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Hutton Uniformity in the Context of Fuzzy Soft Sets

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Abstract. In this paper, we introduce the concept of fuzzy soft uniformity in Hutton's sense. We define topological fuzzy soft remote neighborhood system and use this for investigating the relationship between fuzzy soft cotopology and fuzzy soft (quasi-)uniformity. We show the existence of the initial structure of fuzzy soft uniformities and also we prove the category of fuzzy soft uniform spaces is a topological category over $\mathbf{SET}^3$.

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1. Introduction

In 1999, Molodtsov [12] proposed a completely new concept called soft set theory to model uncertainty, which associates a set with a set of parameters. Later, Maji *et al.* [11] introduced the concept of fuzzy soft set which combines fuzzy sets and soft sets. Soft set and fuzzy soft set theories have a rich potential for applications in several directions. Up till now there are many spectacular and creative works about the theories of soft set and fuzzy soft set in the literature (see [2, 3, 8, 9, 11, 13, 17]). Furthermore, Aygünoğlu *et al.* [4] studied the topological structure of fuzzy soft sets based on the sense of Šostak [16].

It is well-known that uniformity is a very important concept close to topology and a convenient tool for investigating topology. Fuzzy versions of (quasi-)uniformity theory were established by Hutton [7], Lowen [10], Höhle [6] and Shi [14, 15]. Fuzzy (quasi-)uniformity in Hutton's sense has been accepted by many authors and has attracted wide attention in the literature, despite this.

In this paper, we give an approach to the concept of fuzzy soft uniformity in the sense of Hutton which is compatible with the fuzzy soft topology. The structure of this paper is organized as follows. In Section 2, we give some preliminary concepts and properties. In Section 3, we give the definition of fuzzy soft remote neighborhood system and investigate relations

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between fuzzy soft cotopological space and fuzzy soft remote neighborhood system. In Section 4, we define fuzzy soft uniformity in the sense of Hutton and we study the relationship between fuzzy soft cotopology and fuzzy soft uniformity by using fuzzy soft remote neighborhood system. In the last section, we introduce and characterize the initial structure of fuzzy soft uniform spaces.

2. Preliminaries

Throughout this paper, L is a complete lattice, M is a completely distributive lattice and there is an order-reversing involution $'$ on L. Let a, b be elements in L. An element a in L is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprimes of L is denoted by $c(L)$. We say a is way below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$) or $b \gg a$ ($b \rhd a$), if for every directed (arbitrary) subset $D \subseteq L$, $\vee D \geq b$ implies $a \leq d$ for some $d \in D$. Clearly if $a \in L$ is coprime, then $a \ll b$ if and only if $a \lhd b$. A complete lattice L is said to be continuous (completely distributive) if every element in L is the supremum of all elements way below (wedge below) it.

Proposition 1. *[5] Let L be a complete lattice. The following conditions are equivalent:*

(i) L is completely distributive.

(ii) L is distributive continuous lattice with enough coprimes.

(iii) The operator $\vee : Low(L) \to L$ sending every lower set to its supremum has a left adjoint β, and in this case $\beta(a) = \{b \mid b \lhd a\}$.

From (iii) in the above proposition it is easy to see that the wedge below relation has the interpolation property in a completely distributive lattice, this is to say, $a \lhd b$ implies there is some $c \in L$ such that $a \lhd c \lhd b$.

Let E and K be arbitrary nonempty sets viewed on the sets of parameters. A fuzzy soft set f on X, is a mapping from E into L^X, i.e., $f_e := f(e)$ is an L-fuzzy set on X, for each $e \in E$ (see Figure 1). The family of all L-fuzzy soft sets on X is denoted by $(L^X)^E$. By 0_X and 1_X, we denote respectively the null fuzzy soft set and absolute fuzzy soft set. The complement of an L-fuzzy soft set f is denoted by f', where $f'_e(x) = (f_e(x))'$. The set of all coprimes of $(L^X)^E$ is denoted by $c((L^X)^E)$.

Definition 1 ([1, 11]). *(i) We say that f is a fuzzy soft subset of g and write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$.*

(ii) Union of f and g is the fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(iii) Intersection of f and g is the fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

Let $p \mid f$ denote the set $\{g \in (L^X)^E \mid p \not\sqsubseteq g \sqsubseteq f\}$ for $p \in c((L^X)^E)$ and $f \in (L^X)^E$.

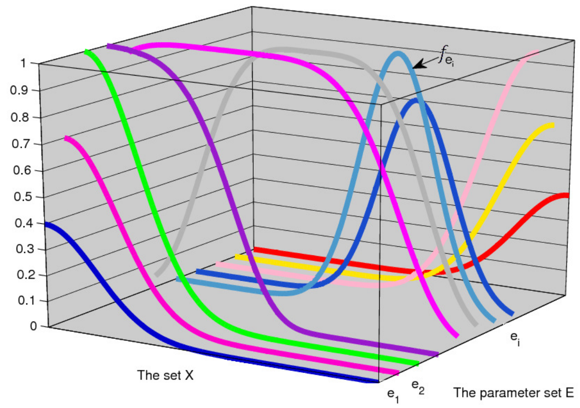

Figure 1: A fuzzy soft set f

Let $\varphi : X_1 \longrightarrow X_2$ and $\psi : E_1 \longrightarrow E_2$ be two functions, where E_1 and E_2 are parameter sets for the crisp sets X_1 and X_2, respectively. Define L-fuzzy soft mapping $\varphi_\psi{}^{\rightarrow} : (L^{X_1})^{E_1} \rightarrow (L^{X_2})^{E_2}$ and its L-fuzzy soft inverse mapping $\varphi_\psi{}^{\leftarrow} : (L^{X_2})^{E_2} \rightarrow (L^{X_1})^{E_1}$ by $(\varphi_\psi{}^{\rightarrow}(f))_{e_2}(y) = \bigvee_{\varphi(x)=y} \bigvee_{\psi(e_1)=e_2} f_{e_1}(x)$, for all $f \in (L^{X_1})^{E_1}$, $y \in X_2$, $e_2 \in E_2$ and $(\varphi_\psi{}^{\leftarrow}(g))_{e_1}(x) = g_{\psi(e_1)}(\varphi(x))$, for all $e_1 \in E_1$, $x \in X_1$ and $g \in (L^{X_2})^{E_2}$.

We refer to [3, 4, 9, 11] for all the basic definitions and notations related to fuzzy soft sets and fuzzy soft mappings.

Definition 2 ([4]). *A mapping $\tau : K \rightarrow M^{(L^X)^E}$ is called an (L,M)-fuzzy (E,K)-soft topology on X if it satisfies the following conditions for each $k \in K$,*

(T1) $\tau_k(0_X) = \tau_k(1_X) = 1_M$.

(T2) $\tau_k(f \sqcap g) \geq \tau_k(f) \wedge \tau_k(g)$ *for each* $f, g \in (L^X)^E$.

(T3) $\tau_k(\bigsqcup_{i\in\Lambda} f_i) \geq \bigwedge_{i\in\Lambda} \tau_k(f_i)$ *for each* $\{f_i\}_{i\in\Lambda} \subseteq (L^X)^E$.

The pair (X,τ) is called an (L,M)-fuzzy (E,K)-soft topological space

Example 1. *Let E be a parameter set, $I = [0,1]$, $K = N$ be the set of natural numbers and $\tau : K \rightarrow I^{(I^X)^E}$ be defined as follows: for all $k \in K$,*

$$\tau_k(f) = \begin{cases} 1, & \text{if } f = 0_X, 1_X, \\ \frac{1}{k}, & \text{otherwise.} \end{cases} \tag{1}$$

It is easy to testify that τ is a fuzzy soft topology on X.

