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The Exact Order of Approximation for Bivariate Complex Bernstein-Schurer Polynomials

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Abstract. In this paper we study the approximation properties of the tensor product kind bivariate complex Bernstein-Schurer polynomials. We obtain the order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for bivariate complex Bernstein-Schurer polynomials attached to analytic functions on compact polydisks.

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1. Introduction

The Bernstein polynomial attached to $f : [0,1] \to \mathbb{R}$ was introduced by Bernstein to give a proof of the Weierstrass Theorem. If f is continuous on $[0,1]$ then $\lim_n B_n(f)(x) = f(x)$ where $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$, $x \in [0,1]$.

If $f : G \to \mathbb{C}$ is an analytic function in the open set $G \subset \mathbb{C}$, with $\overline{D_1} \subset G$ (where $D_1 = \{z \in \mathbb{C} : |z| < 1\}$), then S. N. Bernstein [6] proved that the complex Bernstein polynomials defined by

$$B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right)$$

uniformly converge to f in $\overline{D_1}$. But Bernstein obtained this convergence result without any quantitative estimate. Recently, Voronovskaja- type results with quantitative estimates for the complex Bernstein, complex q-Bernstein, complex Bernstein- Kantorovich, complex Kantorovich - Stancu polynomials attached to analytic functions on compact disks and the exact order of simultaneous approximation by these complex operators were obtained by S. G. Gal [5].

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The complex Bernstein-Schurer polynomials (introduced and studied in the case of real variable in [7]) are defined for any fixed $p = 0, 1, 2, \ldots$ by

$$B_{n,p}(f)(z) = \sum_{k=0}^{n+p} \binom{n+p}{k} z^k (1-z)^{n+p-k} f\,(k/n)\,, z \in \mathbb{C}.$$

The approximation properties of these polynomials are investigated and the exact order of approximation with quantitative estimates were gave in [1]. It is clear that for $p = 0$ these polynomials become the classical complex Bernstein polynomials studied in [5]. In real case the approximation properties of bivariate Berstein-Schurer polynomials is studied by D. Barbasu [2–4].

In this note we would like to extend the approximation results from the univariate case, obtained for the complex Bernstein-Schurer polynomials, to the bivariate case.

First we present a few concepts in the bivariate case which are natural extensions of the usual concepts in the univariate case. Let $D_{R_j} := \left\{z_j \in \mathbb{C} : \left|z_j\right| < R_j, j = 1, 2\right\}$ and $P(0;R) = D_{R_1} \times D_{R_2}$ denotes an open polydisk (of center 0 and radius R) where $R = (R_1, R_2)$ and $\left|z_1\right| \leq r_1$, $\left|z_2\right| \leq r_2, r_1 < R_1$ with $r_2 < R_2$. Let also

$$\overline{P_R} := \overline{P\,(0,R)} = \left\{\left(z_1, z_2\right) \in \mathbb{C}^2 : \left|z_j\right| \leq R_j,\ j = 1, 2\right\}$$

denotes the closed polydisk. For $f\,(z_1, z_2)$ is an analytic function of two complex variables (z_1, z_2) in the polydisk $P(0;R)$ we can define the tensor product kind Bernstein-Schurer polynomials as follows

$$B_{n,m,p,q}(f)\left(z_1, z_2\right) = \sum_{k=0}^{n+p} \sum_{j=0}^{m+q} b_{n,k}\left(z_1\right) b_{m,j}\left(z_2\right) f\left(k/n, j/m\right) \tag{1}$$

where $b_{n,k}\left(z_1\right) = \binom{n+p}{k} z_1^k (1-z_1)^{n+p-k}$, $b_{m,j}\left(z_2\right) = \binom{m+q}{j} z_2^j (1-z_2)^{m+q-j}$, $n, m \in \mathbb{N}$ and $p, q \in \mathbb{N} \cup \{0\}$.

The goal of this paper is to obtain the exact order of approximation for the polynomials given by (1) on compact polydisks. First we give the order of approximation and the Voronovskaja- type theorems with quantitative estimate for the polynomials $B_{n,m}\left(f\right)(z)$ defined by (1). These results allow us to obtain the exact order in approximation by the polynomials $B_{n,m}\left(f\right)(z)$.

