On Class Numbers of Real Quadratic Fields With Certain Fundamental Discriminants

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Abstract: Let N denote the sets of positive integers and $D \in N$ be square free, and let χ_D, $h = h(D)$ denote the non-trivial Dirichlet character, the class number of the real quadratic field $K = Q(\sqrt{D})$, respectively.

ONO, proved the theorem in [8] by applying Sturm's Theorem on the congruence of modular form to Cohen's half integral weight modular forms. Later, Dongho Byeon proved a theorem and corollary in [1] by refining Ono' methods.

In this paper, we will give a theorem for certain real quadratic fields. by considering above mentioned studies. To do this, we shall obtain an upper bound different from current bounds for $L(1, \chi_D)$ and use Dirichlet's class number formula.

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1. Introduction

Let Z_p, N, Q denote the the ring of p-adic integers, positive integers and rational numbers, respectively. Let $R_p(D)$ denote the p-adic regulator of K, $|.|_p$ denote the usual multiplicative p-adic valuation normalized $|p|_p = \frac{1}{p}$, and let $L(s, \chi_D)$ denote the L-function attached to χ_D. Throughout $D \in N$ will be assumed square free, $K = Q(\sqrt{D})$ will denote the real quadratic field, and the class number of K will be denoted by $h = h(D)$. Let Δ denote the discriminant, ε_D the fundamental unit of K.

2. Preliminaries

Theorem 2.1 (ONO) Let $p > 3$ be prime. If there is a fundamental discriminant D_0 coprime to p for which

$$i-)(-1)^{\frac{p-1}{2}} D_0 > 0$$

$$ii-)|B(\frac{p-1}{2}, \chi_{D_0})|$$

Where $B(\frac{p-1}{2}, \chi_{D_0})$ is the $\frac{p-1}{2}$ st generalized Bernoulli number with character χ_{D_0} then

$$\#\left\{0 < D < X| \quad h(D) \not\equiv 0 \pmod{p}, \quad \chi_D(p) = 0, \quad |\frac{R_p(D)}{\sqrt{D}}|_p = 1\right\} \gg_p \frac{\sqrt{X}}{logX}.$$

D. Byeon have given in [] the following theorem and corollary by refining Ono's theorem above mentioned for any prime $p > 3$

Theorem 2.2

Let $p > 3$ be prime.

(a) If $p \equiv 1 \pmod{4}$, then the fundamental discriminant $D_0 > 0$ of the real quadratic field $Q(\sqrt{p-2})$ satisfies the conditions (i) and (ii).

(b) If $p \equiv 3 \pmod{4}$, then the fundamental discriminant $D_0 < 0$ of the real quadratic field $Q(\sqrt{-(p-4)}$ satisfies the conditions (i) and (ii).

Corollary 2.1

Let $p > 3$ be prime. Then

$$\#\left\{0 < D < X| \quad h(D) \not\equiv 0 \pmod p, \quad \chi_D(p) = \delta, \quad |R_p(D)|_p = \frac{1}{p}\right\} \gg_p \frac{\sqrt{X}}{logX}.$$

3. Main Theorem Let $p > 3$ be a prime. If $p \equiv 3 \pmod 4$, then the fundamental discriminant $D_0 > 0$ of the real quadratic fields $K = Q(\sqrt{p^2 - 4})$ and $K = Q(\sqrt{p^2 - 2})$ satisfies the conditions (i) and (ii).

In order to main theorem we need the following lemmas.

Lemma 2.1 i-) If D is a prime with $D \equiv 1 \pmod 4$, we have

$$\varepsilon_D > \begin{cases} \|\sqrt{D \mp 1}\| & , D = n^2 \mp 4, (n \in Z) \\ \|\sqrt{4D \mp 1}\| & , \text{in the other cases} \end{cases}$$

Where "$\| * \|$" represents great value function of a real number.

ii-) If D is a prime with $D \equiv 3 \pmod 4$, we have

$$\varepsilon_D > \begin{cases} 2D \mp 1 & , D = n^2 \mp 2, \\ 8D \mp 1 & , \text{in the other cases} \end{cases}$$

Proof. i-) Let $\varepsilon_D = \dfrac{t + u\sqrt{D}}{2} > 1$ be the fundamental unit of $K = Q(\sqrt{D})$.