Definition 3 ([4]). *A mapping $\mathscr{T} : K \rightarrow M^{(L^X)^E}$ is called an (L,M)-fuzzy (E,K)-soft cotopology on X if it satisfies the following conditions for each $k \in K$,*

(C1) $\mathscr{T}_k(0_X) = \mathscr{T}_k(1_X) = 1_M$.

(C2) $\mathscr{T}_k(f \sqcup g) \geq \mathscr{T}_k(f) \wedge \mathscr{T}_k(g)$ *for all* $f, g \in (L^X)^E$.

(C3) $\mathscr{T}_k(\sqcap_{i\in\Lambda} f_i) \geq \bigwedge_{i\in\Lambda} \mathscr{T}_k(f_i)$ *for all* $\{f_i\}_{i\in\Lambda} \subseteq (L^X)^E$.

The pair $(X, \mathscr{T})$ *is called an* (L,M)*-fuzzy* (E,K)*-soft cotopological space.*

Let $(X_1, \mathscr{T}^1)$ and $(X_2, \mathscr{T}^2)$ be an (L,M)-fuzzy (E_1,K_1)-soft cotopological space and an (L,M)-fuzzy (E_2,K_2)-soft cotopological space, respectively. A fuzzy soft mapping $\varphi_{\psi,\eta} : (X_1, \mathscr{T}^1) \to (X_2, \mathscr{T}^2)$ is said to be continuous if $\mathscr{T}^1_k(\varphi^{\leftarrow}_{\psi}(g)) \geq \mathscr{T}^2_{\eta(k)}(g)$ for each $g \in (L^{X_2})^{E_2}$, $k \in K_1$, where $\varphi : X_1 \to X_2$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ are classical functions.

Let **FSCTOP**(L,M) denote the category of (L,M)-fuzzy (E,K)-soft cotopological spaces and continuous mappings.

If $\mathscr{T}$ is an (L,M)-fuzzy (E,K)-soft cotopology on X, then τ is an (L,M)-fuzzy (E,K)-soft topology on X, where $\tau : K \to M^{(L^X)^E}$ is defined by $\tau_k(f) = \mathscr{T}_k(f')$, for each $k \in K$.

Example 2. *Let* $X = \{x,y\}$ *be a classical set,* $E = \{e_1,e_2\}$, $K = \{k_1,k_2\}$ *be parameter sets,* $L = M = I = [0,1]$. *Define* $h \in (I^X)^E$ *as follows:* $h_{e_1}(x) = 0.6$, $h_{e_1}(y) = 0.5$, $h_{e_2}(x) = 0.8$ *and* $h_{e_2}(y) = 0.6$. *Then the mapping* $\mathscr{T} : K \to I^{(I^X)^E}$ *which is defined as follows is a fuzzy soft cotopology on* X*:*

$$\mathscr{T}_k(f) = \begin{cases} 1, & \text{if } f = 0_X, 1_X, k \in K \\ 0.7, & \text{if } f = h, k = k_1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

3. Fuzzy Soft Remote Neighborhood System

In this section, we define fuzzy soft remote neighborhood system and give the relationships between fuzzy soft remote neighborhood system and fuzzy soft cotopological space. If the parameter sets E and K are both one pointed, then we obtain the results given in the paper of [18].

Definition 4. *A topological fuzzy soft remote neighborhood system is a set* $\mathscr{R} = \{R^p \mid p \in c((L^X)^E)\}$ *of mappings* $R^p : K \to M^{(L^X)^E}$ *such that for each* $k \in K$*:*

(RN1) $R^p_k(1_X) = 0_M$, $R^p_k(0_X) = 1_M$.

(RN2) $R^p_k(f) \neq 0_M$ *implies* $p \not\sqsubseteq f$.

(RN3) $R^p_k(f \sqcup g) = R^p_k(f) \wedge R^p_k(g)$.

(RN4) $R^p_k(f) = \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R^r_k(g)$.

Lemma 1. *Let* $\mathscr{T} : K \to M^{(L^X)^E}$ *be an* (L,M)*-fuzzy* (E,K)*-soft cotopology. Then the followings are valid.*

(1) $\mathscr{R}_{\mathscr{T}} = \{R^p_{\mathscr{T}} \mid p \in c((L^X)^E)\}$ *is a topological fuzzy soft remote neighborhood system, where* $R^p_{\mathscr{T}}$ *is defined by for all* $k \in K$, $p \in c((L^X)^E)$ *and* $f \in (L^X)^E$, *as*

$$(R^p_{\mathscr{T}})_k(f) = \begin{cases} \bigvee_{g \in p|f} \mathscr{T}_k(g), & \text{if } p \not\sqsubseteq f; \\ 0_M, & \text{otherwise}. \end{cases} \tag{3}$$

(2) If $\mathscr{T}$ *and* $\mathscr{S}$ *are two* (L,M)*-fuzzy* (E,K)*-soft cotopologies which determine the same topological fuzzy soft remote neighborhood system, then* $\mathscr{T} = \mathscr{S}$.

Proof. By Definition 4, we need to show (RN1)-(RN4) in the following. First of all, (RN1)-(RN2) are trivial.

(RN3): Let $k \in K$ and $f, g \in (L^X)^E$. From the definition of $R^p_{\mathscr{T}}$, we have $f \sqsubseteq g$ implies $(R^p_{\mathscr{T}})_k(f) \geq (R^p_{\mathscr{T}})_k(g)$. This is to say $(R^p_{\mathscr{T}})_k(f \sqcup g) \leq (R^p_{\mathscr{T}})_k(f) \wedge (R^p_{\mathscr{T}})_k(g)$. Suppose that $\alpha \lhd ((R^p_{\mathscr{T}})_k(f) \wedge (R^p_{\mathscr{T}})_k(g))$, where $\alpha \in c(M)$. Then $\alpha \lhd (R^p_{\mathscr{T}})_k(f)$ and $\alpha \lhd (R^p_{\mathscr{T}})_k(g)$. Then there exist $u \in p \mid f$ and $v \in p \mid g$ such that $\alpha \leq \mathscr{T}_k(u)$ and $\alpha \leq \mathscr{T}_k(v)$. Therefore,
$\alpha \leq \mathscr{T}_k(u) \wedge \mathscr{T}_k(v) \leq \mathscr{T}_k(u \sqcup v)$.

It is clear that $p \not\sqsubseteq (u \sqcup v)$, $f \sqcup g \sqsubseteq u \sqcup v$. Hence by the definition of $R^p_{\mathscr{T}}$, we have
$\alpha \leq (R^p_{\mathscr{T}})_k(f \sqcup g)$.

From the arbitrariness of α, we get for each $k \in K$, $(R^p_{\mathscr{T}})_k(f \sqcup g) \geq (R^p_{\mathscr{T}})_k(f) \wedge (R^p_{\mathscr{T}})_k(g)$.

(RN4): For each $g \in p \mid f$ and $k \in K$, we have

$$\mathscr{T}_k(g) \leq \bigwedge_{p \not\sqsubseteq g} (R^p_{\mathscr{T}})_k(g) \leq (R^p_{\mathscr{T}})_k(g) \leq (R^p_{\mathscr{T}})_k(f).$$

Therefore, $(R^p_{\mathscr{T}})_k(f) = \bigvee_{g \in p|f} \mathscr{T}_k(g) \leq \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{T}})_k(g) \leq (R^p_{\mathscr{T}})_k(f)$.

