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Classical Quasi Primary Elements in Lattice Modules

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Abstract. In this paper we introduce the notion of classical primary and classical quasi primary elements in lattice modules which are the generalization of the concepts in submodules. We obtain some characterizations of classical primary and classical quasi primary elements .We also investigate the decomposition and minimal decomposition into classical quasi primary elements.

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1. Introduction

Multiplicative lattice L is a complete lattice provided with commutative,associative and join distributive multiplication for which the largest element 1 acts as an multiplicative identity. A proper element p of L is called prime element if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ for $a, b \in L$ and is called primary element if $ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$ for some positive integer n. For $a \in L$, $\sqrt{a} = \vee\{x \in L \mid x^n \leqslant a$ for some integer $n\}$. Let L be a multiplicative lattice. A lattice module over L or simply a lattice module is defined to be a complete lattice M with multiplication $L \times M \to M$ satisfying,

(i) $(\bigvee_\alpha a_\alpha)A = \bigvee_\alpha a_\alpha A \quad \forall a_\alpha \in L,\ A \in M$ for some integer

(ii) $a(\bigvee_\alpha A_\alpha) = \bigvee_\alpha aA_\alpha \quad \forall a \in L,\ A_\alpha \in M$

(iii) $(ab)A = a(bA) \quad \forall a, b \in L,\ A \in M$

(iv) $IA = A \quad \forall A \in M$

(v) $OA = O_M \quad \forall A \in M$ where $O_M = glb(M)$.

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Elements of L will generally be denoted by $a, b, c, \ldots$ except that the least element of L will be denoted by O and greatest element of L will be denoted by 1. The elements of M will generally be denoted by $A, B, C, \ldots$ except that the least element and the greatest element of M will be denoted by O_M and I_M. Here after L will be a multiplicative lattice and M will be a lattice module over L. As in the case of commutative rings, there are residuation operations in lattice module. For $a, b \in L$ and $A, B \in M$,

- $a : b$ is the join of all elements c in L such that $cb \leqslant a$,
- $A : b$ is the join of all elements C in M such that $bC \leqslant A$ and
- $A : B$ is the join of all elements a in L such that $aB \leqslant A$.

An element $N \neq I_M$ of a lattice module M is called a prime element if whenever $aA \leqslant N$ where $a \in L, A \in M$ implies either $a \leqslant (N : I_M)$ or $A \leqslant N$. An element $N \neq I_M$ of a lattice module M is called a primary element if whenever $aA \leqslant N$ where $a \in L, A \in M$ implies either $A \leqslant N$ or $a^n \leqslant (N : I_M)$ for some positive integer n. A lattice module M is called a multiplication lattice module if for any element N of M there exists an element a of L such that $N = aI_M$.An element $N \neq I_M$ of a lattice module M is said to have primary decomposition if there exist primary elements $Q_1, Q_2, \ldots, Q_k$ such that $N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$. If some Q_i contains the meet of remaining ones then this Q_i can be dropped from the primary decomposition. Similarly any other primary component which contains the meet of remaining ones can be dropped from the primary decomposition. If such primary components are removed and the primary components with same associated primes are combined then we get a reduced primary decomposition in which distinct primary components are associated with distinct primes such a primary decomposition is called a normal decomposition. This decomposition is also said to be reduced. This study is carried out by D. D. Anderson [1] and for multiplicative lattices this work is done by R. P. Dilworth [2].

2. Classical Primary and Classical Quasi Primary Elements

The notion of a quasi primary ideal was defined by Fuchs [4] which is a generalization of the notions of a primary ideal. The notions of quasi primary, classical primary, classical quasi primary sub modules are studied by M Behboodi *et al.* [1]. We generalize there notions for multiplicative lattices and lattice modules.

Definition 1. *An element q of a multiplicative lattice L is called a classical primary element if* $abr \leqslant q$ *where* $a, b \in L$, $r \in L$ *implies that either* $ar \leqslant q$ *or* $b^k r \leqslant q$ *for some integer k.*

Definition 2. *An element q of a multiplicative lattice L is called a classical quasi primary element if* $abr \leqslant q$ *where* $a, b \in L$, $r \in L$ *implies that either* $a^k r \leqslant q$ *or* $b^k r \leqslant q$ *for some integer k.*

Definition 3. *An element q of a multiplicative lattice L is called a quasi primary element if radical of q is a prime element that is q is called quasi primary if* $ab \leqslant \sqrt{q}$ *where* $a, b \in L$ *implies that either* $a^k \leqslant q$ *or* $b^k \leqslant q$ *for some integer k.*

Definition 4. *Let M be a lattice module over a multiplicative lattice L A proper element Q of M is called a classical primary element in M if $abN \leqslant Q$ where $a, b \in L, N \in M$ then either $aN \leqslant Q$ or $b^k N \leqslant Q$ for some integer k.*

Definition 5. *A proper element Q of M is called a classical quasi primary element in M if $abN \leqslant Q$ where $a, b \in L$, $N \in M$ then either $a^k N \leqslant Q$ or $b^k N \leqslant Q$ for some integer k.*

Definition 6. *A proper element Q of M is called a quasi primary element if $\sqrt{(Q : I_M)}$ is a prime element of L.*

Example 1. *Let R be a integral domain and L(R) denote the set of all ideals of R. Then L(R) is a multiplicative lattice. Let $F = \bigoplus_{\lambda \in \wedge} R_\lambda$ be a free R-module and let $M = L(F)$ denote the set of all submodules of F. Then M is a lattice module over a multiplicative lattice L(R). Assume that, P is a nonzero prime ideal in R. Let $N = \bigoplus_{\curlywedge \in \wedge} A_\curlywedge$ be a proper submodule of F such that for every $\curlywedge \in \wedge$ either $A_\curlywedge = P$ or $A_\curlywedge = (0)$. Then N is a classical primary element of M. It can be verified that, if there exist $\curlywedge_1, \curlywedge_2 \in \wedge$ such that $A_{\curlywedge_1} = P$ and $A_{\curlywedge_2} = (0)$ then N is not a primary element of M, see [1].*

Example 2. *Let L(Z) denote the set of all ideals of Z, the set of integers. If p is a prime integer then $Z(p^\infty) = \{\frac{a}{p^k} + Z \mid a, k$ are integers and$k \in Z_+\}$ is a module over Z. Let $M = L(Z(p^\infty))$ denote the set of all submodules of $Z(p^\infty)$. Then M is a lattice module over a multiplicative lattice $L(Z)$. Every nonzero proper submodule of $Z(p^\infty)$ is a classical primary but not a primary element of M, see [1].*

