge x]^+$.

Proof: (i). Let $a,b\in I$ and suppose $(a]^+\subseteq(b]^+$. Let $t\in(a]^*$. Then $t\wedge a=0$ and $t\in R\Rightarrow t\wedge x\wedge a=0$ and $t\wedge x\in I$, since $x\in I$. Which implies $t\wedge x\in(a]^+\subseteq(b]^+\Rightarrow t\wedge x\wedge b=0\Rightarrow t\wedge b=0$, since $t\in I=[0,x]$. Hence $t\in(b]^*$.

(ii). Let $t\in(z]^*\cap I$. Then $t\in(z]^*$ and $t\in I$. Hence $t\wedge z=0$ and $t\in I$. Thus $t\wedge z\wedge x=0$ and $t\in I\Rightarrow t\wedge(z\wedge x)=0$ and $t\in I\Rightarrow t\in(z\wedge x]^+$.

Therefore $(z]^*\cap I\subseteq(z\wedge x]^+$. Again, let $t\in(z\wedge x]^+$, then $t\wedge z\wedge x=0$ and $t\in I\Rightarrow z\wedge t\wedge x=0$ and $t\in I\Rightarrow z\wedge t=0$ and $t\in I\Rightarrow t\in(z]^*$ and $t\in I$. Hence $t\in(z]^*\cap I$. Thus $(z\wedge x]^+\subseteq(z]^*\cap I$. Therefore $(z]^*\cap I=(z\wedge x]^+$. □

We now prove the characterization theorem of a sectionally $\star$-ADL in terms of it's annulets. Before proving it, we can observe that if R is an ADL with 0 and $I=[0,x],0<x$ for some $x\in R$, then $\mathscr{A}_0(I)$ is a bounded distributive lattice (with respect to the operations given in the theorem 3.3) with the greatest element $I=(0]^+$ and the least element $(x]^+$.

Theorem 3.19. *Let R be an ADL with 0. Then $\mathscr{A}_0(R)$ is relatively complemented if and only if R is sectionally $\star$-ADL.*

Proof: Assume that $\mathscr{A}_0(R)$ is relatively complemented.

We have to prove that each interval $I = [0, x]$ in R is a $\star - ADL$. By theorem 3.12, it is enough to prove that $\mathscr{A}_0(I)$ is relatively complemented.

Since $\mathscr{A}_0(I)$ is a distributive lattice with the greatest element $I = (0]^+$, it is enough to prove that each interval $[J, I], J \in \mathscr{A}_0(I)$ is complemented.

Choose $a, b \in I$ such that $(b]^+ \in \left[(a]^+, I\right] \subseteq \mathscr{A}_0(I)$. Then $(a]^+ \subseteq (b]^+ \subseteq I$.

By lemma 3.18(i), $(a]^* \subseteq (b]^* \subseteq R$.

Since $\mathscr{A}_0(R)$ is relatively complemented and $(b]^* \in [(a]^*, R]$, there exists an element $c \in R$ such that $(c]^* \in [(a]^*, R]$ and $(b]^* \cap (c]^* = (a]^*$ and $(b]^* \underline{\vee} (c]^* = R$.

Now $(b]^* \cap (c]^* = (a]^* \Rightarrow (b]^* \cap (c]^* \cap I = (a]^* \cap I \Rightarrow [(b]^* \cap I] \cap [(c]^* \cap I] = (a]^* \cap I \Rightarrow (b]^+ \cap (c]^+ = (a]^+ \quad \longrightarrow (1)$

Secondly, $(b]^* \underline{\vee} (c]^* = R \Rightarrow \left[(b]^* \underline{\vee} (c]^*\right] \cap I = R \cap I \Rightarrow [(b]^* \cap I] \,\underline{\vee}\, [(c]^* \cap I] = I \Rightarrow (b]^+ \underline{\vee} (c]^+ = I \quad \longrightarrow (2)$

From (1) and (2), we get that $(c]^+$ is the complement of $(b]^+$ in $\left[(a]^+, I\right]$.

Hence $\left[(a]^+, I\right]$ is relatively complemented.

Conversely assume that R is sectionally $\star$-ADL.

Since $\mathscr{A}_0(R)$ is a distributive lattice with the greatest element R, it is enough to prove that each interval $[(a]^*, R], (a]^* \in \mathscr{A}_0(R)$ is complemented.

Let $(b]^* \in [(a]^*, R]$. Therefore $(a]^* \subseteq (b]^* \subseteq R$.

Consider the interval $I = [0, b \vee a]$. Then by the hypothesis, I is a $\star - ADL$.

So by theorem 3.12, $\mathscr{A}_0(I)$ is complemented.

Hence each interval $\left[(a]^+, I\right], (a]^+ \in \mathscr{A}_0(I)$, where $a \in I$ is complemented.

We have by the lemma 3.18(ii), $(a]^* \cap I = (a \wedge (b \vee a)]^+$ and

$(b]^* \cap I = (b \wedge (b \vee a)]^+ = (b]^+ \subseteq I$, that is $(b]^+ \in \left[(a \wedge (b \vee a)]^+, I\right]$.

Since $\mathscr{A}_0(I)$ is complemented, there exists an element $c \in I$ such that

$(b]^+ \cap (c]^+ = (a \wedge (b \vee a)]^+$ and $(b]^+ \underline{\vee} (c]^+ = I \qquad \longrightarrow (3)$

Now our claim is $(b]^* \cap (c]^* = (a]^*$ and $(b]^* \underline{\vee} (c]^* = R$.