This means that for each $k \in K$, $(R^p_{\mathscr{T}})_k(f) = \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{T}})_k(g)$.

(2) For the proof of the second claim of Lemma 1, it is sufficient to show the validity of the following equality; $\mathscr{T}_k(f) = \bigwedge_{p \not\sqsubseteq f} (R^p_{\mathscr{T}})_k(f)$ for all $f \in (L^X)^E$ and $k \in K$.

Obviously, $\mathscr{T}_k(f) \leq \bigwedge_{p \not\sqsubseteq f} (R^p_{\mathscr{T}})_k(f)$ for all $f \in (L^X)^E$ and $k \in K$. So it is enough to prove $\mathscr{T}_k(f) \geq \bigwedge_{p \not\sqsubseteq f} (R^p_{\mathscr{T}})_k(f)$. In fact, we have

$$\bigwedge_{p \not\sqsubseteq f} (R^p_{\mathscr{T}})_k(f) = \bigwedge_{p \not\sqsubseteq f} \bigvee_{g \in p|f} \mathscr{T}_k(g) = \bigvee_{A \in \Pi_{p \not\sqsubseteq f} p|f} \bigwedge_{p \not\sqsubseteq f} \mathscr{T}_k(A(p)) \leq \bigvee_{A \in \Pi_{p \not\sqsubseteq f} p|f} \mathscr{T}_k(\sqcap_{p \not\sqsubseteq f} A(p)) = \mathscr{T}_k(f).$$

The last equality is due to $\sqcap_{p \not\sqsubseteq f} A(p) = f$ for every $A \in \Pi_{p \not\sqsubseteq f} p \mid f$. □

Lemma 2. *Let* $\mathscr{R} = \{R^p \mid p \in c((L^X)^E)\}$ *be a topological fuzzy soft remote neighborhood system and* $\mathscr{T} : K \to M^{(L^X)^E}$ *be defined by for all* $k \in K$ *and* $f \in (L^X)^E$,

$$\mathscr{T}_k(f) = \bigwedge_{p \not\sqsubseteq f} R^p_k(f).$$

Then $\mathscr{T}$ *is an* (L,M)*-fuzzy* (E,K)*-soft cotopology on* X. *Furthermore, if* $\mathscr{R}$ *and* $\mathscr{P}$ *are two topological fuzzy soft remote neighborhood systems which determine the same* (L,M)*-fuzzy* (E,K)*-soft cotopology,then* $\mathscr{R} = \mathscr{P}$.

Proof. By Definition 3, (C1) is trivial.
(C2) is proved by the following equations: for each $k \in K$,

$$\mathscr{T}_k(f \sqcup g) = \bigwedge_{p \not\sqsubseteq (f \sqcup g)} R_k^p(f \sqcup g) \geq (\bigwedge_{p \not\sqsubseteq f} R_k^p(f)) \wedge (\bigwedge_{p \not\sqsubseteq g} R_k^p(g)) = \mathscr{T}_k(f) \wedge \mathscr{T}_k(g).$$

Finally, (C3) is shown by the following computation: for each $k \in K$,

$$\mathscr{T}_k(\sqcap_{j\in J} f_j) = \bigwedge_{p \not\sqsubseteq \sqcap_{j\in J} f_j} R_k^p(\sqcap_{j\in J} f_j) = \bigwedge_{j\in J} \bigwedge_{p \not\sqsubseteq f_j} R_k^p(\sqcap_{j\in J} f_j) \geq \bigwedge_{j\in J} \bigwedge_{p \not\sqsubseteq f_j} R_k^p(f_j) = \bigwedge_{j\in J} \mathscr{T}_k(f_j).$$

This completes the proof.

Moreover, it is obvious that $\mathscr{R} = \mathscr{P}$ if $\mathscr{R}$ and $\mathscr{P}$ are two topological fuzzy soft remote neighborhood systems which determine the same fuzzy soft cotopology. □

Lemma 3. *Let $\mathscr{R} = \{R^p \mid p \in c((L^X)^E)\}$ be a set satisfying (RN1)-(RN3). Then the following statements are equivalent:*

(RN4) $R_k^p(f) = \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R_k^r(g)$.

(RN4)* $R_k^p(f) = \bigvee_{g \in p|f} (R_k^p(g) \wedge \bigwedge_{r \not\sqsubseteq g} R_k^r(f))$.

Proof. Suppose (RN4*) holds, i.e., $R_k^p(f) = \bigvee_{g \in p|f} (R_k^p(g) \wedge \bigwedge_{r \not\sqsubseteq g} R_k^r(f))$. Let $\alpha \in c(M)$ such that $\alpha \lhd R_k^p(f) = \bigvee_{g \in p|f} (R_k^p(g) \wedge \bigwedge_{r \not\sqsubseteq g} R_k^r(f))$. Then there exists some $g \in p \mid f$ such that

(1) $\alpha \lhd R_k^p(g)$;

(2) $\alpha \lhd R_k^r(f)$, for each $r \not\sqsubseteq g$.

It is clear that the meet of fuzzy soft sets containing f and fulfilling (1) and (2) is still of such kind. So we can define $g*$ to be the minimal fuzzy soft set containing f and fulfilling (1), (2), i.e., $\alpha \lhd R_k^p(g*)$ and $\alpha \lhd R_k^r(f)$ for all $r \not\sqsubseteq g*$. Thus, for each $r \not\sqsubseteq g*$, it follows from $\alpha \lhd R_k^r(f)$ that there exists $h^r \in r \mid f$ such that

(3) $R_k^r(h^r) \rhd \alpha$;

(4) $R_k^r(f) \rhd \alpha$, for each $g \not\sqsubseteq h^r$.

It is easy to check that $g * \sqcap h^r$ satisfies (1) and (2). Hence, by the minimality of $g*$, it follows that $g* \sqsubseteq g * \sqcap h^r$. Therefore, $g* \sqsubseteq h^r$. Then we get that $\alpha \lhd R_k^r(h^r) \leq R_k^r(g*)$ for all $r \not\sqsubseteq g*$. Thus, $\alpha \leq \bigwedge_{r \not\sqsubseteq g*} R_k^r(g*)$. Therefore, $\alpha \leq \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R_k^r(g)$.

From the arbitrariness of α, we have $R_k^p(f) \leq \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R_k^r(g)$, for each $k \in K$. Since for each $k \in K$, $R_k^p(f) \geq \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R_k^r(g)$ is obvious. We have $R_k^p(f) = \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R_k^r(g)$, as desired. □

4. Fuzzy Soft Uniform Spaces

In this section, we introduce the concept of fuzzy soft uniformity as a parameterized family of Hutton uniformity in the spirit of fuzzy soft topology. Also, we consider the categorical relationship between the fuzzy soft remote neighborhood system and fuzzy soft uniform space.

Let $\mathscr{H}(X,E)$ denote the family of all mappings $\lambda : (L^X)^E \to (L^X)^E$ such that:

(1) $f \sqsubseteq \lambda(f)$ for all $f \in (L^X)^E$.

(2) $\lambda(\bigsqcup_{j\in J} f_j) = \bigsqcup_{j\in J} \lambda(f_j)$ for all $\{f_j\}_{j\in J} \subseteq (L^X)^E$.