Example 3. *Let $R = Z$ and $M = Q$ where Q is a module over $R = Z$. Let $L(R)$ denote the set of all ideals of Z and $L(Q)$ denote the set of submodules of Q. Then $L(Q)$ is a lattice module over $L(R)$. Each proper submodule N of M is a quasi primary element since, $\sqrt{(N : Q)} = (0)$. If $N = Z + Z.(\frac{1}{5})$, the submodule of M generated by $\{1, \frac{1}{5}\}$, then $2.3\langle\frac{1}{2.3}\rangle \subseteq N$, but for each $k \geq 1$, $2^k\langle\frac{1}{2.3}\rangle \nsubseteq N$ and $3^k\langle\frac{1}{2.3}\rangle \nsubseteq N$. Thus, N is not a classical quasi primary element of $L(Q)$ see [1].*

Example 4. *Let $R = Z, M = Z \bigoplus Z$ where M is a module over $R = Z$.Let $L(R)$ denote the set of all ideals of R and $L(M)$ denote the set of all submodules of M. Then $L(M)$ is a lattice module over $L(R)$. Let $Q = pZ \bigoplus (0)$, for some prime number p. Then Q is a classical quasi primary element of $L(M)$ but it is not a primary element of $L(M)$ see [1].*

In the next result we obtain characterizations of primary elements, classical primary elements and classical quasi primary elements of a multiplicative lattice.

Theorem 1. *Consider the following statements for a proper element q of L,*

(i) q is a primary element

(ii) q is a classical primary element

(iii) $(q : c)$ is a primary element for each element c of L such that $c \notin q$

(iv) q is a classical quasi primary element

(v) $\sqrt{(q:c)}$ *is a prime element for each element c of L such that* $c \notin q$

(vi) *q is a quasi primary element that is* $\sqrt{q} = \sqrt{(q:1)}$ *is a prime element*

(vii) *q is a power of prime element.*

Then $(i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (iv), (iv) \Rightarrow (v), (v) \Rightarrow (vi), (v) \Rightarrow (iv), (vii) \Rightarrow (vi)$.

Proof. $(i) \Rightarrow (ii)$
Suppose, q is a primary element. Let $a, b \in L$, $r \in L$ such that $abr \leqslant q$ and $br \nleqslant q$. Since, q is a primary element, $abr \leqslant q, br \nleqslant q$ implies $a^k \leqslant q$ for some positive integer k. Hence, $a^k r \leqslant q$. Thus, q is a classical primary element of L.
$(ii) \Rightarrow (iii)$
Suppose, q is a classical primary element of L and let $ab \leqslant (q:c)$, $a, b \in L$. Then $(ab)c \leqslant q$. Hence, $ac \leqslant q$ or $b^k c \leqslant q$, that is $a \leqslant (q:c)$ or $b^k \leqslant (q:c)$ for some positive integer k. Therefore, $(q:c)$ is a primary element for each $c \nleqslant q$.
$(iii) \Rightarrow (i)$
Suppose, $(q:c)$ is a primary element of L, for each $c \nleqslant q$. Take $c = 1$. Then, $(q:1)$ is primary. Let $ab \leqslant q$. So, $(ab) \leqslant (q:1)$ and $a \leqslant (q:1)$ or $b^k \leqslant (q:1)$. Hence, $a \leqslant q$ or $b^k \leqslant q$ and q is a primary element.
That is $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.
$(iii) \Rightarrow (v)$
Suppose, $(q:c)$ is primary, for any $c \nleqslant q$. But $(q:c)$ is a primary implies $\sqrt{(q:c)}$ is prime.
$(iv) \Rightarrow (v)$
Suppose, q is a classical quasi primary element and let $ab \leqslant \sqrt{(q:c)}$ where $c \nleqslant q$. So, $(ab)^k \leqslant (q:c)$ for some positive integer k. That is $a^k b^k c \leqslant q$. As q is classical primary, $(a^k)^t c \leqslant q$ or $(b^k)^t c \leqslant q$. That is $a^m c \leqslant q$ or $b^m c \leqslant q$ where $m = kt \in Z_+$. This shows that, $a \leqslant \sqrt{(q:c)}$ or $b \leqslant \sqrt{(q:c)}$. Therefore, $\sqrt{(q:c)}$ is prime.
$(v) \Rightarrow (iv)$
Suppose, $\sqrt{(q:c)}$ is prime for any $c \nleqslant q$. Let $abr \leqslant q$ where $a, b, r \in L$. We have,
$ab \leqslant (q:r) \leqslant \sqrt{(q:r)}$. Let $r \nleqslant q$. Then $\sqrt{(q:r)}$ is prime implies $a \leqslant \sqrt{(q:r)}$ or
$b \leqslant \sqrt{(q:r)}$. This implies, $a^n r \leqslant q$ or $b^n r \leqslant q$ for some positive integer n. Thus, q is a classical quasi primary element.
$(v) \Rightarrow (vi)$
Suppose, $\sqrt{(q:c)}$ is prime, $c \nleqslant q$. Let $ab \leqslant \sqrt{q}$. Then $(ab)^k \leqslant q$ for some positive integer k. Take $c = 1$. Then, $\sqrt{(q:1)}$ is prime and $(ab)^k \leqslant (q:1)$, that is $ab \leqslant \sqrt{(q:1)}$ which is prime. Hence, $a \leqslant \sqrt{(q:1)}$ or $b \leqslant \sqrt{(q:1)}$. Therefore, $a \leqslant \sqrt{q}$ or $b \leqslant \sqrt{q}$ and $\sqrt{q}$ is prime.
$(vii) \Rightarrow (vi)$
Suppose, q is a power of a prime element and $q = p^k$ where p is prime and k is positive integer. We prove that, $\sqrt{q}$ is prime. Let $ab \leqslant \sqrt{q}$. Then $(ab)^m \leqslant q = p^k \leqslant p$, for some positive integer m. Then $a^m \leqslant p$ or $b^m \leqslant p$. So, $a \leqslant p$ or $b \leqslant p$ and hence, $a^k \leqslant p^k$ or $b^k \leqslant p^k = q$. This shows that $a \leqslant \sqrt{q}$ or $b \leqslant \sqrt{q}$ and $\sqrt{q}$ is a prime element. □ □

We now prove the characterizations of a classical primary element and a classical quasi primary element of a lattice module M. These results establish the relation between a classical primary element of a lattice module and a primary element of a multiplicative lattice and also the relation between a classical quasi primary element of a lattice module and a quasi primary element of a multiplicative lattice.

Theorem 2. *Let M be a lattice module and Q be a proper element of M. Then*

(i) Q is a classical primary element if and only if for every element N of M such that $N \nleqslant Q, (Q : N)$ is a primary element of L.

(ii) Q is a classical quasi primary if and only if for every element $N \in M$ such that $N \nleqslant Q$, $(Q : N)$ is quasi primary element of L.