2. The Convergence Results with Quantitative Estimates

Theorem 1. *For fixed $p, q \in \mathbb{N} \cup \{0\}$ and $R_1 > p+1$, $R_2 > q+1$ suppose that $f : P(0;R) \to \mathbb{C}$ is analytic in $P(0;R) = D_{R_1} \times D_{R_2}$, that is $f\,(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $(z_1, z_2) \in P(0;R)$, $R = (R_1; R_2)$. Then we have*

(i) For all $\left|z_1\right| \leq r_1$, $\left|z_2\right| \leq r_2$ with $1 < r_1$, $(p+1)r_1 < R_1$, $1 < r_2$, $(q+1)r_2 < R_2$ and $n, m \in \mathbb{N}$

$$\left|B_{n,m,p,q}(f)\left(z_1, z_2\right) - f\,(z_1; z_2)\right| \leq M^{p,q}_{r_1, r_{2,n,m}}(f)$$

where

$$M^{p,q}_{r_1,r_{2,n,m}}(f)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right| r_2^j\left[\frac{3k(k-1)}{(n+p)}\left[(p+1)r_1\right]^k+\frac{1}{n}\left[(p+1)r_1\right]^k-\frac{r_1^k}{n}\right]$$
$$+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right| r_1^k\left[\frac{3j(j-1)}{(m+q)}\left[(q+1)r_2\right]^j+\frac{1}{m}\left[(q+1)r_2\right]^j-\frac{r_2^j}{m}\right]$$

and $M^{p,q}_{r_1,r_{2,n,m}}(f)<\infty$.

(ii) Let $k_1,k_2\in\mathbb{N}$ *be with* $k_1+k_2\geq 1$, $1\leq r_1<r_1^*\leq(1+p)r_1<R_1$, $1\leq r_2<r_2^*\leq(1+q)r_2<R_2$. *Then for all* $\left|z_1\right|\leq r_1$, $\left|z_2\right|\leq r_2$ *and* $n,m\in\mathbb{N}$, $p,q\in\mathbb{N}\cup\{0\}$ *we have*

$$\left|\frac{\partial^{k_1+k_2}B_{n,m,p,q}(f)}{\partial^{k_1}z_1\partial^{k_2}z_2}\left(z_1,z_2\right)-\frac{\partial^{k_1+k_2}f}{\partial^{k_1}z_1\partial^{k_2}z_2}\left(z_1,z_2\right)\right|\leq M^{p,q}_{r_1^*,r^*_{2,n,m}}(f).\frac{k_1!}{\left(r_1^*-r_1\right)^{k_1+1}}\frac{k_2!}{\left(r_2^*-r_2\right)^{k_2+1}}$$

where $M^{p,q}_{r_1^*,r^*_{2,n,m}}(f)$ *is given as at the above point (i).*

Proof. (i) Denote $e_{k,j}(z_1,z_2)=e_k(z_1)e_j(z_2)$ where $e_k(u)=u^k$. Since $f(z_1,z_2)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_{k,j}e_{k,j}(z_1,z_2)$ and by the definition of the operator (1) we get

$$\left|B_{n,m,p,q}(f)\left(z_1,z_2\right)-f\left(z_1,z_2\right)\right|\leq\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{k,j}\right|\left|B_{n,m,p,q}(e_{k,j})\left(z_1,z_2\right)-e_{k,j}(z_1,z_2)\right|.$$

By the simple calculation we can write

$$\begin{aligned}\left|B_{n,m,p,q}(e_{k,j})\left(z_1,z_2\right)-e_{k,j}(z_1,z_2)\right| &=\left|B_{n,p}(e_k)(z_1).B_{m,q}(e_j)(z_2)-z_1^kz_2^j\right|\\ &\leq\left|z_2^j\right|\left|B_{n,p}(e_k)(z_1)-z_1^k\right|+\left|B_{n,p}(e_k)(z_1)\right|\left|B_{m,q}(e_j)(z_2)-z_2^j\right|\\ &\leq r_2^jA+\left|B_{n,p}(e_k)(z_1)\right|B, \qquad (2)\end{aligned}$$

say. By the Stirling numbers of second kind $S\left(k,j\right)$, we can write

$$B_{n,p}(e_k)(z_1)=\sum_{j=1}^{k}S\left(k,j\right)\frac{\left(n+p\right)\ldots\left(n+p-\left(j-1\right)\right)}{\left(n+p\right)^j}z^j$$

(for complex Bernstein polynomials in one variable see [5, page 27]). Since $S\left(k,j\right)\geqslant 0$, for all $n,k\in\mathbb{N}$, $p\in\mathbb{N}\cup\{0\}$ and $\sum_{j=1}^{k}S\left(k,j\right)\left(n+p\right)\ldots\left(n+p-\left(j-1\right)\right)=\left(n+p\right)^k$ it follows

$$\left|B_{n,p}(e_k)(z_1)\right|\leqslant\sum_{j=1}^{k}S\left(k,j\right)\frac{\left(n+p\right)\ldots\left(n+p-\left(j-1\right)\right)}{\left(n+p\right)^k}r^j\leqslant r^k$$

for all $\left|z_1\right| \leq r_1$, with $1 < r_1$, $(p+1)r_1 < R_1$. To estimate A and B for fixed $n, m \in \mathbb{N}$, we should consider two possible cases:

(1) $0 \leq k \leq n+p$, $0 \leq j \leq m+q$ and (2) $k > n+p$, $j > m+q$.