λ^* denotes the biggest element of $\mathscr{H}(X,E)$, i.e., $\lambda^*(f) = 0_X$ when $f = 0_X$ and $\lambda^*(f) = 1_X$ otherwise.

For $\lambda, \mu \in \mathscr{H}(X,E)$, we have that $\lambda\Delta\mu \in \mathscr{H}(X,E)$ and $\lambda \circ \mu \in \mathscr{H}(X,E)$, where $\lambda\Delta\mu(f) = \sqcap\{\lambda(g) \sqcup \mu(h) \mid f = g \sqcup h\}$ and $\lambda \circ \mu(f) = \lambda(\mu(f))$. For each $\lambda \in \mathscr{H}(X,E)$, let $\lambda^{\lhd}(g) = \sqcap\{h \in (L^X)^E \mid \lambda(h') \sqsubseteq g'\}$.

Proposition 2.

(i) $\lambda^{\lhd} \in \mathscr{H}(X,E)$.

(ii) $(\lambda^{\lhd})^{\lhd} = \lambda$.

(iii) $(\lambda \circ \mu)^{\lhd} = \mu^{\lhd} \circ \lambda^{\lhd}$.

(iv) $\lambda \leq \mu$ *implies* $\lambda^{\lhd} \leq \mu^{\lhd}$.

(v) $(\lambda\Delta\mu)^{\lhd} = \lambda^{\lhd}\Delta\mu^{\lhd}$.

(vi) $(\bigvee_{i\in\Gamma} \lambda_i)^{\lhd} = \bigvee_{i\in\Gamma} \lambda_i^{\lhd}$.

(vii) *If* $\lambda_1 \leq \lambda_2$ *and* $\mu_1 \leq \mu_2$, *then* $\lambda_1\Delta\mu_1 \leq \lambda_2\Delta\mu_2$.

Suppose $\varphi_\psi : (L^X)^E \to (L^Y)^F$ be a fuzzy soft mapping and $\lambda \in \mathscr{H}(Y,F)$, define $\varphi_\psi^{\Leftarrow}(\lambda) : (L^X)^E \to (L^X)^E$ by $\varphi_\psi^{\Leftarrow}(\lambda)(f) = \varphi_\psi^{\leftarrow} \circ \lambda \circ \varphi_\psi^{\rightarrow}(f)$ for all $f \in (L^X)^E$.

Proposition 3.

(i) $\varphi_\psi^{\Leftarrow}(\lambda) \in \mathscr{H}(X,E)$.

(ii) $\lambda \leq \mu$ *implies* $\varphi_\psi^{\Leftarrow}(\lambda) \leq \varphi_\psi^{\Leftarrow}(\mu)$.

(iii) $\varphi_\psi^{\Leftarrow}(\lambda^{\lhd}) = (\varphi_\psi^{\Leftarrow}(\lambda))^{\lhd}$.

(iv) $\varphi_\psi^{\Leftarrow}(\lambda \circ \mu) \leq \varphi_\psi^{\Leftarrow}(\lambda) \circ \varphi_\psi^{\Leftarrow}(\mu)$.

Definition 5. *An* (L,M)*-fuzzy* (E,K)*-soft quasi-uniformity is a mapping* $\mathscr{U} : K \to M^{\mathscr{H}(X,E)}$ *which satisfies the following conditions: for each* $k \in K$,

(U1) $\mathscr{U}_k(\lambda^*) = 1_M$.

(U2) $\mathscr{U}_k(\lambda\Delta\mu) \geq \mathscr{U}_k(\lambda) \wedge \mathscr{U}_k(\mu)$ *for each* $\lambda, \mu \in \mathscr{H}(X, E)$.

(U3) *If* $\lambda \geq \mu$, *then* $\mathscr{U}_k(\lambda) \geq \mathscr{U}_k(\mu)$.

(U4) $\mathscr{U}_k(\lambda) \leq \bigvee\{\mathscr{U}_k(\mu) \mid \mu \circ \mu \leq \lambda\}$ *for all* $\lambda \in \mathscr{H}(X, E)$.

The pair $(X, \mathscr{U})$ *is called an* (L, M)*-fuzzy* (E, K)*-soft quasi-uniform space. An* (L, M)*-fuzzy* (E, K)*-soft quasi-uniform space* $(X, \mathscr{U})$ *is said to be an* (L, M)*-fuzzy* (E, K)*-soft uniform space if* $\mathscr{U}$ *provides the condition:*

(U) $\mathscr{U}_k(\lambda) \leq \bigvee\{\mathscr{U}_k(\mu) \mid \mu \leq \lambda^{\triangleleft}\}$ *for each* $k \in K, \lambda \in \mathscr{H}(X, E)$.

Given two $\mathscr{U}^1$ and $\mathscr{U}^2$ uniformities on X, we say $\mathscr{U}^1$ is finer than $\mathscr{U}^2$ (or $\mathscr{U}^2$ is coarser than $\mathscr{U}^1$) iff $\mathscr{U}^1_k(\lambda) \geq \mathscr{U}^2_k(\lambda)$ for each $k \in K$ and $\lambda \in \mathscr{H}(X, E)$.

A fuzzy soft mapping $\varphi_{\psi,\eta} : (X_1, \mathscr{U}^1) \to (X_2, \mathscr{U}^2)$ is called (quasi-) uniformly continuous if $\mathscr{U}^1_k(\varphi^{\Leftarrow}_{\psi}(\mu)) \geq \mathscr{U}^2_{\eta(k)}(\mu)$ for all $\mu \in \mathscr{H}(X_2, E_2)$, $k \in K_1$, where $(X_1, \mathscr{U}^1)$ and $(X_2, \mathscr{U}^2)$ is an (L, M)-fuzzy (E_1, K_1)-soft uniform space and an (L, M)-fuzzy (E_2, K_2)-soft uniform space, respectively.

Theorem 1. *Let* $(X_1, \mathscr{U}^1)$, $(X_2, \mathscr{U}^2)$ *and* $(X_3, \mathscr{U}^3)$ *be* (L, M)*-fuzzy* (E_i, K_i)*-soft uniform spaces, respectively for* $i = 1, 2, 3$. *If* $\varphi_{\psi,\eta} : (X_1, \mathscr{U}^1) \to (X_2, \mathscr{U}^2)$ *and* $\varphi^*_{\psi^*,\eta^*} : (X_2, \mathscr{U}^2) \to (X_3, \mathscr{U}^3)$ *are uniformly continuous, then the composition is uniformly continuous.*

The category of (L, M)-fuzzy (E, K)-soft quasi-uniform spaces and continuous mappings is denoted by $\mathbf{HFSU}(L, M)$.

Theorem 2. *Let* $(X, \mathscr{U})$ *be an* (L, M)*-fuzzy* (E, K)*-soft quasi-uniform space and* $R^p_{\mathscr{U}} : K \to M^{(L^X)^E}$ *be defined by for all* $f \in (L^X)^E$,

$$(R^p_{\mathscr{U}})_k(f) = \bigvee_{p \not\sqsubseteq h} \bigvee_{\lambda(h') \sqsubseteq f'} \mathscr{U}_k(\lambda).$$

Then $\mathscr{R}_{\mathscr{U}} = \{R^p_{\mathscr{U}} \mid p \in c((L^X)^E)\}$ *is a topological fuzzy soft remote neighborhood system.*

Proof. We need to check (RN1)-(RN4). (RN1), (RN2) and (RN3) are straightforward, what remains is to prove.