Proof. (i) Suppose, Q is a classical primary element of M and $N \nleqslant Q$. Let $ab \leqslant (Q : N)$. Since, Q is a classical primary element of M,$aN \leqslant Q$ or $b^kN \leqslant Q$ for some positive integer k. That is $a \leqslant (Q : N)$ or $b^k \leqslant (Q : N)$. Hence, $(Q : N)$ is a primary element of L. Conversely, suppose $(Q : N)$ is a primary element of L for any $N \in M$ such that $N \nleqslant Q$. Let $abN \leqslant Q$. Then, $ab \leqslant (Q : N)$, which is primary. So, $a \leqslant (Q : N)$ or $b^k \leqslant (Q : N)$. That is $aN \leqslant Q$ or $b^kN \leqslant Q$. Suppose, $N \leqslant Q$. Then for any $a \in L$, $aN \leqslant N \leqslant Q$ implies $(Q : N) = 1$ and hence $ab \leqslant (Q : N) = 1$. So, $a \leqslant (Q : N)$, $b \leqslant (Q : N)$ which shows that $aN \leqslant Q$ and $bN \leqslant Q$, when $abN \leqslant Q$. Hence, Q is a classical primary element of M.
(ii) Let Q be a classical quasi primary element and let $ab \leqslant \sqrt{(Q : N)}$ where $N \nleqslant Q$. Then, $(ab)^k \leqslant (Q : N)$ for some positive integer k. Since, Q is a classical quasi primary element of M, $a^kb^kN \leqslant Q$ implies $(a^k)^lN \leqslant Q$ or $(b^k)^lN \leqslant Q$. That is $a^nN \leqslant Q$ or $b^nN \leqslant Q$ for some $n \in Z_+$. Hence, $a \leqslant \sqrt{(Q : N)}$ or $b \leqslant \sqrt{(Q : N)}$ and $\sqrt{(Q : N)}$ is prime. Thus, $(Q : N)$ is a quasi primary element of L. Suppose, $(Q : N)$ is a quasi primary element. Let $abN \leqslant Q$ where $a, b \in L$ and for each element $N \nleqslant Q$. Then $ab \leqslant (Q : N) \leqslant \sqrt{(Q : N)})$. Since, $(Q : N)$ is quasi primary, $\sqrt{(Q : N)}$ is prime. So that $a^k \leqslant (Q : N)$ or $b^k \leqslant (Q : N)$. That is, $a^kN \leqslant Q$ or $b^kN \leqslant Q$. Hence, Q is a classical quasi primary element. If $N \leqslant Q, (Q : N) = 1$. In this case $abN \leqslant Q$ implies $ab \leqslant (Q : N) = 1$ and obviously $a \leqslant (Q : N)$, $b \leqslant (Q : N)$. Consequently, $aN \leqslant Q$, $bN \leqslant Q$ and Q is a classical quasi primary element. □ □

The following theorem is obvious.

Theorem 3. *Let M be a L-module and Q be a proper element of M.*

(i) The following statements are equivalent,

  (a) Q is a classical primary element.

  (b) For every $a, b \in L$ and $N \in M$, $abN \leqslant Q$ implies that either $aN \leqslant Q$ or $b^kN \leqslant Q$ for some $k \in Z_+$

  (c) For every $N \in M$ where $N \nleqslant Q$, $(Q : N)$ is a primary element of L.

(ii) The following statements are equivalent.

  (a) Q is a classical quasi primary element.

  (b) For every $a, b \in L$ and $N \in M$, $abN \leqslant Q$ implies either $a^kN \leqslant Q$ or $b^kN \leqslant Q$ for some $k \in Z_+$

(c) For every $N \in M$, where $N \nleqslant Q$, $(Q : N)$ is a quasi primary element of L.

Theorem 4. *Let M be a lattice module and Q be a classical primary (or classical quasi primary) element of M. Then $\{\sqrt{(Q:N)} \mid N \nleqslant Q\}$ is a chain of prime element of L.*

Proof. For each $M_1, M_2 \nleqslant Q$, we show that $\sqrt{(Q:M_1)} \wedge \sqrt{(Q:M_2)} \leqslant \sqrt{(Q:(M_1 \vee M_2))}$. Let $x \leqslant \sqrt{(Q:M_1)} \wedge \sqrt{(Q:M_2)}$. Then $x \leqslant \sqrt{(Q:M_1)}$ and $x \leqslant \sqrt{(Q:M_2)}$. So $x^{n_1}M_1 \leqslant Q$, $x^{n_2}M_2 \leqslant Q$ for some integers n_1, n_2 and let $n = max\{n_1, n_2\}$. Hence $x^n(M_1 \vee M_2) \leqslant Q$. That is $x \leqslant \sqrt{(Q:(M_1 \vee M_2))}$ and we have

$$\sqrt{(Q:M_1)} \wedge \sqrt{(Q:M_2)} \leqslant \sqrt{(Q:(M_1 \vee M_2))}.$$

Since Q is classical primary and $M_1 \vee M_2 \nleqslant Q$, $\sqrt{(Q:(M_1 \vee M_2))}$ is prime by Theorem 1, we conclude that, $\sqrt{(Q:M_1)} \leqslant \sqrt{(Q:(M_1 \vee M_2))}$ or $\sqrt{(Q:M_2)} \leqslant \sqrt{(Q:(M_1 \vee M_2))}$. Suppose $\sqrt{(Q:M_1)} \leqslant \sqrt{(Q:(M_1 \vee M_2))}$. Then

$$\begin{aligned}\vee\{x \in L \mid x^{k_1}M_1 \leqslant Q, k_1 \in Z_+\} &\leqslant \vee\{x \in L \mid x^{k_2}M_1 \vee x^{k_2}M_2 \leqslant Q\}\\ &= \vee\{x \in L \mid x^{k_2}M_1 \leqslant \text{ and } x^{k_2}M_2 \leqslant Q\}.\end{aligned}$$

Hence $x^{k_1}M_1 \leqslant Q$ implies $x^{k_2}M_2 \leqslant Q$ for some integers k_1, k_2. Therefore,

$$\vee\{x \in L \mid x^{k_1}M_1 \leqslant Q\} \leqslant \vee\{x \in L \mid x^{k_2}M_2 \leqslant Q\}.$$

Consequently, $\sqrt{(Q:M_1)} \leqslant \sqrt{(Q:M_2)}$. Similarly, $\sqrt{(Q:M_2)} \leqslant \sqrt{(Q:(M_1 \vee M_2))}$ implies $\sqrt{(Q:M_2)} \leqslant \sqrt{(Q:M_1)}$. Thus $\sqrt{(Q:M_1)} \leqslant \sqrt{(Q:M_2)}$or $\sqrt{(Q:M_2)} \leqslant \sqrt{(Q:M_1)}$. Hence $\{\sqrt{(Q:N)} \mid N \nleqslant Q\}$ is a chain of prime elements of L. □ □

We now obtain the characterizations of a classical primary element, a primary element of a lattice module and a primary element of a multiplicative lattice.

