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On Class Numbers of Real Quadratic Fields with Certain Fundamental Discriminants

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Abstract. Let N denote the sets of positive integers and $D \in N$ be square free, and let χ_D, $h = h(D)$ denote the non-trivial Dirichlet character, the class number of the real quadratic field $K = Q(\sqrt{D})$, respectively.
Ono proved the theorem in [2] by applying Sturm's Theorem on the congruence of modular form to Cohen's half integral weight modular forms. Later, Dongho Byeon proved a theorem and corollary in [1] by refining Ono's methods.
In this paper, we will give a theorem for certain real quadratic fields by considering above mentioned studies. To do this, we shall obtain an upper bound different from current bounds for $L(1,\chi_D)$ and use Dirichlet's class number formula.

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1. Introduction

Let Z_p, N, Q denote the the ring of p-adic integers, positive integers and rational numbers, respectively. Let $R_p(D)$ denote the p-adic regulator of K, $|.|_p$ denote the usual multiplicative p-adic valuation normalized $|p|_p = \frac{1}{p}$, and let $L(s,\chi_D)$ denote the L-function attached to χ_D. Throughout $D \in N$ will be assumed square free, $K = Q(\sqrt{D})$ will denote the real quadratic field, and the class number of K will be denoted by $h = h(D)$. Let Δ denote the discriminant, ε_D the fundamental unit of K.

2. Preliminaries

Theorem 1 ([2]). *Let $p > 3$ be prime. If there is a fundamental discriminant D_0 coprime to p for which*

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(i) $(-1)^{\frac{p-1}{2}} D_0 > 0$

(ii) $|B(\frac{p-1}{2}, \chi_{D_0})| = 1$

then

$$\#\left\{0 < D < X| \quad h(D) \not\equiv 0 \pmod p, \quad \chi_D(p) = 0, \quad |\frac{R_p(D)}{\sqrt{D}}|_p = 1\right\} \gg_p \frac{\sqrt{X}}{logX}.$$

Here $B(\frac{p-1}{2}, \chi_{D_0})$ *is the* $\frac{p-1}{2}$*st generalized Bernoulli number with character* χ_{D_0}.

D. Byeon given in [1] the following theorems by refining Ono's theorem above mentioned for any prime $p > 3$.

Theorem 2. *Let* $p > 3$ *be prime.*

(a) If $p \equiv 1 \pmod 4$*, then the fundamental discriminant* $D_0 > 0$ *of the real quadratic field* $Q(\sqrt{p-2})$ *satisfies the conditions (i) and (ii).*

(b) If $p \equiv 3 \pmod 4$*, then the fundamental discriminant* $D_0 < 0$ *of the real quadratic field* $Q(\sqrt{-(p-4})$ *satisfies the conditions (i) and (ii).*

Theorem 3. *Let* $p > 3$ *be prime. Then*

$$\#\left\{0 < D < X| \quad h(D) \not\equiv 0 \pmod p, \quad \chi_D(p) = \delta, \quad |R_p(D)|_p = \frac{1}{p}\right\} \gg_p \frac{\sqrt{X}}{logX}.$$

3. Main Theorem

Main Theorem. *Let* $p > 3$ *be a prime. If* $p \equiv 3 \pmod 4$*, then the fundamental discriminant* $D_0 > 0$ *of the real quadratic fields* $K = Q(\sqrt{p^2-4})$ *and* $K = Q(\sqrt{p^2-2})$ *satisfies the conditions (i) and (ii).*

In order to prove the main theorem we need the following lemmas.

Lemma 1. *Assume* D *is a prime.*

(i) If $D \equiv 1 \pmod 4$*, we have*

$$\varepsilon_D > \begin{cases} ||\sqrt{D \mp 1}|| & \text{if } D = n^2 \mp 4, (n \in Z), \\ ||\sqrt{4D \mp 1}|| & \text{otherwise,} \end{cases}$$

where "$|| * ||$" represents great value function of a real number.