We start with case (1). If $k=0$, $j=0$ then obviously we get $\left|B_{n,p}(e_k)(z_1)-(e_k)(z_1)\right| = 0$ and $\left|B_{m,q}(e_j)(z_2)-(e_j)(z_2)\right| = 0$. Therefore, let us suppose that $1 \leq k \leq n+p$, $1 \leq j \leq m+q$. Denoting by Δ^k the finite difference of order k, as in the case of the classical Bernstein polynomials we easily can write the representation formulas

$$B_{n,p}(f)(z_1) = \sum_{v=0}^{n+p} \Delta^v_{1/n} f(0) e_v(z_1),\ B_{m,q}(f)(z_2) = \sum_{w=0}^{m+q} \Delta^w_{1/m} f(0) e_w(z_2).$$

For simplicity, we use the following notations

$$\begin{aligned} C^p_{n,v,k} &= \binom{n+p}{v} \Delta^v_{1/n} e_k(0) = \binom{n+p}{v} \left[0, \frac{1}{n} \ldots, \frac{j}{n}; e_k\right] v!/n^v, \\ C^q_{n,w,j} &= \binom{m+q}{w} \Delta^w_{1/m} e_j(0) = \binom{m+q}{w} \left[0, \frac{1}{m}, \ldots, \frac{j}{m}; e_j\right] w!/m^w. \end{aligned}$$

Since e_k, e_j are convex of any order, it follows that all $C^p_{n,v,k} \geq 0$, $C^q_{n,w,j} \geq 0$ and taking into account that $B_{n,p}(f)(1) = f\left((n+p)/p\right)$, $B_{m,q}(f)(1) = f\left(\left(m+q\right)/m\right)$ we get

$$\sum_{v=0}^{n+p} C^p_{n,v,k} = B_{n,p}(e_1)(1) = \left(\frac{n+p}{n}\right)^k, \quad \sum_{w=0}^{m+q} C^q_{n,w,j} = B_{n,p}(e_2)(1) = \left(\frac{m+q}{m}\right)^j. \tag{3}$$

Using the result in the proof of Theorem 2.1 in [1] for any $\left|z_1\right| \leq r_1$, $\left|z_2\right| \leq r_2$ with $1 \leq r_1 \leq (p+1)r_1 < R_1$, $1 \leq r_2 \leq (q+1)r_2 < R_2$, directly we can write

$$A = \left|B_{m,q}(e_j)(z_2) - z_2^j\right| \leq \frac{j\left(j-1\right)}{\left(m+q\right)} \left[\left(q+1\right) r_2\right]^j + \frac{1}{m} \left[\left(q+1\right) r_2\right]^j - \frac{r_2^j}{m}$$

and

$$B = \left|B_{n,p}(e_k)(z_1) - e_k(z_1)\right| \leq \frac{k\left(k-1\right)}{\left(n+p\right)} \left[\left(p+1\right) r_1\right]^k + \frac{1}{n} \left[\left(p+1\right) r_1\right]^k - \frac{r_1^k}{n}.$$

We now consider second case. For $k > n+p$, $\left|z_1\right| \leq r_1$ with $1 \leq r_1 \leq (p+1)r_1 < R_1$ and for $j > m+q$, $\left|z_2\right| \leq r_2$ with $1 \leq r_2 \leq (q+1)r_2 < R_2$, and considering (3) we get

$$\begin{aligned} B &= \left|B_{n,p}(e_k)(z_1) - e_k(z_1)\right| \leq \left|B_{n,p}(e_k)(z_1)\right| + r_1^k \\ &\leq \left(\frac{n+p}{n}\right)^k r_1^{n+p} + r_1^k \\ &\leq 2\left[\left(p+1\right) r_1\right]^k \leq \frac{2\left(k-1\right)}{n+p} \left[\left(p+1\right) r_1\right]^k, \end{aligned}$$

and in similar way, $A = \left|B_{m,q}(e_j)(z_2) - (e_j)(z_2)\right| \leq \frac{2\left(j-1\right)}{m+q} \left[\left(q+1\right) r_2\right]^j$.

Combining all of the results obtained for A and B, we have the desired inequality.