Theorem 5. *Let M be a multiplication L-module and Q be a proper element of M. The following statements are equivalent,*

(i) Q is a classical primary element of M.

(ii) Q is a primary element

(iii) $(Q : I_M)$ is a primary element of L

(iv) $Q = qI_M$ where q is primary element of L, is maximal with respect to this property i.e. $aI_M = Q$ implies $a \leqslant q$, $a \in L$.

Proof. $(i) \Rightarrow (ii)$
Assume that Q is a classical primary element of M. Let, $aN \leqslant Q$, $a \in L$, $N \in M$ and $N \nleqslant Q$. Since, M is a multiplication module, $N = bI_M$ for some $b \in L$. Hence, $abI_M \leqslant Q$ and $bI_M \nleqslant Q$, i.e. $ab \leqslant (Q : I_M)$ and $b \nleqslant (Q : I_M)$. By Theorem 3, $(Q : I_M)$ is a primary element of L. Hence, $a^kI_M \leqslant Q$ for some $k \in Z_+$. Thus, Q is primary element of M.

$(ii) \Rightarrow (iii)$
Assume that, Q is a primary element of M. Let $ab \leqslant (Q : I_M)$. Then, $abI_M \leqslant Q$. Since, Q is a primary element of M, either $bI_M \leqslant$ or $a^k I_M \leqslant Q$. So, $b \leqslant (Q : I_M)$ or $a^k \leqslant (Q : I_M)$ and $(Q : I_M)$ is a primary element of L.
$(iii) \Rightarrow (iv)$
Assume that, $q = (Q : I_M)$ is a primary element of L. Since, M is a multiplication module, $Q = aI_M$ for some$a \in L$. We have, $qI_M \leqslant Q$ and $a \leqslant (Q : I_M) = q$. So $Q = aI_M \leqslant qI_M$. Thus, $Q = qI_M, q$ is primary and $bI_M = Q \Rightarrow b \leqslant q$.
$(iv) \Rightarrow (i)$
Suppose, $Q = qI_M$ where q is primary element which is maximal w.r.t. this property. Let, $abN \leqslant Q$ where $a, b \in L$ and $N \in$ such that $bN \nleqslant Q$. Since, M is a multiplication L-module $N = cI_M$ for some$c \in L$. Thus $abcI_M \leqslant Q$ and hence $abc \leqslant (Q : I_M) \leqslant q$. Since, $bcI_M \nleqslant Q$, we have, $bc \nleqslant (Q : I_M) \leqslant q$. As, q is primary and $abc \leqslant q$ with$bc \nleqslant q$ we have, $a^k \leqslant q$ for some $k \in Z_+$. This implies that, $a^k N \leqslant qI_M = Q$ and so Q is a classical primary element of M. □ □

In the next result we have the important relation between a classical quasi primary element of a lattice module and a classical quasi primary element of a multiplicative lattice.

Theorem 6. *Let M be a multiplication lattice module and Q be a proper element of M. The following statements are equivalent,*

(i) *Q is a classical quasi primary element.*

(ii) $q = (Q : I_M)$ *is a classical quasi primary element of L.*

(iii) $Q = qI_M$ *where q is a classical quasi primary element which is maximal w.r.t this property. (i.e.* $aI_M = Q \Rightarrow a \leqslant q$*)*

Proof. $(i) \Rightarrow (ii)$
Suppose, Q is a classical quasi primary element of M. Let, $abr \leqslant q$, $a, b, r \in L$. This gives, $abrI_M \leqslant Q$. By classical quasi primality of Q, we have, either $(ab)^k I_M \leqslant Q$ or $r^k I_M \leqslant Q$ for some $k \in Z_+$. i.e. $(ab)^k \leqslant (Q : I_M)$ or $r^k \leqslant (Q : I_M)$. As, Q is a classical quasi primary element and $I_M \nleqslant Q$, $(Q : I_M)$ is a quasi primary element of L i.e. $\sqrt{(Q : I_M)}$ is a prime element of L, by Theorem 5. Hence $abrI_M \leqslant Q$ implies $ab \leqslant \sqrt{(Q : I_M)}$ or $r \leqslant \sqrt{(Q : I_M)}$. Thus again, $a \leqslant \sqrt{(Q : I_M)}$ or $b \leqslant \sqrt{Q : I_M}$. So $a^k \leqslant (Q : I_M)$ or $b^k \leqslant (Q : I_M)$. Consequently, $a^k r \leqslant (Q : I_M)$ or $b^k r \leqslant (Q : I_M)$ i.e. $a^k r \leqslant q$ or $b^k r \leqslant q$. Hence, q is classical quasi primary.
$(ii) \Rightarrow (iii)$
Suppose, $q = (Q : I_M)$ is a classical quasi primary element of L. Since, M is a multiplication module, $Q = aI_M$ for some $a \in L$. We have, $qI_M \leqslant Q$ and $a \leqslant (Q : I_M) = q$. Thus, $aI_M = Q$ gives a $\leqslant q$. So, q is maximal w.r.t. $Q = qI_M$, $(q \in L)$
$(iii) \Rightarrow (i)$
Suppose, $Q = qI_M$ where q is classical quasi primary element which is maximal w.r.t. this property. So, $aI_M = Q$ implies $a \leqslant q$. We show that, Q is a classical quasi primary element of M. Let $abN \leqslant Q$ where $a, b \in L, N \in M$. Suppose, $b^k N \nleqslant Q$ for any integer k. Since M is a

multiplication L-module. $N = cI_M$ for some $c \in L$. Thus, $abcI_M \leqslant Q$ i.e. $abc \leqslant (Q : I_M) \leqslant q$ where q is a classical quasi primary element of L. Now, $abc \leqslant q$ and $b^k cI_M \nleqslant Q$ implies $b^k c \nleqslant (Q : I_M) = q$ for any k. Therefore we have $a^k c \leqslant q$ for some integer k. So, $a^k cI_M \leqslant qI_M$, i.e. $a^k N \leqslant qI_M = Q$. Consequently, Q is a classical quasi primary element of M. □ □

3. Classical Quasi Primary Decomposition of Elements

The study of decomposition into classical primary submodules was introduced in [3]. Further investigation of decomposition of submodules into classical primary submodules is carried out by Behboodi *et al.* [1]. We carry out this study for lattice modules.

The next result shows when the element having a primary decomposition is classical quasi primary element.