(ii) If $D \equiv 3 \pmod 4$*, we have*

$$\varepsilon_D > \begin{cases} 2D \mp 1 & \text{if } D = n^2 \mp 2, \\ 8D \mp 1 & \text{otherwise.} \end{cases}$$

Proof. (i) Let $\varepsilon_D = \frac{t+u\sqrt{D}}{2} > 1$ be the fundamental unit of $K = Q(\sqrt{D})$. Since ε_D is equal to the fundamental solution of the Pell's equation $x^2 - Dy^2 = \mp 4$, then we can write

$$\varepsilon_D{}^2 = (\frac{t+u\sqrt{D}}{2})^2 = \frac{1}{4}(\sqrt{Du^2 \mp 4} + u\sqrt{D})^2 > \frac{Du^2 \mp 1}{\sqrt{2}} \geq \begin{cases} \frac{D\mp 1}{\sqrt{2}} & \text{if } u = 1, \\ \frac{4D\mp 1}{\sqrt{2}} & \text{if } u > 1. \end{cases}$$

and

$$\varepsilon_D > \begin{cases} \|\sqrt{D \mp 1}\| & \text{if } D = n^2 \mp 4 \\ \|\sqrt{4D \mp 1}\| & \text{otherwise.} \end{cases}$$

(ii) If $D \equiv 3 \pmod 4$, then we have $\varepsilon_D^2 = (\frac{t+u\sqrt{D}}{2})^2$ from the least positive integer solution $(x, y) = (t, u)$ of Pell's equation $x^2 - Dy^2 = \mp 1$ [4]. Similarly, we can write

$$\varepsilon_D > \begin{cases} 2D \mp 1 & \text{if } D = n^2 \mp 2 \text{ for some odd integer } n, \\ 8D \mp 1 & \text{in the other cases.} \end{cases}$$

□

Lemma 2. *Let $D \in N$ be square free, then $h(D) < \sqrt{D}$.*

In order to prove this we need the following Lemma [3].

Lemma 3. *Let γ be Euler's constant, then*

$$|L(1, \chi_D)| \leq \begin{cases} \frac{1}{4}(log\Delta + 2 + \gamma - log\pi) & \textit{if } 2 \mid \Delta, \\ \frac{1}{2}(log\Delta + 2 + \gamma - log4\pi) & \textit{otherwise.} \end{cases}$$

Proof. [Lemma 2] By Dirichlet's class number formula, we have

$$h(D) = \frac{\sqrt{\Delta}}{2log\varepsilon_D}|L(1, \chi_D)|$$

where Δ is a fundamental discriminant of a quadratic field defined by

$$\Delta = \begin{cases} 4D & \text{if } D \equiv 2, 3 \pmod 4, \\ D & \text{if } D \equiv 1 \pmod 4). \end{cases}$$

First, we consider the case $D \equiv 1 \pmod 4$ and $D = n^2 \mp 4$. Thus, by the upper bound for $L(1, \chi_D)$ in Lemma 3, and from Lemma 1 we have that

$$h(D) < \frac{\sqrt{D}(logD + 1,478)}{4log\|\sqrt{D \mp 1}\|} = \frac{\sqrt{D}(logD + 1,478)}{2log\|(D \mp 1)\|} < \sqrt{D}, \quad (D > 5).$$

Moreover, we can write $h(D) \leq \|\frac{\sqrt{D}(logD+1,478)}{2log(D\mp 1)}\|$ where "$\|x\|$" is the greatest integer less than or equal to x. It is also $h(D) < \sqrt{D}$ for $D \neq n^2 \mp 4$.

Now, we consider the case $D \equiv 3 \pmod 4$ and $D = n^2 \mp 2$ ($n \in Z$ is odd). Similarly, by applying Lemmas 1, 3 and using class number formula, we get

$$h(D) < \frac{\sqrt{D}(log4D + 1,478)}{2log(2D \mp 1)} < \sqrt{D} \text{ and } h(D) \leq \| \frac{\sqrt{D}(log4D + 1,478)}{2log(2D \mp 1)} \|$$

It is also true for $D \neq n^2 \mp 2$. □

4. Proof of Main Theorem

Specially, if we write Ds depend on prime p in the forms of $D = p^2 - 4$, $D = p^2 - 2$ we can immediately prove that $h(D) < p$ for the class numbers of real quadratic fields $K = Q(\sqrt{p^2 - 4})$, $K = Q(\sqrt{p^2 - 2})$ from Lemma 2. Therefore we have $h(D) \not\equiv 0 \pmod p$ and it is clear that $|\frac{R_p(D)}{\sqrt{D}}|_p = 1$ for above mentioned real quadratic fields. □

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