(ii) Now we give the rate of convergence in simultaneous approximation. Let $1 \leq r_1 < r_1^* < \frac{R_1}{2}$, $1 \leq r_2 < r_2^* < \frac{R_2}{2}$ and $\gamma_1 = \left|u_1 - z_1\right| = r_1^*$, $\gamma_2 = \left|u_2 - z_2\right| = r_2^*$. By the Cauchy's formula

$$\frac{\partial^{k_1+k_2} B_{n,m,p,q}(f)}{\partial^{k_1} z_1 \partial^{k_2} z_2}\left(z_1, z_2\right) - \frac{\partial^{k_1+k_2} f}{\partial^{k_1} z_1 \partial^{k_2} z_2}\left(z_1, z_2\right) = \frac{k_1! k_2!}{(2\pi i)^2} \int\limits_{\gamma_2} \int\limits_{\gamma_1} \frac{\left[B_{n,m,p,q}(u_1, u_2) - f(u_1, u_2)\right] du_1 du_2}{\left(u_1 - z_1\right)^{k_1+1}\left(u_2 - z_2\right)^{k_2+1}}$$

passing to absolute value with $\left|z_1\right| \leq r_1$, $\left|z_2\right| \leq r_2$ and taking into account that $\left|u_1 - z_1\right| \geq r_1^* - r_1$, $\left|u_2 - z_2\right| \geq r_2^* - r_2$, by applying the estimate in (i) we easily obtain

$$\left|\frac{\partial^{k_1+k_2} B_{n,m,p,q}(f)}{\partial^{k_1} z_1 \partial^{k_2} z_2}\left(z_1, z_2\right) - \frac{\partial^{k_1+k_2} f}{\partial^{k_1} z_1 \partial^{k_2} z_2}\left(z_1, z_2\right)\right| \leq M^{p,q}_{r_{1,}^* r_{2,n,m}^*}(f). \frac{k_1!}{\left(r_1^* - r_1\right)^{k_1+1}} \frac{k_2!}{\left(r_2^* - r_2\right)^{k_2+1}}$$

which proves the theorem. □

In what follows a Voronovskaja's result for $B_{n,m,p,q}(f)$ is presented. It will be the product of the parametric extensions generated by the Voronovskaja's formula in univariate case in Theorem 2.2 in [1]. Indeed, for $f(z_1, z_2)$ defining the parametric extensions of the Voronovskaja's formula by

$$\begin{aligned}
{}_{z_1}L_n(f)(z_1, z_2) &:= B_{n,p}\left(f(., z_2)\right)(z_1) - f(z_1, z_2) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}\left(z_1, z_2\right) - \frac{z_1(1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2), \\
{}_{z_2}L_m(f)(z_1, z_2) &:= B_{m,q}\left(f(z_1, .)\right)(z_2) - f(z_1, z_2) - \frac{q}{m} z_2 \frac{\partial f}{\partial z_2}\left(z_1, z_2\right) - \frac{z_2(1 - z_2)}{2m} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2),
\end{aligned}$$

their product (composition) gives

$$\begin{aligned}
&{}_{z_2}L_m(f)(z_1, z_2) o_{z_1} L_n(f)(z_1, z_2) \\
&= B_{m,q}\left[B_{n,p}\left(f(., .)\right)(z_1) - f(z_1, .) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}\left(z_1, .\right) - \frac{z_1(1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, .)\right]\left(z_2\right) \\
&\quad - \left[B_{n,p}\left(f(., z_2)\right)(z_1) - f(z_1, z_2) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}\left(z_1, z_2\right) - \frac{z_1(1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2)\right] \\
&\quad - \frac{q}{m} z_2 \left[B_{n,p}\left(\frac{\partial f}{\partial z_2}(., z_2)\right)(z_1) - \frac{\partial f}{\partial z_2}(z_1, z_2)\right. \\
&\quad \left. - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}\left[\frac{\partial f}{\partial z_2}\right]\left(z_1, z_2\right) - \frac{z_1(1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}\left[\frac{\partial f}{\partial z_2}\right]\left(z_1, z_2\right)\right] \\
&\quad - \frac{z_2(1 - z_2)}{2m}. \left[B_{n,p}\left(\frac{\partial^2 f}{\partial z_2^2}(., z_2)\right)(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2) - \right. \\
&\quad \left. - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}\left[\frac{\partial^2 f}{\partial z_2^2}\right]\left(z_1, z_2\right) - \frac{z_1(1 - z_1)}{2n} \frac{\partial^2}{\partial z_1^2}\left[\frac{\partial^2 f}{\partial z_2^2}\right]\left(z_1, z_2\right)\right]
\end{aligned}$$