Theorem 7. *Let M be an L-module and let $Q = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$ be a primary decomposition of Q with $p_i = \sqrt{(Q_i : I_M)}$. If $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ then Q is a classical p_1-quasi primary element.*

Proof. Assume that, $abN \leqslant Q$, where $a, b \in L, N \nleqslant Q$. Then, $N \nleqslant Q_i$, for some i $(1 \leqslant i \leqslant n)$. Suppose, t $(1 \leqslant t \leqslant n)$ is the smallest number such that $N \nleqslant Q_t$. Thus, $N \leqslant Q_1 \wedge Q_2 \wedge \cdots \wedge Q_{t-1}$. We have $abN \leqslant Q_t$ and Q_t is p_t primary. Hence, $(ab)^{k_1} I_M \leqslant Q_t$ for some $k_1 \in Z_+$ implies $ab \leqslant \sqrt{(Q_t : I_M)} = p_t$. Thus, $a \leqslant p_t$ or $b \leqslant p_t$. Now $p_t \leqslant p_{t+1} \leqslant \cdots \leqslant p_n$. Suppose, $a \leqslant p_t = \sqrt{(Q_t : I_M)}$. Consequently $a^k I_M \leqslant Q_t$, for some integer k. If $b \leqslant p_t = \sqrt{(Q_t : I_M)}$ we have $b^k I_M \leqslant Q_t$. Thus $a^k I_M \leqslant Q_t$ or $b^k I_M \leqslant Q_t$ for some $k \in Z_+$ where t is the smallest positive integer such that $N \nleqslant Q_t$. We have $N \nleqslant Q_{t+1}, \ldots, N \nleqslant Q_n$, $a^k I_M \leqslant Q_t, Q_{t+1}, \ldots, Q_n$ or $b^k I_M \leqslant Q_t, Q_{t+1}, \ldots, Q_n$. Hence $a^k I_M \leqslant Q_t \wedge Q_{t+1} \wedge \cdots \wedge Q_n$ or $b^k I_M \leqslant Q_t \wedge Q_{t+1} \wedge \cdots \wedge Q_n$ for some $k \in Z_+$. But, $N \leqslant Q_1 \wedge Q_2 \wedge \cdots \wedge Q_{t-1}$. It follows that, $a^k N \leqslant Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n = Q$ or $b^k N \leqslant Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$. If $N \leqslant Q$, $abN \leqslant Q$ implies $aN \leqslant Q$, $bN \leqslant Q$. Also, $\sqrt{(Q : I_M)} = \sqrt{(Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n) : I_M} = p_1$. Therefore, Q is a classical p_1- quasi primary element. □ □

Remark 1. *The following example shows that the above theorem is not necessarily true if $Q_1, Q_2, \ldots, Q_n$ are only assured to be classical (quasi) primary elements.*

Example 5. *Let $R = Z, M = Z_2 \oplus Z_3 \oplus Z$ Then M is a z-module. Let $L(R)$ denote the set of all ideals of R and $L(M)$ denote the set of all submodules of M. Then $L(M)$ is a module over $L(R)$ where $L(R)$ is a multiplicative lattice. Let $Q_1 = Z_2 \oplus (0) \oplus (0)$. $Q_2 = (0) \oplus Z_3 \oplus (0)$. Then Q_1, Q_2, are elements of $L(M)$. It can be shown that Q_1 and Q_2 are classical (quasi) primary elements of $L(M)$. Also, $(0) = Q_1 \cap Q_2$ and $\sqrt{(Q_1 : M)} = \sqrt{(Q_2 : M)} = (0)$. Obviously, $2 \times 3(Z_2 \oplus Z_3 \oplus (0)) = (0)$. Also, for each $k \geq 1, 2^k(Z_2 \oplus Z_3 \oplus (0)) \nsubseteq (0)$ and $3^k(Z_2 \oplus Z_3 \oplus (0)) \nsubseteq (0)$.*

Thus, $(0) \subset M$ is not a classical (quasi) primary element.

Remark 2. *We shall show that the converse of Theorem 7 is also true when decomposition $Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$ is a reduced primary decomposition.*

Theorem 8. *Let Q be a p-primary element of a lattice module M and N be an element of M. If $N \nleqslant Q$ then $(Q : N)$ is a p-primary element.*

Proof. First we show that $(Q : N)$ is a primary element. Let $a, b \in L$, $ab \leqslant (Q : N)$ and suppose a $\nleqslant (Q : N)$. Also as $ab \leqslant (Q : N)$, $abN \leqslant Q$ with $aN \nleqslant Q$ and Q is a primary element implies that $b^n \leqslant (Q : I_M)$ for some integer n. But $b^n I_M \leqslant Q$ implies $b^n N \leqslant Q$ and we have $b \leqslant \sqrt{(Q : N)}$. Therefore, $(Q : N)$ is a primary element of L. Now since $N \nleqslant Q$, there exists $A \leqslant I_M$ and $A \leqslant N$ such that $A \nleqslant Q$. Let $a \leqslant \sqrt{(Q : N)}$. Then $a^n N \leqslant Q$ and $a^n A \leqslant Q$. But $A \nleqslant Q$ and Q is primary implies that $(a^n)^k = a^m \leqslant (Q : I_M)$ for some integer m. That is $a \leqslant \sqrt{(Q : I_M)} = p$ and $\sqrt{(Q : N)} \leqslant p$. Conversely, let $a \leqslant \sqrt{(Q : I_M)} = p$. Hence, $a^n I_M \leqslant Q$ for some integer$n$. So $a^n N \leqslant Q$ for some integer n. Thus $a^n \leqslant (Q : N)$ and $a \leqslant \sqrt{(Q : N)}$. This shows that $p \leqslant \sqrt{(Q : N)}$ and we have $\sqrt{(Q : N)} = p$. Therefore, $(Q : N)$ is a p-primary element. □ □

We now establish the characterization of an associated prime of an element of a lattice module having a primary decomposition.

Theorem 9. *Let $N \neq I_M$ be an element of a lattice module M and assume that N has a primary decomposition. Let $N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$ be a reduced primary decomposition of N and p be prime element of L. Then following statements are equivalent,*

(i) $p = \sqrt{Q_i}$ *for some i*

(ii) $(N : X)$ *is a p-primary element of L for some* $X \nleqslant N$.