(RN4): From Lemma 3, we know that it is equivalent to check (RN4*). Since $(R^p_{\mathscr{U}})_k(f) \geq \bigvee_{g \in p|f} \left((R^p_{\mathscr{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{U}})_k(f)\right)$, for all $k \in K$, is obvious.

It is sufficient to show that $(R^p_{\mathscr{U}})_k(f) \leq \bigvee_{g \in p|f} \left((R^p_{\mathscr{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{U}})_k(f)\right)$, for each $k \in K$. Let $k \in K$ and $\alpha \in c(M)$ such that $\alpha \lhd (R^p_{\mathscr{U}})_k(f)$, that is,

$$\alpha \lhd (R^p_{\mathscr{U}})_k(f) = \bigvee_{p \not\sqsubseteq h} \bigvee_{\lambda(h') \sqsubseteq f'} \mathscr{U}_k(\lambda) \leq \bigvee_{p \not\sqsubseteq h} \bigvee_{\lambda(h') \sqsubseteq f'} \bigvee_{\mu \circ \mu \leq \lambda} \mathscr{U}_k(\mu).$$

Then there exist $h \in (L^X)^E$, $\lambda \in \mathscr{H}(X, E)$ and $\mu \in \mathscr{H}(X, E)$ such that

$$p \not\sqsubseteq h \sqsupseteq (\mu(h'))' \sqsupseteq ((\mu \circ \mu)(h'))' \sqsupseteq (\lambda(h'))' \sqsupseteq f$$

and $\alpha \leq \mathscr{U}_k(\mu)$.

Let $g = (\mu(h'))'$. Then $g \in p \mid f$. Furthermore, we have

$$(R^p_{\mathscr{U}})_k(g) = \bigvee_{p \not\sqsubseteq d} \bigvee_{\nu(d') \sqsubseteq g'} \mathscr{U}_k(\nu) \geq \bigvee_{\nu(h') \sqsubseteq g'} \mathscr{U}_k(\nu) \geq \mathscr{U}_k(\mu) \geq \alpha$$

and

$$\bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{U}})_k(f) = \bigwedge_{r \not\sqsubseteq g} \bigvee_{r \not\sqsubseteq d} \bigvee_{\nu(d') \sqsubseteq f'} \mathscr{U}_k(\nu) \geq \bigwedge_{r \not\sqsubseteq g} \bigvee_{\nu(g') \sqsubseteq f'} \mathscr{U}_k(\nu) \geq \bigwedge_{r \not\sqsubseteq g} \mathscr{U}_k(\mu) \geq \alpha.$$

Then $\alpha \leq (R^p_{\mathscr{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{U}})_k(f)$. Therefore, $\alpha \leq \bigvee_{g \in p \mid f} \Big((R^p_{\mathscr{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{U}})_k(f) \Big)$.

From the arbitrariness of α, we have $(\mathscr{R}^p_{\mathscr{U}})_k(f) \leq \bigvee_{g \in p \mid f} \Big((R^p_{\mathscr{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathscr{U}})_k(f) \Big)$. □

Theorem 3. *Let $(X, \mathscr{U})$ be an (L, M)-fuzzy (E, K)-soft quasi-uniform space. Then, $\mathscr{R}^p_{\mathscr{U}}$ can also be written as follows:*

(i) $(\mathscr{R}^p_{\mathscr{U}})_k(f) = \bigvee_{p \not\sqsubseteq h} \bigvee_{\lambda \circ \lambda(h') \sqsubseteq f'} \mathscr{U}_k(\lambda)$.

(ii) $(\mathscr{R}^p_{\mathscr{U}})_k(f) = \bigvee_{h \in (L^X)^E} \bigvee_{p \not\sqsubseteq \lambda(h') \sqsupseteq (\lambda \circ \lambda(h'))' \sqsupseteq f} \mathscr{U}_k(\lambda)$.

(iii) $(\mathscr{R}^p_{\mathscr{U}})_k(f) = \bigvee_{p \not\sqsubseteq \lambda^{\triangleleft}(f)} \mathscr{U}_k(\lambda)$.

Proof. (i) and (ii) are trivial. (iii) can be obtained by the definition of $\lambda^{\triangleleft}$. □

From Lemma 2, we know that $\mathscr{T}_{\mathscr{U}}$ is an (L, M)-fuzzy (E, K)-soft cotopology on X and call it the generated (L, M)-fuzzy (E, K)-soft cotopology by $\mathscr{U}$.

Theorem 4. *Let $(X, \mathscr{T})$ be an (L, M)-fuzzy (E, K)-soft cotopological space. Then there is one (L, M)-fuzzy (E, K)-soft quasi uniformity $\mathscr{U}_{\mathscr{T}}$ on X such that the generated (L, M)-fuzzy (E, K)-soft cotopology by $\mathscr{U}_{\mathscr{T}}$ is just $\mathscr{T}$, i.e., $\mathscr{T} = \mathscr{T}_{\mathscr{U}_{\mathscr{T}}}$. This is to say that each (L, M)-fuzzy (E, K)-soft cotopological space is (L, M)-fuzzy (E, K)-soft quasi-uniformizable.*

Proof. Let $g \in (L^X)^E$ and $\lambda_g : (L^X)^E \to (L^X)^E$ be defined as follows:

$$\lambda_g(f) = \begin{cases} 1_X, & \text{if } f \not\sqsubseteq g; \\ g, & \text{if } 0_X \neq f \sqsubseteq g; \\ 0_X, & \text{otherwise.} \end{cases}$$

Then $\lambda_f \in \mathscr{H}(X, E)$ and $\lambda_f \circ \lambda_f = \lambda_f$. Define $\mathscr{U}_{\mathscr{T}} : K \to M^{\mathscr{H}(X,E)}$ by

$$(\mathscr{U}_{\mathscr{T}})_k(\lambda) = \bigvee \{ \bigwedge_{i=1}^{n} \mathscr{T}_k(g_i) \mid \lambda \geq \Delta^n_{i=1} \lambda_{g'_i}, n \in \mathbb{N} \}.$$

It is easy to verify that $\mathscr{U}_{\mathscr{T}}$ is an (L, M)-fuzzy (E, K)-soft quasi uniformity on X. Now we prove that $\mathscr{T} = \mathscr{T}_{\mathscr{U}_{\mathscr{T}}}$. Noting that $\lambda^{\triangleleft}_{g'_i}(f) = f$, from the definition of $\mathscr{T}_{\mathscr{U}_{\mathscr{T}}}$, we have for $k \in K$

$$(\mathscr{T}_{\mathscr{U}_{\mathscr{T}}})_k(f) = \bigwedge_{p \not\sqsubseteq f} \bigvee_{p \not\sqsubseteq \lambda^{\triangleleft}(f)} \bigvee \{ \bigwedge_{i=1}^{n} \mathscr{T}_k(g_i) \mid \lambda \geq \Delta^n_{i=1} \lambda_{g'_i}, n \in \mathbb{N} \} \geq \bigwedge_{p \not\sqsubseteq f} \mathscr{T}_k(f) = \mathscr{T}_k(f).$$