$$:= E_1 - E_2 - E_3 - E_4.$$

After simple calculation, we can write

$$\begin{aligned}
{}_{z_2}L_m(f)(z_1,z_2)o_{z_1}L_n(f)(z_1,z_2) &= B_{n,m,p,q}\left(f\right)\left(z_1,z_2\right) - B_{m,q}\left(f\left(z_1,.\right)\right)\left(z_2\right) \\
&- \frac{p}{n}z_1.B_{m,q}\left(\frac{\partial f}{\partial z_1}\left(z_1,.\right)\right)\left(z_2\right) - \frac{z_1(1-z_1)}{2n}.B_{m,q}\left(\frac{\partial^2 f}{\partial z_1^2}(z_1,.)\right)(z_2) \\
&- B_{n,p}\left(f\left(.,z_2\right)\right)\left(z_1\right) - f\left(z_1,z_2\right) - \frac{p}{n}z_1\frac{\partial f}{\partial z_1}\left(z_1,z_2\right) - \frac{z_1(1-z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) \\
&- \frac{q}{m}z_2\, B_{n,p}\left(\frac{\partial f}{\partial z_2}(.,z_2)\right)(z_1) + \frac{q}{m}z_2\frac{\partial f}{\partial z_2}(z_1,z_2) + \frac{p}{n}z_1\frac{q}{m}z_2\frac{\partial^2 f}{\partial z_1 \partial z_2}\left(z_1,z_2\right) \\
&+ \frac{q}{m}z_2\frac{z_1(1-z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}\left[\frac{\partial f}{\partial z_2}\right]\left(z_1,z_2\right) \\
&- \frac{z_2(1-z_2)}{2m}.B_{n,p}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1) + \frac{z_2(1-z_2)}{2m}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) \\
&+ \frac{z_2(1-z_2)}{2m}\frac{p}{n}z_1\frac{\partial f}{\partial z_1}\left[\frac{\partial^2 f}{\partial z_2^2}\right]\left(z_1,z_2\right) + \frac{z_1(1-z_1)}{2n}\frac{z_2(1-z_2)}{2m}\frac{\partial^4 f}{\partial z_1^2 \partial z_2^2}(z_1,z_2)
\end{aligned}$$

from which immediately can be derived the commutativity property

$${}_{z_2}L_m(f)(z_1,z_2)o_{z_1}L_n(f)(z_1,z_2) = {}_{z_1}L_n(f)(z_1,z_2)o_{z_2}L_m(f)(z_1,z_2).$$

Now we can give the Voronovskaja- type theorem.

Theorem 2. *For fixed $p,q \in \mathbb{N} \cup \{0\}$ and $R_1 > p+1$, $R_2 > q+1$ suppose that $f : P(0;R) \to \mathbb{C}$ is analytic in $P(0;R) = D_{R_1} \times D_{R_2}$, that is $f(z_1;z_2) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $\left(z_1, z_2\right) \in P(0;R)$, $R = (R_1,R_2)$. For all $\left|z_1\right| \le r_1$, $\left|z_2\right| \le r_2$ with $1 < r_1$, $(p+1)r_1 < R_1$, $1 < r_2$, $(q+1)r_2 < R_2$ and $n,m \in \mathbb{N}$ we have*

$$\left|{}_{z_2}L_m(f)(z_1,z_2)o_{z_1}L_n(f)(z_1,z_2)\right| \le M_{r_1,r_2}^{p,q}(f)\left[\frac{1}{n^2} + \frac{1}{m^2}\right],$$

where

$$M_{r_1,r_2}^{p,q}(f) = \max\left\{\sum_{k=2}^{\infty}\sum_{j=0}^{\infty} c_{k,j}\left[\left(q+1\right)r_2\right]^j D_{k,p,r_1}, \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right| j\left[\left(q+1\right)r_2\right]^{j-1} D_{k,p,r_1}, \right.$$
$$\left.\sum_{k=2}^{\infty}\sum_{j=2}^{\infty} c_{k,j} j(j-1)\left[\left(q+1\right)r_2\right]^{j-2} D_{k,p,r_1}\right\},$$

$$D_{k,p,r_1} = (k-1)A_k\left[\left(p+1\right)r_1\right]^{k-1} + C_{k,p}\left[\left(p+1\right)r_1\right]^{k-2}, A_k = (k-1)\left[4\left(k-1\right)\left(k-2\right)+2\right]$$

and

$$C_{k,p} = (k-1)\left[p\left(5k-4\right) + p^2 + k\left(4k-7\right)\right].$$

Proof. Since $f(z_1;z_2)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_{k,j}z_1^kz_2^j$ for all $(z_{1,}z_2)\in P(0;R)$, we can write $f(z_1;z_2)=\sum_{k=0}^{\infty}f_k(z_2)z_1^k$ with $f_k(z_2)=\sum_{j=0}^{\infty}c_{k,j}z_2^j$. It follows $\frac{\partial f}{\partial z_1}(z_1,z_2)=\sum_{k=1}^{\infty}f_k(z_2)kz_1^{k-1}$, $\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)=\sum_{k=2}^{\infty}f_k(z_2)k(k-1)z_1^{k-2}$ and $\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)=\sum_{k=0}^{\infty}\frac{\partial^2}{\partial z_2^2}f_k(z_2)z_1^k$ where $\frac{\partial^2}{\partial z_2^2}f_k(z_2)=\sum_{j=2}^{\infty}c_{k,j}j(j-1)z_2^{j-2}$. Hence $B_{n,p}(f(.,z_2))(z_1)=\sum_{k=0}^{\infty}f_k(z_2)B_{n,p}(e_1^k)(z_1)$ and

$$
\begin{aligned}
&B_{n,p}(f(.,z_2))(z_1)-f(z_1,z_2)-\frac{p}{n}z_1\frac{\partial f}{\partial z_1}(z_1,z_2)-\frac{z_1(1-z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\\
&=\sum_{k=2}^{\infty}f_k(z_2)\left[B_{n,p}(e_1^k)(z_1)-(e_1^k)(z_1)-\frac{p}{n}kz_1^k-\frac{z_1^{k-1}(1-z_1)k(k-1)}{2n}\right].
\end{aligned}
$$