Proof. $(i) \Rightarrow (ii)$
Let $N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$ be a reduced primary decomposition of N. First suppose that, $p = \sqrt{Q_i}$ for some i. Without loss of generality we can assume that $p = \sqrt{(Q_1 : I_M)}$ where $p_i = \sqrt{(Q_i : I_M)}$ $i = 1, 2, \ldots, k$. We prove that, $(N : X)$ is a p-primary element of L for some $X \nleqslant N$. Since the decomposition is reduced $Q_i \ngeqslant Q_1 \wedge Q_2 \wedge \cdots \wedge Q_{i-1} \wedge Q_{i+1} \wedge \cdots \wedge Q_k$ for $i = 1, 2, \ldots, k$. In particular, $Q_1 \ngeqslant Q_2 \wedge Q_3 \wedge \cdots \wedge Q_k$. So there exists $X \leqslant Q_2 \wedge Q_3 \wedge \cdots \wedge Q_k$ such that $X \nleqslant Q_1$ and hence $X \nleqslant N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$. Also

$$(N : X) = (Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k) : X = (Q_1 : X) \wedge (Q_2 : X) \wedge \cdots \wedge (Q_k : X).$$

For $i = 2, 3, \ldots, k$ we show that $(Q_i : X) = 1$. Since $X \leqslant Q_2 \wedge Q_3 \cdots \wedge Q_k$, we have $X \leqslant Q_i$ for all $i = 2, \ldots, k$. Then $aX \leqslant Q_i$ for all $a \in L$ and for all $i = 2, 3, \ldots, k$. That is $a \leqslant (Q_i : X)$ for all $i = 2, 3, \ldots, k$. So $1 \leqslant (Q_i : X)$. But $(Q_i : X) \leqslant 1$ implies $(Q_i : X) = 1$ for $i = 2, 3, \ldots, k$. Hence, $(N : X) = (Q_1 : X) \wedge 1 \wedge \cdots \wedge 1 = (Q_1 : X)$. So by the above result, $(Q_1 : X)$ is p-primary element implies $(N : X)$ is a p-primary element of L where $X \nleqslant N$.
$(ii) \Rightarrow (i)$
Assume that $(N : X)$ is a p-primary element of L for some $X \nleqslant N, X \in M$. We prove that $\sqrt{Q_i} = p$ for some i. We have,

$$p = \sqrt{(N : X)} = \sqrt{[(Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k) : X]} = \sqrt{(Q_1 : X)} \wedge \sqrt{(Q_2 : X)} \wedge \cdots \wedge \sqrt{(Q_k : X)}.$$

We claim that for each i, $\sqrt{(Q_i : X)} = p_i$ or 1 and equal to p_i for at least one i. We have, $X \nleqslant N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$ implies $X \nleqslant Q_i$ for at least one i $(1 \leqslant i \leqslant k)$. Suppose, $X \nleqslant Q_r$ $(1 \leqslant r \leqslant k)$ and $X \leqslant Q_1 \wedge Q_2 \wedge \cdots \wedge Q_{r-1} \wedge Q_{r+1} \wedge \ldots Q_k$ that is $X \leqslant \wedge Q_i$, where $(i \neq r)$.

Let $a \leqslant (Q_i : X), a \in L$. Since, $X \leqslant \wedge Q_i$. We have, $aX \leqslant Q_i$ for all $i \neq r$ and $a \in L$. Hence, $a \leqslant \sqrt{(Q_i : X)}$ for all $a \in L$. In particular, $1 \leqslant \sqrt{(Q_i : X)}$ for all $i \neq r$. But, $\sqrt{(Q_i : X)} \leqslant 1$ for all $i \neq r$. Therefore, $\sqrt{(Q_i : X)} = 1$ for all $i \neq r$. For $i = r, X \nleqslant Q_r$. Let $a \leqslant \sqrt{(Q_r : X)}$. Hence, $a^n X \leqslant Q_r$, for some positive integer n, where $X \nleqslant Q_r$. As Q_r is primary, $a^n \leqslant \sqrt{(Q_r : I_M)} = p_r$. Thus, $a \leqslant p_r$, since p_r is prime and we have, $\sqrt{(Q_r : X)} \leqslant p_r$. On the other hand, let $a \leqslant p_r = \sqrt{Q_r} = \sqrt{(Q_r : I_M)}$. Hence, $a^n \leqslant (Q_r : I_M)$ for some positive integer n. That is $a^n I_M \leqslant Q_r$ and therefore, $a^n X \leqslant Q_r$, for some positive integer n. Consequently, $a^n \leqslant (Q_r : X)$ and hence $a \leqslant \sqrt{(Q_r : X)}$. This gives $p_r \leqslant \sqrt{(Q_r : X)}$. Hence, $\sqrt{(Q_r : X)} = p_r$ where $X \nleqslant Q_r$. We have shown that for each i, $\sqrt{(Q_i : X)} = p_i$ or 1 and is equal to p_i for at least one i, since $X \nleqslant N$. Then, $p = \sqrt{(N : X)} = \sqrt{(Q_1 : X)} \wedge \cdots \wedge \sqrt{(Q_k : X)}$ is the meet of some of the prime elements $p_1, p_2, \ldots, p_l$ $(1 \leqslant l \leqslant k)$. That is $p = \sqrt{(N : X)} = p_1 \wedge p_2 \wedge \cdots \wedge p_l$. We show that $p = p_i$ for some i. We have, $p \leqslant p_i$ $i = 1, 2, \ldots, l$. If for each i $p \neq p_i$ then $p_i \nleqslant p$ for all $i = 1, 2, \ldots, l$. This implies that there exist $x_i \leqslant p_i$ such that $x_i \nleqslant p$ for all $i = 1, 2, \ldots, l$. Then, $x_1 x_2 \ldots x_l \leqslant p_1 \wedge p_2 \wedge \cdots \wedge p_l = p$. This shows that $x_i \leqslant p$ for at least one i $(1 \leqslant i \leqslant k)$ a contradiction. Hence, $p = p_i$ for at least one i. □ □

We have the characterization of a classical quasi primary element of a lattice module in terms of a chain of its associated primes.

Theorem 10. *Let M be a L-module and Q be a proper element of M. Let $Q = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$ with $p_i = \sqrt{(Q_i : I_M)}$ be a reduced primary decomposition of Q. Then Q is a classical quasi primary element if and only if $\{p_1, p_2, \ldots, p_n\}$ is a chain of prime elements of L. In that case, radical of $(Q : I_M)$ is the smallest of the primes $p_1, p_2, \ldots, p_n$.*