This is to say $\mathscr{T}_{\mathscr{U}_{\mathscr{T}}} \geq \mathscr{T}$. On the other hand, we have

$$\begin{aligned}(\mathscr{T}_{\mathscr{U}_{\mathscr{T}}})_k(f) &= \bigwedge_{p\not\sqsubseteq f} \bigvee_{p\not\sqsubseteq \lambda^{\triangleleft}(f)} \bigvee\{\bigwedge_{i=1}^{n} \mathscr{T}_k(g_i) \mid \lambda \geq \Delta_{i=1}^{n}\lambda_{g_i'}, n\in\mathbb{N}\}\\
&\leq \bigwedge_{p\not\sqsubseteq f} \bigvee_{p\not\sqsubseteq \lambda^{\triangleleft}(f)} \bigvee\{\bigwedge_{i=1}^{n} \mathscr{T}_k(g_i) \mid \lambda^{\triangleleft} \geq \Delta_{i=1}^{n}\lambda^{\triangleleft}_{g_i'}, n\in\mathbb{N}\}\\
&\leq \bigwedge_{p\not\sqsubseteq f} \bigvee_{p\not\sqsubseteq \lambda^{\triangleleft}(f)} \bigvee\{\bigwedge_{i=1}^{n} \mathscr{T}_k(g_i) \mid \lambda^{\triangleleft}(f) \geq \Delta_{i=1}^{n}\lambda_{g_i'}(f), n\in\mathbb{N}\}\\
&\leq \bigwedge_{p\not\sqsubseteq f} \bigvee\{\mathscr{T}_k(\sqcap_{j=1}^{m} g_j) \mid p \not\sqsubseteq \sqcap_{j=1}^{m} g_j \sqsupseteq f, m\in\mathbb{N}\}\\
&\leq \bigwedge_{p\not\sqsubseteq f} \bigvee\{\mathscr{T}_k(g) \mid p\not\sqsubseteq g \sqsupseteq f\} = \mathscr{T}_k(f).\end{aligned}$$

This completes the proof. □

Theorem 5. *If* $\varphi_{\psi,\eta} : (X_1, \mathscr{U}^1) \to (X_2, \mathscr{U}^2)$ *is quasi uniformly continuous, then* $\varphi_{\psi,\eta} : (X_1, \mathscr{T}_{\mathscr{U}^1}) \to (X_2, \mathscr{T}_{\mathscr{U}^2})$ *is fuzzy soft continuous.*

Proof. Let $g \in (L^{X_2})^{E_2}$, $k \in K_1$ and $\alpha \lhd (\mathscr{T}_{\mathscr{U}^2})_{\eta(k)}(g)$. Since $\varphi_{\psi,\eta} : (X_1, \mathscr{U}^1) \to (X_2, \mathscr{U}^2)$ is quasi uniformly continuous, we have $\mathscr{U}^1_k(\varphi_\psi^{\Leftarrow}(\mu)) \geq \mathscr{U}^2_{\eta(k)}(\mu)$ for all $\mu \in \mathscr{H}(X_2, E_2)$, $k \in K_1$. Hence,

$$\alpha \lhd (\mathscr{T}_{\mathscr{U}^2})_{\eta(k)}(g) = \bigwedge_{p\not\sqsubseteq g} \bigvee_{p\not\sqsubseteq\lambda^{\triangleleft}(g)} \mathscr{U}^2_{\eta(k)}(\lambda) \leq \bigwedge_{p\not\sqsubseteq g} \bigvee_{p\not\sqsubseteq\lambda^{\triangleleft}(g)} \mathscr{U}^1_k(\varphi_\psi^{\Leftarrow}(\lambda)).$$

Noting that $\varphi_\psi^{\rightarrow}(h) \not\sqsubseteq g$ when $h \not\sqsubseteq \varphi_\psi^{\leftarrow}(g)$, we can find some $\lambda_{(h)} \in \mathscr{H}(X_2, E_2)$ such that $\varphi_\psi^{\rightarrow}(h) \not\sqsubseteq \lambda_{(h)}(g)$ and $\alpha \leq \mathscr{U}^1_k(\varphi_\psi^{\Leftarrow}(\lambda_{(h)}))$. Now let $\nu_{(h)} = \varphi_\psi^{\Leftarrow}(\lambda_{(h)})$. Then $\nu_{(h)} \in \mathscr{H}(X_1, E_1)$ and $h \not\sqsubseteq \nu^{\triangleleft}_{(h)}(\varphi_\psi^{\leftarrow}(g))$. Hence,

$$\alpha \leq \bigwedge_{h\not\sqsubseteq\varphi_\psi^{\leftarrow}(g)} \mathscr{U}^1_k(\nu_{(h)}) \leq \bigwedge_{h\not\sqsubseteq\varphi_\psi^{\leftarrow}(g)} \bigvee_{h\not\sqsubseteq\nu^{\triangleleft}(\varphi_\psi^{\leftarrow}(g))} \mathscr{U}^1_k(\nu) = (\mathscr{T}_{\mathscr{U}^1})_k(\varphi_\psi^{\leftarrow}(g)).$$

Therefore, $(\mathscr{T}_{\mathscr{U}^2})_{\eta(k)}(g) \leq (\mathscr{T}_{\mathscr{U}^1})_k(\varphi_\psi^{\leftarrow}(g))$ from the arbitrariness of α. So, $\varphi_{\psi,\eta} : (X_1, \mathscr{T}_{\mathscr{U}^1}) \to (X_2, \mathscr{T}_{\mathscr{U}^2})$ is fuzzy soft continuous. □

Theorem 6. *If* $\varphi_{\psi,\eta} : (X_1, \mathscr{T}^1) \to (X_2, \mathscr{T}^2)$ *is fuzzy soft continuous, then* $\varphi_{\psi,\eta} : (X_1, \mathscr{U}_{\mathscr{T}^1}) \to (X_2, \mathscr{U}_{\mathscr{T}^2})$ *is quasi uniformly continuous.*

Proof. Let $\varphi_{\psi,\eta} : (X_1, \mathscr{T}^1) \to (X_2, \mathscr{T}^2)$ be continuous. From the definition of $\mathscr{U}_{\mathscr{T}^1}$, we know that for each $k \in K$, $(\mathscr{U}_{\mathscr{T}^2})_{\eta(k)}(\lambda) = \bigvee\{\bigwedge_{i=1}^{n} \mathscr{T}^2_{\eta(k)}(g_i) \mid \lambda \geq \Delta_{i=1}^{n}\lambda_{g_i'},\ n \in \mathbb{N}\}$. Moreover, if $\lambda \geq \Delta_{i=1}^{n}\lambda_{g_i'}$, then we have

$$\varphi_\psi^{\Leftarrow}(\lambda) \geq \varphi_\psi^{\Leftarrow}(\Delta_{i=1}^{n}\lambda_{g_i'}) = \bigwedge_{i=1}^{n}(\varphi_\psi^{\Leftarrow}(\lambda(g_i'))) = \bigwedge_{i=1}^{n}\lambda_{\varphi_\psi^{\leftarrow}(g_i)'}.$$

Since $\varphi_{\psi,\eta} : (X_1, \mathscr{T}^1) \to (X_2, \mathscr{T}^2)$ is continuous, we have $\bigwedge_{i=1}^n \mathscr{T}^2_{\eta(k)}(g_i) \leq \bigwedge_{i=1}^n \mathscr{T}^1_k(\varphi_\psi^{\leftarrow}(g_i))$. Hence, $(\mathscr{U}_{\mathscr{T}^2})_{\eta(k)}(\lambda) \geq (\mathscr{U}_{\mathscr{T}^1})_k(\varphi_\psi^{\Leftarrow}(\lambda))$.