Since ε_D is equal to the fundamental solution of the Pell's equation $x^2 - Dy^2 = \mp 4$, then we can write

$$\varepsilon_D{}^2 = (\frac{t + u\sqrt{D}}{2})^2 = \frac{1}{4}(\sqrt{Du^2 \mp 4} + u\sqrt{D})^2 > \frac{Du^2 \mp 1}{\sqrt{2}} \geq \begin{cases} \frac{D\mp 1}{\sqrt{2}} & , u = 1 \\ \frac{4D\mp 1}{\sqrt{2}} & , u > 1 \end{cases}$$

and

$$\varepsilon_D > \begin{cases} \|\sqrt{D \mp 1}\| & , D = n^2 \mp 4 \\ \|\sqrt{4D \mp 1}\| & , \text{in the other cases} \end{cases}$$

ii-) If $D \equiv 3 \pmod 4$, then we have $\varepsilon_D{}^2 = (\dfrac{t + u\sqrt{D}}{2})^2$ from the least positive integer solution $(x, y) = (t, u)$ of the Pell's equation $x^2 - Dy^2 = \mp 2$ [7]. Similary, we can write

$$\varepsilon_D > \begin{cases} 2D \mp 1 & , D = n^2 \mp 2, \quad (n: \text{ odd integer }) \\ 8D \mp 1 & , \text{in the other cases} \end{cases}$$

Lemma 2.2 Let $D \in N$ be square free, then $h(D) < \sqrt{D}$.

In order to prove this we need the following Lemma [5].

Lemma 2.3 Let γ be Euler's constant, then

$$|L(1, \chi_D)| \leq \begin{cases} \frac{1}{4}(log\Delta + 2 + \gamma - log\pi) & , 2 \mid \Delta, \\ \frac{1}{2}(log\Delta + 2 + \gamma - log4\pi) & , \text{otherwise} \end{cases}$$

Proof of Lemma 2.2. By Dirichlet's class number formula, we have

$$h(D) = \frac{\sqrt{\Delta}}{2log\varepsilon_D}|L(1, \chi_D)|$$

where Δ is a fundamental discriminant of a quadratic field define by

$$\Delta = \begin{cases} 4D & , D \equiv 2, 3 \pmod 4 \\ D & , D \equiv 1 \pmod 4 \end{cases}$$

First, we consider the case $D \equiv 1 \pmod 4$ and $D = n^2 \mp 4$. Thus, by the upper bound for $L(1, \chi_D)$ in Lemma 2.3,and from Lemma 2.1 we have that

$$h(D) < \frac{\sqrt{D}(logD + 1,478)}{4log\|\sqrt{D \mp 1}\|} = \frac{\sqrt{D}(logD + 1,478)}{2log\|(D \mp 1)\|} < \sqrt{D}, \quad (D > 5).$$

Moreover, we can write $h(D) \leq \|\frac{\sqrt{D}(logD + 1,478)}{2log(D \mp 1)}\|$ where"$\|x\|$"is the greatest integer less than or equal to x . It is also $h(D) < \sqrt{D}$ for $D \neq n^2 \mp 4$.

Now, we consider the case $D \equiv 3 \pmod 4$ and $D = n^2 \mp 2$ ($n \in Z$ is odd). Similarly, by applying Lemmas 2.1, 2.3 and using class number formula, we get

$$h(D) < \frac{\sqrt{D}(log4D + 1,478)}{2log(2D \mp 1)} < \sqrt{D} \quad \text{and} \quad h(D) \leq \|\frac{\sqrt{D}(log4D + 1,478)}{2log(2D \mp 1)}\|$$

It is also true for $D \neq n^2 \mp 2$.

4. Proof of Theorem

Specially, if we write Ds depend on prime p in the forms of $D = p^2 - 4$, $D = p^2 - 2$ we can immediately see that $h(D) < p$ for the class numbers of real quadratic fields $K = Q(\sqrt{p^2 - 4})$, $K = Q(\sqrt{p^2 - 2})$ from Lemma 2.2.. Therefore we have $h(D) \not\equiv 0 \pmod p$ for above mentioned real quadratic fields.

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