Proof. Since, $Q = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$ is a reduced primary decomposition of Q by Theorem 7 for each i $(1 \leqslant i \leqslant n)$, $p_i = \sqrt{(Q : A_i)}$ for some $A_i \nleqslant Q$ where $p_i = \sqrt{(Q_i : I_M)}$. Assume that, Q is a classical quasi primary element. We show that $\{p_1, p_2, \ldots, p_n\}$ is the chain of prime elements of L. Assume that, it is not a chain. Then $p_i \nleqslant p_j$ and $p_j \nleqslant p_i$ for some $i \neq j$. Select $a \leqslant p_i$ such that $a \nleqslant p_j$ and $b \leqslant p_j$ such that $b \nleqslant p_i$. i.e. $a \leqslant \sqrt{(Q_i : I_M)}$, $a \nleqslant \sqrt{(Q_j : I_M)}$ and $b \leqslant \sqrt{(Q_j : I_M)}$, $b \nleqslant \sqrt{(Q_i : I_M)}$. Since $p_i = \sqrt{(Q : A_i)}, A_i \nleqslant Q$ for each $i = 1, 2, \ldots, n$, we have, $a \leqslant \sqrt{(Q_i : I_M)} = \sqrt{(Q : A_i)}$ and $a \nleqslant \sqrt{(Q : A_j)}$. Similarly, $b \leqslant \sqrt{(Q_j : I_M)} = \sqrt{(Q : A_j)}$ and $b \nleqslant \sqrt{(Q : A_i)}$. This shows that $a^{k_i} A_i \leqslant Q$ for some integer k_i and $a^k A_j \nleqslant Q$ for any k. Similarly, $b^{k_j} A_j \leqslant Q$ for some integer k_j and $b^k A_i \nleqslant Q$ for any k. Thus $a^{k_i} A_i \vee b^{k_j} A_j \leqslant Q$ and $a^k A_j \vee b^k A_i \nleqslant Q$ for any k. Let $k = max\{k_i, k_j\}$. Then $a^k b^k A_i \vee b^k a^k A_j = a^k b^k (A_j \vee A_i) \leqslant Q$. Since, Q is a classical quasi primary element of M, it follows that $(a^k)^l (A_i \vee A_j) \leqslant Q$ or $(b^k)^l (A_i \vee A_j) \leqslant Q$ i.e. $a^s (A_i \vee A_j) \leqslant Q$ or $b^s (A_i \vee A_j) \leqslant Q$ for some integer s. Therefore, $a^s \leqslant (Q : A_j)$ or $b^s \leqslant (Q : A_i)$ for some integer s. Hence, $a \leqslant p_j$ or $b \leqslant p_i$, which is a contradiction. Thus, $\{p_1, p_2, \ldots, p_n\}$ is a chain of prime elements of L. Converse follows by Theorem 7. □ □

Remark 3. *We note that Theorem 8 is not necessarily true if the primary decomposition $Q = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$ is not minimal.*

Example 6. *Let $R = Z$, $M = Z \oplus Z$. Let $L(R)$ denote the set of all ideals of R and $L(M)$ denote the set of all sub modules of a module M over $R = Z$. Then $L(M)$ is an L-module over $L(R)$. Let*

$Q_1 = 2Z \oplus Z$, $Q_2 = Z \oplus 3Z$, $Q_3 = Z \oplus (0)$, $Q_4 = (0) \oplus Z$. Then, Q_1, Q_2, Q_3, Q_4 are primary sub modules of M and hence, these are the elements of a lattice module $L(M)$ with $\sqrt{(Q_i : M)} = 2Z$, $\sqrt{(Q_2 : M)} = 3Z$ and $\sqrt{(Q_i : M)} = \sqrt{(Q_4 : M)} = (0)$. Also, $(0) = Q_1 \cap Q_2 \cap Q_3 \cap Q_4$ and (0) is a classical quasi primary sub module of M and hence a classical quasi primary element of $L(M)$. But $\{(0), 2Z, 3Z\}$ is not a chain of prime ideals of R i.e. $(0), 2Z, 3Z$ is not a chain of prime elements of $L(R)$.

Definition 7. *Let N be a proper element of a lattice module M over L. A classical primary (respectively classical quasi primary) decomposition of N is an expression $N = \bigwedge_{i=1}^{n} Q_i$ where each Q_i is classical primary (respectively classical quasi primary) element of M.The decomposition is called reduced if it satisfies the following two conditions,*

(i) No $Q_{i1} \wedge Q_{i2} \wedge \cdots \wedge Q_{it}$ is classical primary (respectively classical quasi primary) element where $(i_1, i_2, \ldots, i_t) \subseteq \{1, 2, \ldots, n\}$ for $t \geqslant 2$ with $i_1 < i_2 < \cdots < i_t$

(ii) for each j, $Q_j \ \bigwedge_{i \neq j} Q_i$

Corresponding to the above definition by Theorem 2 we have a list of prime elements $\sqrt{(Q_1 : I_M)}, \ldots \sqrt{(Q_n : I_M)}$. Among reduced classical primary (resp classical quasi primary) decomposition, any one that has the least number of distinct primes will be called minimal. It is clear that, every primary decomposition of an element N of M is called classical primary but the converse is not always true.

The next result is useful in proving the uniqueness of associated primes of an element of a lattice module with a primary decomposition.

Theorem 11. *Let $N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k$ be a reduced primary decomposition of a lattice module M over L. Then every minimal prime divisor of N is a prime divisor of N.*

Proof. Let $p_i = \sqrt{(Q_i : I_M)}$. Then $p_1, p_2, \ldots, p_k$ are the prime elements which are called prime divisors of N or associated primes of N. We have

$$\begin{aligned}\sqrt{(N : I_M)} &= \sqrt{(Q_1 \wedge Q_2 \wedge \cdots \wedge Q_k) : I_M} \\ &= \sqrt{(Q_1 : I_M)} \wedge \sqrt{(Q_2 : I_M)} \wedge \cdots \wedge \sqrt{(Q_k : I_M)} \\ &= p_1 \wedge p_2 \wedge \cdots \wedge p_k.\end{aligned}$$

If $a \leqslant p_1 p_2 p_3 \ldots p_k$ then $a \leqslant p_i$ for each $i = 1, 2, \ldots, n$. Hence $a \leqslant \sqrt{(Q_i : I_M)}$ for each i. That is $a \leqslant \sqrt{(Q_1 : I_M)} \wedge \sqrt{(Q_2 : I_M)} \wedge \ldots \sqrt{(Q_k : I_M)} = \sqrt{(N : I_M)}$. So $a^n \leqslant (N : I_M)$ for some positive integer n. If p is any prime element of L containing $(N : I_M)$ then $a^n \leqslant p$ and hence $a \leqslant p$. Then $p_1 p_2 \ldots p_k \leqslant p$ whenever $(N : I_M) \leqslant p$. This shows that $p_i \leqslant p$ for some i. If p is a minimal prime element containing $(N : I_M)$ then $p_i \leqslant p$ for some i. Also p_i contains $(N : I_M)$, but p is minimal such element. Hence $p_i = p$. Thus every minimal prime divisors of N is a prime divisor of N and is minimal in the set of prime divisors of N. □ □

The next result we prove the important property of uniqueness of associated primes of an element having classical quasi primary decomposition.