Therefore, $\varphi_{\psi,\eta} : (X_1, \mathscr{U}_{\mathscr{T}^1}) \to (X_2, \mathscr{U}_{\mathscr{T}^2})$ is quasi uniformly continuous. □

Theorem 7. *Let G : **FSCTOP**$(L,M) \to$ **HFSU**(L,M) be defined by $G((X,\mathscr{T})) = (X, \mathscr{U}_{\mathscr{T}})$. Then G is an embedding functor from **FSCTOP**(L,M) to **HFSU**(L,M).*

5. Category of Fuzzy Soft Uniform Spaces

In this section, we will show that the category **HFSU**(L,M) of (L,M)-fuzzy (E,K)-soft uniform spaces and continuous functions is a topological category over **SET**^{3}.

Theorem 8. *Let $\{(X_i, \mathscr{U}^i)\}_{i\in\Gamma}$ be a family of (L,M)-fuzzy (E_i,K_i)-soft uniform spaces, X be a set, E,K be the parameter sets and for each $i \in \Gamma$, $\varphi_i : X \to X_i, \psi_i : E \to E_i$ and $\eta_i : K \to K_i$ be a function. We define the mapping $\mathscr{U} : K \to M^{\mathscr{H}(X,E)}$ by:*

$$\mathscr{U}_k(\lambda) = \bigvee\{\bigwedge_{j=1}^n \mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\lambda_{i_j}) \mid \Delta^n_{j=1}(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j}) \leq \lambda\}, \text{ for each } k \in K,$$

where $\vee$ is taken over the finite index set $\{i_1, \ldots, i_n\} \subseteq \Gamma$. Then the following items are satisfied.

(1) $\mathscr{U}$ is the coarsest (L,M)-fuzzy (E,K)-soft uniformity on X for which each $(\varphi_{\psi,\eta})_i$ is uniformly continuous function.

(2) A function $\varphi_{\psi,\eta} : (Z, \mathscr{V}) \to (X, \mathscr{U})$ is uniformly continuous iff $(\varphi_{\psi,\eta})_i \circ \varphi_{\psi,\eta} : (Z, \mathscr{V}) \to (X_i, \mathscr{U}^i)$ is uniformly continuous for all $i \in \Gamma$.

Proof. (1) Firstly, we will prove that $\mathscr{U}$ is an (L,M)-fuzzy (E,K)-soft uniformity on X.

(U1) and (U3) are clear.

(U2): Suppose there exist $\lambda, \mu \in \mathscr{H}(X,E)$ and $k \in K$ s.t. $\mathscr{U}_k(\lambda\Delta\mu) \ngeq \mathscr{U}_k(\lambda) \wedge \mathscr{U}_k(\mu)$.

By the definition of $\mathscr{U}$, there exist finite index sets $\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_m\} \subseteq \Gamma$ such that $\mathscr{U}_k(\lambda\Delta\mu) \ngeq \left(\bigwedge_{r=1}^n \mathscr{U}^{i_r}_{\eta_{i_r}(k)}(\lambda_{i_r})\right) \wedge \left(\bigwedge_{s=1}^m \mathscr{U}^{j_s}_{\eta_{j_s}(k)}(\mu_{j_s})\right)$ where $\Delta^n_{r=1}(\varphi_\psi)^{\Leftarrow}_{i_r}(\lambda_{i_r}) \leq \lambda$ and $\Delta^m_{s=1}(\varphi_\psi)^{\Leftarrow}_{j_s}(\mu_{j_s}) \leq \mu$.

Since $(\Delta^n_{r=1}(\varphi_\psi)^{\Leftarrow}_{i_r}(\lambda_{i_r}))\Delta(\Delta^m_{s=1}(\varphi_\psi)^{\Leftarrow}_{j_s}(\mu_{j_s})) \leq \lambda\Delta\mu$. By Proposition 2 (vii), we have $\mathscr{U}_k(\lambda\Delta\mu) \geq \left(\bigwedge_{r=1}^n \mathscr{U}^{i_r}_{\eta_{i_r}(k)}(\lambda_{i_r})\right) \wedge \left(\bigwedge_{s=1}^m \mathscr{U}^{j_s}_{\eta_{j_s}(k)}(\mu_{j_s})\right)$.

This is a contradiction. Hence for each $k \in K$ and $\lambda, \mu \in \mathscr{H}(X,E)$, we obtain $\mathscr{U}_k(\lambda\Delta\mu) \geq \mathscr{U}_k(\lambda) \wedge \mathscr{U}_k(\mu)$.

(U4): Suppose there exist $k \in K$ and $\lambda \in \mathscr{H}(X,E)$ such that $\mathscr{U}_k(\lambda) \nleq \bigvee\{\mathscr{U}_k(\mu) \mid \mu \circ \mu \leq \lambda\}$.

By the definition of $\mathscr{U}_k(\lambda)$, there exists a finite index set $J = \{i_1, \ldots, i_n\} \subseteq \Gamma$ such that $\bigwedge_{j=1}^n \mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\lambda_{i_j}) \nleq \bigvee\{\mathscr{U}_k(\mu) \mid \mu \circ \mu \leq \lambda\}$, where $\Delta^n_{j=1}(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j}) \leq \lambda$. Since $(X_{i_j}, \mathscr{U}^{i_j})$ is an

(L,M)-fuzzy (E_{i_j},K_{i_j})-soft uniformity for each $i_j \in \{i_1,\ldots,i_n\}$, by Definition 5,
$\mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\lambda_{i_j}) \le \bigvee\{\mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\nu) \mid \nu\circ\nu \le \lambda_{i_j}\}$.

For each $i_j \in \{i_1,\ldots,i_n\}$, there exists $\nu_{i_j} \in \mathscr{H}(X_{i_j},E_{i_j})$ with $\nu_{i_j}\circ\nu_{i_j} \le \lambda_{i_j}$ such that $\bigwedge_{j=1}^n \mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\nu_{i_j}) \not\le \bigvee\{\mathscr{U}_k(\mu) \mid \mu\circ\mu \le \lambda\}$. Put $\nu^* = \Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\nu_{i_j})$. For each $i_j \in J$, we have $\nu^*\circ\nu^* = (\Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\nu_{i_j}))\circ(\Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\nu_{i_j}))$. Hence,

$$\nu^*\circ\nu^* \le \Delta_{j=1}^n((\varphi_\psi)^{\Leftarrow}_{i_j}(\nu_{i_j})\circ(\varphi_\psi)^{\Leftarrow}_{i_j}(\nu_{i_j})) \le \Delta_{j=1}^n((\varphi_\psi)^{\Leftarrow}_{i_j}(\nu_{i_j}\circ\nu_{i_j})) \le \Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j}) \le \lambda.$$

Then we have $\nu^*\circ\nu^* \le \lambda$ and $\mathscr{U}_k(\nu^*) \ge \bigwedge_{j=1}^n \mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\nu_{i_j})$. This is a contradiction.

(U4): Let $\{(X_i,\mathscr{U}^i)\}_{i\in\Gamma}$ be a family of (L,M)-fuzzy (E_i,K_i)-soft uniform spaces. Suppose there exists $\lambda \in \mathscr{H}(X,E)$ and $k\in K$ such that $\mathscr{U}_k(\lambda) \not\le \bigvee\{\mathscr{U}_k(\mu) \mid \mu \le \lambda^\triangleleft\}$. By using the definition of $\mathscr{U}_k(\lambda)$, there exists a finite index set $J=\{j_1,\ldots,j_n\}$ of Γ such that $\bigvee\{\mathscr{U}_k(\mu) \mid \mu \le \lambda^\triangleleft\} \not\ge \bigwedge_{i=1}^n \mathscr{U}^{j_i}_{\eta_{j_i}(k)}(\lambda_{j_i})$, where $\Delta_{i=1}^n(\varphi_\psi)^{\Leftarrow}_{j_i}(\lambda_{j_i}) \le \lambda$.