Applying $B_{m,q}$ to the last expression with respect to z_2, we obtain

$$
\begin{aligned}
E_1&=\sum_{k=2}^{\infty}B_{m,q}(f_k)(z_2)\left[B_{n,p}(e_1^k)(z_1)-(e_1^k)(z_1)-\frac{p}{n}kz_1^k-\frac{z_1^{k-1}(1-z_1)k(k-1)}{2n}\right]\\
&=\sum_{k=2}^{\infty}\left(\sum_{j=0}^{\infty}c_{k,j}B_{m,q}(e_1^j)(z_2)\right)\left[B_{n,p}(e_1^k)(z_1)-(e_1^k)(z_1)-\frac{p}{n}kz_1^k-\frac{z_1^{k-1}(1-z_1)k(k-1)}{2n}\right].
\end{aligned}
$$

Passing now to absolute value with $|z_1|\le r_1$, $|z_2|\le r_2$ and considering the estimates in the proof of Theorem 2.1 and Theorem 2.2 in [1], we can write

$$
|E_1|\le\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}|c_{k,j}|[(1+q)r_2]^j\left[\frac{(k-1)A_k[(p+1)r_1]^{k-1}+C_{k,p}[(p+1)r_1]^{k-2}}{n^2}\right]
$$

where $A_k=(k-1)[4(k-1)(k-2)+2]$ and $C_{k,p}=(k-1)[p(5k-4)+p^2+k(4k-7)]$. Similarly,

$$
\begin{aligned}
|E_2|&\le\sum_{k=2}^{\infty}|f_k(z_2)|\left[\frac{(k-1)A_k[(p+1)r_1]^{k-1}+C_{k,p}[(p+1)r_1]^{k-2}}{n^2}\right]\\
&\le\sum_{k=2}^{\infty}\sum_{j=0}^{\infty}|c_{k,j}|[(1+q)r_2]^j\left[\frac{(k-1)A_k[(p+1)r_1]^{k-1}+C_{k,p}[(p+1)r_1]^{k-2}}{n^2}\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
B_{n,p}\left(\frac{\partial f}{\partial z_2}(.,z_2)\right)(z_1)&=\sum_{k=0}^{\infty}\frac{\partial f_k}{\partial z_2}(z_2)B_{n,p}(e_1^k)(z_1)\\
&=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}c_{k,j}jz_2^{j-1}B_{n,p}(e_1^k)(z_1)
\end{aligned}
$$

and

$$\left[B_{n,p}\left(\frac{\partial f}{\partial z_2}(.,z_2)\right)(z_1)-\frac{\partial f}{\partial z_2}(z_1,z_2)-\frac{p}{n}z_1\frac{\partial f}{\partial z_1}\left[\frac{\partial f}{\partial z_2}\right](z_1,z_2)\right.$$
$$\left.-\frac{z_1(1-z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}\left[\frac{\partial f}{\partial z_2}\right](z_1,z_2)\right]$$
$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}c_{k,j}jz_2^{j-1}\left[B_{n,p}\left(e_1^k\right)(z_1)-\left(e_1^k\right)(z_1)-\frac{p}{n}kz_1^k-\frac{z_1^{k-1}(1-z_1)k(k-1)}{2n}\right],$$

in the same way, we get

$$\left|E_3\right|\leq\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|c_{k,j}\right|j\left[(q+1)r_2\right]^{j-1}\left[\frac{(k-1)A_k\left[(p+1)r_1\right]^{k-1}+C_{k,p}\left[(p+1)r_1\right]^{k-2}}{n^2}\right].$$