Theorem 12. *Let L be a Noetherian lattice and M be a module over L. Let N be a proper element of M and* $N = Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n$ *with* $\sqrt{(Q_i : I_M)} = p_i$ *for* $i = 1, 2, \ldots, n$ *be a reduced classical quasi primary decomposition of N. Then,* $\{p_i \mid i = 1, 2, \ldots, n\} = min(N : I_M)\}$ *and the set* $\{p_i \mid i = 1, 2, \ldots, n\}$ *is uniquely determined.*

Proof. First we show that, $min(N : I_M) \subseteq \{p_i \mid i = 1, 2, 3, \ldots, n\}$. Let p be a minimal prime of $(N : I_M)$. Then p is a minimal member of associated primes of N. But $p = p_i$ for some i if and only if there exists $A \nleqslant N$ in M such that $(N : A)$ p-primary. Thus, $p = \sqrt{(N : A)} = \sqrt{(Q : I_M)}$ for some $A \nleqslant N$. Renumber the Q_i such that $A \nleqslant Q_i$ for $1 \leqslant i \leqslant j$ and $A \leqslant Q_i$ for $j + 1 \leqslant i \leqslant n$. Since $p_i = \sqrt{(Q_i : I_M)}$, ${p_i}^{k_i} I_M \leqslant Q_i$ for some integer k_i, $(1 \leqslant i \leqslant n)$. Therefore, $(\overset{j}{\underset{i=1}{\wedge}} {p_i}^{k_i})A \leqslant \overset{n}{\underset{i=1}{\wedge}} Q_i = N$ and so $\overset{j}{\underset{i=1}{\wedge}} {p_i}^{k_i} \leqslant (N : A) \leqslant p$. Since p is prime, $p_t \leqslant p$ for some $t \leqslant j$. As, $(N : I_M) \leqslant \sqrt{(N : I_M)} \leqslant \sqrt{(Q_t : I_M)} = p_t$ and p is a minimal prime of $(N : I_M)$, we conclude that $p = p_t$. Now it is sufficient to show that each $p_i (1 \leqslant i \leqslant n)$ is a minimal prime of $(N : I_M)$. Without loss of generality, we may take $i = 1$. Clearly

$$(N : I_M) \leqslant \sqrt{(N : I_M)} = \sqrt{(Q_1 \wedge Q_2 \wedge \cdots \wedge Q_n : M)} = \overset{n}{\underset{i=1}{\wedge}} \sqrt{(Q_i : I_M)} \leqslant p_1.$$

On the contrary, suppose that p_1 is not a minimal prime of $(N : I_M)$. Thus $\exists$ an $i \in \{1, 2, \ldots, n\}$ such that p_i is minimal prime of $(N : I_M)$ with $p_i < p_1$ (since $min(N : I_M) \subseteq \{p_i \mid i = 1, 2, 3, \ldots, n\}$). Again without loss of generality, we may take $i = 2$. Thus $(N : I_M) \leqslant p_2 < p_1$. By [5], each Q_i has a minimal primary decomposition. Suppose that, $Q_1 = Q_{11} \wedge \cdots \wedge Q_{1s}$ with $\sqrt{(Q_{1j} : I_M)} = p_{1j}$ $(1 \leqslant j \leqslant s)$ and $Q_2 = Q_{21} \wedge \cdots \wedge Q_{2t}$ with $\sqrt{(Q_{2j} : I_M)} = p_{2j}$ $(1 \leqslant j \leqslant t)$ are a reduced primary decompositions of Q_1 and Q_2 respectively. By Theorem 8 $\{p_{1j} \mid 1 \leqslant j \leqslant s\}$ and $\{p_{2j} \mid 1 \leqslant j \leqslant t\}$ are chains of prime elements. Without loss of generality, we can assume that, $p_{11} \subseteq p_{12} \subseteq \cdots \subseteq p_{1s}$ and $p_{21} \subseteq p_{22} \subseteq \cdots \subseteq p_{2t}$. We thus get $p_1 = p_{11}$ and $p_2 = p_{21}$, since

$$p_1 = \sqrt{(Q_1 : I_M)} = \sqrt{(Q_{11} \wedge \cdots \wedge Q_{1s}) : I_M} = \overset{s}{\underset{i=1}{\wedge}} \sqrt{(Q_{1i} : I_M)} = p_{11}$$

and

$$p_2 = \sqrt{(Q_2 : I_M)} = \sqrt{(Q_{21} \wedge \cdots \wedge Q_{2t}) : I_M} = \overset{t}{\underset{i=1}{\wedge}} \sqrt{(Q_{2i} : I_M)} = p_{21}.$$

It follows that, $p_{21} \subseteq p_{11} \subseteq p_{12} \subseteq \cdots \subseteq p_{1s}$ and so by Theorem 8, ${Q_1}' = Q_{21} \wedge Q_{11} \wedge \cdots \wedge Q_{1s}$ is a classical quasi primary element of M with $\sqrt{({Q_1}' : I_M)} = p_{21} = p_2$. On the other hand,

$$\begin{aligned} N = & Q_1 \wedge \cdots \wedge Q_n = (Q_{11} \wedge \cdots \wedge Q_{1s}) \wedge (Q_{21} \wedge \cdots \wedge Q_{2t}) \wedge Q_3 \cdots \wedge Q_n \\ = & (Q_{21} \wedge Q_{11} \wedge \cdots \wedge Q_{1s}) \wedge (Q_{21} \wedge \cdots \wedge Q_{2t}) \wedge (Q_3 \wedge \cdots \wedge Q_n). \end{aligned}$$

Thus, $N = Q_1' \wedge Q_2 \wedge \cdots \wedge Q_n$ is a classical quasi primary decomposition of N with $\sqrt{(Q_1' : I_M)} = \sqrt{(Q_2 : I_M)} = p_2$ and $\sqrt{(Q_i : I_M)} = p_i$ for $i = 3, \ldots n$. We note that if $\exists$ another Q_i $(3 \leqslant i \leqslant n)$ such that $\sqrt{(Q_i : I_M)} = p_i = p_1$ then by similar arguments we can replace it by $Q_i^{'}$ such that $\sqrt{(Q_i^{'} : I_M)} = \sqrt{(Q_2 : I_M)} = p_2$. Now, by using this decomposition we can obtain a reduced classical quasi primary decomposition, $N = Q_1^{''} \wedge Q_2^{''} \wedge \cdots \wedge Q_k^{''}$ such that $p_1 \notin \{\sqrt{(Q_i^{''} : I_M)} \mid i = 1,2,3,4,\ldots,k\} \subseteq \{p_i \mid i = 2,\ldots,n\}$, contrary with the irrenduntness of the decomposition $N = Q_1, \wedge Q_2 \wedge \ldots, \wedge Q_n$ with

$$\{\sqrt{(Q_i : I_M)} \mid i = 1,2,3,\ldots,n\} = \{p_i \mid i = 1,2\ldots,n\}.$$

Thus, $\{p_i \mid i = 1,2,\ldots,n\} = min(N : I_M)$. □ □

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