Let $x \in (b]^* \cap (c]^*$. Then $x \in (b]^*$ and $x \in (c]^*$, implies $b \wedge x = 0$ and $c \wedge x = 0$

$\Rightarrow\ x \wedge (b \vee a) \wedge b = 0$ and $x \wedge (b \vee a) \wedge c = 0$.

$\Rightarrow\ x \wedge (b \vee a) \in (b]^+$ and $x \wedge (b \vee a) \in (c]^+$, since $x \wedge (b \vee a) \in I$.

$\Rightarrow\ x \wedge (b \vee a) \in (b]^+ \cap (c]^+$

$\Rightarrow\ x \wedge (b \vee a) \in (a \wedge (b \vee a)]^+$, by (3)

$\Rightarrow\ x \wedge (b \vee a) \wedge a \wedge (b \vee a) = 0$

$\Rightarrow\ x \wedge a \wedge (b \vee a) \wedge (b \vee a) = 0$

$\Rightarrow\ (x \wedge a) \wedge (b \vee a) = 0$

$\Rightarrow\ (b \vee a) \wedge (x \wedge a) = 0$

$\Rightarrow\ x \wedge (b \vee a) \wedge a = 0$

$\Rightarrow\ x \wedge a = 0$

$\Rightarrow\ x \in (a]^*$

Hence $(b]^* \cap (c]^* \subseteq (a]^* \quad \longrightarrow (4)$

Conversely, let $x \in (a]^*$. Then $x \wedge a = 0$

$\Rightarrow\ x \wedge a \wedge (b \vee a) \wedge (b \vee a) = 0$

$\Rightarrow\ x \wedge (b \vee a) \wedge a \wedge (b \vee a) = 0$

$\Rightarrow\ x \wedge (b \vee a) \in (a \wedge (b \vee a)]^+$, since $x \wedge (b \vee a) \in I$.

$\Rightarrow\ x \wedge (b \vee a) \in (b]^+ \cap (c]^+$, by (3)

$\Rightarrow\ x \wedge (b \vee a) \in (b]^+$ and $x \wedge (b \vee a) \in (c]^+$

$\Rightarrow\ x \wedge (b \vee a) \wedge b = 0$ and $x \wedge (b \vee a) \wedge c = 0$.

$\Rightarrow\ x \wedge b = 0$ and $x \wedge c = 0$, since $c \in I = [0, b \vee a]$.

$\Rightarrow\ x \in (b]^*$ and $x \in (c]^*$

$\Rightarrow\ x \in (b]^* \cap (c]^*$

Hence $(a]^* \subseteq (b]^* \cap (c]^*$. $\longrightarrow (5)$

From (4) and (5), we can obtain $(b]^* \cap (c]^* = (a]^*$.

Again from (3), we have $(b]^+ \underline{\vee} (c]^+ = I$

$\Rightarrow\ (b \wedge c]^+ = (b]^+ \underline{\vee} (c]^+ = I$

$\Rightarrow\ (b \wedge c]^+ = I$

$\Rightarrow\ b \wedge c = 0$

$$\Rightarrow\ (b\wedge c]^{*} = (0]^{*} = R$$

$$\Rightarrow\ (b]^{*}\underline{\vee}(c]^{*} = R$$

Hence $(c]^{*}$ is the complement of $(b]^{*}$ in $[(a]^{*}, R]$.

Thus $\mathscr{A}_0(R)$ is relatively complemented. □

ACKNOWLEDGEMENTS. The authors would like to thank the referee for his comments and valuable suggestions.

References

[1] Birkhoff. G. : *Lattice Theory*, Amer.Math.Soc.Colloq. XXV, Providence, (1967), U.S.A.

[2] Burris. S., Sankappanavar, H.P. : *A Cource in Univerasal Algebra*, Springer Verlag, (1981).

[3] Cornish.W.H. : *Normal Lattices*, J.Austral.Math.Soc., **14** (1972), 200-215.

[4] Cornish.W.H.: *Annulets and α- ideals in Distributive Lattices*, J.Austral.Math.Soc., **15** (1973), 70-77.

[5] Mandelker. M : *Relative annihilators in lattices*, Duke Math. J, **37** (1970), 377-386.

[6] Rao.G.C. : *Almost Distributive Lattices*, Doctoral Thesis, Andhra University, Waltair, (1980).

[7] Rao.G.C.and Ravikumar.S : *Normal Almost Distributive Lattices*, Southeast Asian Bulletin of mathematics,(to appear).

[8] Swamy.U.M., Rao.G.C. : *Almost Distributive Lattices*, J.Austral.Math.Soc. (Series A), **31** (1981), 77-91.

[9] Swamy.U.M., Rao.G.C., Nanaji Rao.G. : *Pseudo-Complementation on Almost Distributive Lattices*, Southeast Asian Bulletin of mathematics, **24** (2000), 95-104.

[10] Swamy.U.M., Rao.G.C., Nanaji Rao.G. : *Stone Almost Distributive Lattices*, Southeast Asian Bulletin of mathematics, **27** (2003), 513-526.

[11] Swamy.U.M., Rao.G.C., Nanaji Rao.G. : *Dense Elements in Almost Distributive Lattices*, Southeast Asian Bulletin of mathematics, **27** (2004), 1081-1088.