Similarly

$$B_{n,p}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1)=\sum_{k=0}^{\infty}\frac{\partial^2 f_k}{\partial z_2^2}(z_2)B_{n,p}\left(e_1^k\right)(z_1)$$
$$=\sum_{k=0}^{\infty}\sum_{j=2}^{\infty}c_{k,j}j(j-1)z_2^{j-2}B_{n,p}\left(e_1^k\right)(z_1)$$

and

$$B_{n,p}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1)-\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)-\frac{p}{n}z_1\frac{\partial f}{\partial z_1}\left[\frac{\partial^2 f}{\partial z_2^2}\right](z_1,z_2)-\frac{z_1(1-z_1)}{2n}\frac{\partial^2}{\partial z_1^2}\left[\frac{\partial^2 f}{\partial z_2^2}\right](z_1,z_2)$$
$$=\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}c_{k,j}j(j-1)z_2^{j-2}\left[B_{n,p}\left(e_1^k\right)(z_1)-\left(e_1^k\right)(z_1)-\frac{p}{n}kz_1^k-\frac{z_1^{k-1}(1-z_1)k(k-1)}{2n}\right]$$

hence by similar opinion we have

$$\left|E_4\right|\leq\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\left|c_{k,j}\right|j(j-1)\left[(q+1)r_2\right]^{j-2}\left[\frac{(k-1)A_k\left[(p+1)r_1\right]^{k-1}+C_{k,p}\left[(p+1)r_1\right]^{k-2}}{n^2}\right].$$

Interchanging above the places of n and m we obtain a similar order of approximation for $\left|{}_{z_1}L_m(f)(z_1,z_2)o_{z_2}L_n(f)(z_1,z_2)\right|$ therefore

$$\left|{}_{z_2}L_m(f)(z_1,z_2)o_{z_1}L_n(f)(z_1,z_2)\right|\leq\left|E_1\right|+\left|E_2\right|+\left|E_3\right|+\left|E_4\right|$$
$$\leq M_{r_1,r_2}^{p,q}(f)\left[\frac{1}{n^2}+\frac{1}{m^2}\right]$$

with $M_{r_1,r_2}^{p,q}(f)$ given by the statement. □

The Voronovskaja- type theorem will be used to find the exact order in approximation by $B_{n,n,p,p}(f)$. We present the following Theorem.

Theorem 3. *For fixed $p,q\in\mathbb{N}\cup\{0\}$ and $R_1>p+1$, $R_2>q+1$ suppose that $f:P(0;R)\to\mathbb{C}$ is analytic in $P(0;R)=D_{R_1}\times D_{R_2}$, that is $f(z_1,z_2)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_{k,j}z_1^kz_2^j$ for all $(z_1,z_2)\in P(0;R)$, $R=(R_1;R_2)$. Denoting $\left\|f\right\|_{r_1,r_2}=\sup\left\{\left|f(z_1;z_2)\right|:\left|z_1\right|\le r_1,\left|z_2\right|\le r_2\right\}$, if f is not a solution of the complex partial differential equation*

$$pz_1\frac{\partial f}{\partial z_1}+\frac{z_1(1-z_1)}{2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)+qz_2\frac{\partial f}{\partial z_2}+\frac{z_2(1-z_2)}{2}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)=0, \tag{4}$$

for any $(z_1,z_2)\in P(0;R)$, then we have

$$\left\|B_{n,n,p,p}-f\right\|_{r_1,r_2}\ge\frac{K_{r_1,r_2,f}}{n},\ \text{for all } n\in\mathbb{N} \tag{5}$$

where $K_{r_1,r_2,f}$ is independent on n.

Proof. We can write

$$\begin{aligned}
&B_{n,n,p,p}\left(f\right)\left(z_1,z_2\right)-f\left(z_1,z_2\right)\\
&=\frac{2}{n}\left\{\frac{p}{2}z_1\frac{\partial f}{\partial z_1}+\frac{z_1(1-z_1)}{4}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)+\frac{p}{2}z_2\frac{\partial f}{\partial z_2}+\frac{z_2(1-z_2)}{4}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)\right.\\
&\quad\left.+\frac{2}{n}\left[\frac{n^2}{4}\left({}_{z_2}L_n(f)o_{z_1}L_n(f)\right)(z_1,z_2)\right]+R_n\left(f\right)(z_1,z_2)\right\}
\end{aligned}$$

where

$$\begin{aligned}
&R_n\left(f\right)(z_1,z_2)\\
&=\frac{n}{2}\left(B_{n,p}\left(f(z_1,.)\right)(z_2)-f(z_1,z_2)-\frac{p}{n}z_2\frac{\partial f}{\partial z_2}\left(z_1,z_2\right)-\frac{z_2(1-z_2)}{2n}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)\right)\\
&\quad+\frac{n}{2}\left(B_{n,p}\left(f(.,z_2)\right)(z_1)-f(z_1,z_2)-\frac{p}{n}z_1\frac{\partial f}{\partial z_1}\left(z_1,z_2\right)-\frac{z_1(1-z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right)\\
&\quad+\frac{p}{2n}\left(\begin{array}{c} z_1\left(B_{n,p}\left(\frac{\partial f}{\partial z_1}\left(z_1,.\right)\right)(z_2)-\frac{\partial f}{\partial z_1}\left(z_1,z_2\right)\right)\\ +z_2\left(B_{n,p}\left(\frac{\partial f}{\partial z_2}(.,z_2)\right)(z_1)-\frac{\partial f}{\partial z_2}\left(z_1,z_2\right)\right)\end{array}\right)\\
&\quad+\frac{z_2(1-z_2)}{4}\left(B_{n,p}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1)-\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)-\frac{p}{n}z_1\frac{\partial f}{\partial z_1}\left[\frac{\partial^2 f}{\partial z_2^2}\right](z_1,z_2)\right)\\
&\quad+\frac{z_1(1-z_1)}{4}\left(B_{n,p}\left(\frac{\partial^2 f}{\partial z_1^2}(z_1,.)\right)(z_2)-\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)-\frac{p}{n}z_2\frac{\partial^2 f}{\partial z_1^2}\left[\frac{\partial f}{\partial z_2}\right](z_1,z_2)\right)\\
&\quad+\frac{z_1(1-z_1)z_2(1-z_2)}{8n}\frac{\partial^4 f}{\partial z_1^2\partial z_2^2}(z_1,z_2)
\end{aligned}$$

From Theorem 2.2 in [1] and by the reasonings in the above Theorem 2, it is immediate that $\left\|R_n\left(f\right)\right\|_{r_1,r_2}\to 0$ as $n\to\infty$. Also, by Theorem 2 we obtain

$$\frac{n^2}{4}\left\|{}_{z_2}L_m(f)o_{z_1}L_n(f)\right\|_{r_1,r_2}\le\frac{M_{r_1,r_2}^{p,q}(f)}{2}$$

which implies

$$\left\| \frac{2}{n} \left[\frac{n^2}{4} \left({}_{z_2}L_m(f) o_{z_1} L_n(f) \right) \right] + R_n\left(f\right) \right\|_{r_1,r_2} \to 0, \text{ as } n \to \infty.$$

Denoting

$$H\left(z_1, z_2\right) = \frac{p}{2} z_1 \frac{\partial f}{\partial z_1} + \frac{z_1(1-z_1)}{4} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2) + \frac{q}{2} z_2 \frac{\partial f}{\partial z_2} + \frac{z_2(1-z_2)}{4} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2)$$

and from the inequalities

$$\|F+G\|_{r_1,r_2} \geq \left| \|F\|_{r_1,r_2} - \|G\|_{r_1,r_2} \right| \geq \|F\|_{r_1,r_2} - \|G\|_{r_1,r_2}$$

it follows

$$\begin{aligned} \left\| B_{n,n,p,p} - f \right\|_{r_1,r_2} &\geq \frac{2}{n} \left\{ \|H\|_{r_1,r_2} - \left\| \frac{2}{n} \left[\frac{n^2}{4} \left({}_{z_2}L_m(f) o_{z_1} L_n(f) \right) \right] + R_n\left(f\right) \right\|_{r_1,r_2} \right\} \\ &\geq \frac{2}{n} \frac{1}{2} \|H\|_{r_1,r_2} = \frac{1}{n} \|H\|_{r_1,r_2}, \end{aligned}$$

for all $n \geq n_0$, with n_0 depending only on f, r_1 and r_2. We used here that by hypothesis we have $\|H\|_{r_1,r_2} > 0$.

For $n \in \left\{1, 2, \ldots n_0 - 1\right\}$ it is easily seen that $\left\| B_{n,n,p,p} - f \right\|_{r_1,r_2} \geq \frac{A_{r_1,r_2,n,p}(f)}{n}$ with $A_{r_1,r_2,n,p}\left(f\right) = n \left\| B_{n,n,p,p} - f \right\|_{r_1,r_2}$ which finally implies (5) where

$$K_{r_1,r_2,f} = \max\left\{ A_{r_1,r_2,1,p}\left(f\right), \ldots, A_{r_1,r_2,n_0-1,p}\left(f\right), \frac{1}{2} \|H\|_{r_1,r_2} \right\}.$$

This completes the proof. □

Combining now Theorem 1 with Theorem 3 we immediately obtain the following exact order.

Corollary 1. *For fixed $p, q \in \mathbb{N} \cup \{0\}$ and $R_1 > p+1$, $R_2 > q+1$ suppose that $f : P(0;R) \to \mathbb{C}$ is analytic in $P(0;R) = D_{R_1} \times D_{R_2}$, that is $f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $(z_1, z_2) \in P(0;R)$, $R = (R_1; R_2)$. Assume that $1 < r_1$, $(p+1)r_1 < R_1$, $1 < r_2$, $(q+1)r_2 < R_2$. If f is not a solution of the equation* (4) *then we have*

$$\left\| B_{n,n,p,p} - f \right\|_{r_1,r_2} \sim \frac{1}{n}$$

for all $n \in \mathbb{N}$.

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