Since $\mathscr{U}^{j_i}$ is an (L,M)-fuzzy (E_{j_i},K_{j_i})-soft uniformity on X_{j_i}, then
$\bigvee\{\mathscr{U}^{j_i}_{\eta_{j_i}(k)}(\nu) \mid \nu \le \lambda^\triangleleft_{j_i}\} \ge \mathscr{U}^{j_i}_{\eta_{j_i}(k)}(\lambda_{j_i})$. For each $j_i \in J$, there exists $\nu^*_{j_i} \in \mathscr{H}(X_{j_i},E_{j_i})$ with $\nu^*_{j_i} \le \lambda^\triangleleft_{j_i}$ such that $\bigvee\{\mathscr{U}_k(\mu) \mid \mu \le \lambda^\triangleleft\} \not\ge \bigwedge_{i=1}^n \mathscr{U}^{j_i}_{\eta_{j_i}(k)}(\nu^*_{j_i})$.

On the other hand, we have

$$\Delta_{i=1}^n(\varphi_\psi)^{\Leftarrow}_{j_i}(\nu^*_{j_i}) \le \Delta_{i=1}^n(\varphi_\psi)^{\Leftarrow}_{j_i}(\lambda^\triangleleft_{j_i}) = \Delta_{i=1}^n((\varphi_\psi)^{\Leftarrow}_{j_i}(\lambda_{j_i}))^\triangleleft = (\Delta_{i=1}^n(\varphi_\psi)^{\Leftarrow}_{j_i}(\lambda_{j_i}))^\triangleleft \le \lambda^\triangleleft.$$

Put $\nu^* = \Delta_{i=1}^n(\varphi_\psi)^{\Leftarrow}_{j_i}(\nu^*_{j_i})$. Then there exists $\nu^* \in \mathscr{H}(X,E)$ such that $\nu^* \le \lambda^\triangleleft$ and
$\mathscr{U}_k(\nu^*) \ge \bigwedge_{i=1}^n \mathscr{U}^{j_i}_{\eta_{j_i}(k)}(\nu^*_{j_i})$. Thus

$$\bigwedge_{i=1}^n \mathscr{U}^{j_i}_{\eta_{j_i}(k)}(\nu^*_{j_i}) \le \mathscr{U}_k(\nu^*) \le \bigvee\{\mathscr{U}_k(\mu) \mid \mu \le \lambda^\triangleleft\}.$$

This is a contradiction. Hence for each $k\in K$ and $\lambda \in \mathscr{H}(X,E)$, we have $\mathscr{U}_k(\lambda) \le \bigvee\{\mathscr{U}_k(\mu) \mid \mu \le \lambda^\triangleleft\}$.

Secondly by using the definition of $\mathscr{U}$, we have $\mathscr{U}_k((\varphi_\psi)^{\Leftarrow}_i(\lambda_i)) \ge \mathscr{U}^i_{\eta_i(k)}(\lambda_i)$ for each $k\in K, i\in\Gamma$ and $\lambda_i \in \mathscr{H}(X_i,E_i)$. Hence $(\varphi_{\psi,\eta})_i$ is uniformly continuous function.

Finally, if $(\varphi_{\psi,\eta})_i : (X,\mathscr{V}) \to (X_i,\mathscr{U}^i)$ is uniformly continuous, i.e.,
$\mathscr{V}_k((\varphi_\psi)^{\Leftarrow}_i(\lambda_i)) \ge \mathscr{U}^i_{\eta_i(k)}(\lambda_i)$ for each $k\in K, i\in\Gamma$ and $\lambda_i \in \mathscr{H}(X_i,E_i)$. Then for $k\in K$, we have

$$\begin{aligned}\mathscr{U}_k(f) &= \bigvee\{\bigwedge_{j=1}^n \mathscr{U}^{i_j}_{\eta_{i_j}(k)}(\lambda_{i_j}) \mid \Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j}) \le \lambda\}\\ &\le \bigvee\{\bigwedge_{j=1}^n \mathscr{V}_k((\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j})) \mid \Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j}) \le \lambda\}\\ &\le \bigvee\{\mathscr{V}_k(\Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j})) \mid \Delta_{j=1}^n(\varphi_\psi)^{\Leftarrow}_{i_j}(\lambda_{i_j}) \le \lambda\} \le \mathscr{V}_k(\lambda).\end{aligned}$$

(2) Necessity of the composition condition is clear. Suppose that for (L,M)-fuzzy (E^*,K^*) -soft uniform space $(Z,\mathscr{V})$, $\varphi_{\psi,\eta}:(Z,\mathscr{V})\to(X,\mathscr{U})$ is not uniformly continuous. Then there exist $k^*\in K^*$ and $\lambda\in\mathscr{H}(X,E)$ such that $\mathscr{V}_{k^*}((\varphi_\psi)^{\Leftarrow}(\lambda))\not\geq\mathscr{U}_{\eta(k^*)}(\lambda)$. By the definition of $\mathscr{U}$, there exists a finite index set $J=\{j_1,\ldots,j_n\}$ of Γ such that $\mathscr{V}_{k^*}((\varphi_\psi)^{\Leftarrow}(\lambda))\not\geq\bigwedge_{i=1}^n\mathscr{U}^{j_i}_{\eta_{j_i}(\eta(k^*))}(\lambda_{j_i})$, where $\Delta_{i=1}^n(\varphi_\psi)^{\Leftarrow}_{j_i}(\lambda_{j_i})\leq\lambda$.

On the other hand, since $(\varphi_{\psi,\eta})_{j_i}\circ\varphi_{\psi,\eta}$ is uniformly continuous, we have

$$\begin{aligned}\bigwedge_{i=1}^n\mathscr{U}^{j_i}_{\eta_{j_i}(\eta(k^*))}(\lambda_{j_i})\leq&\bigwedge_{i=1}^n\mathscr{V}_{k^*}(\varphi^{\Leftarrow}_{\psi,\eta}\circ(\varphi_{\psi,\eta})^{\Leftarrow}_{j_i}(\lambda_{j_i}))\\ \leq&\mathscr{V}_{k^*}(\Delta_{i=1}^n\varphi^{\Leftarrow}_{\psi,\eta}((\varphi_{\psi,\eta})^{\Leftarrow}_{j_i}(\lambda_{j_i})))\\ =&\mathscr{V}_{k^*}(\varphi^{\Leftarrow}_{\psi,\eta}(\Delta_{i=1}^n(\varphi_{\psi,\eta})^{\Leftarrow}_{j_i}(\lambda_{j_i})))\\ \leq&\mathscr{V}_{k^*}(\varphi^{\Leftarrow}_{\psi,\eta}(\lambda)).\end{aligned}$$

This is a contradiction. □

Definition 6. *Let $\{(X_i,\mathscr{U}^i)\}_{i\in\Gamma}$ be a family of (L,M)-fuzzy (E_i,K_i)-soft uniform spaces, X be a set, E,K be the parameter sets and $\varphi_i:X\to X_i$, $\psi_i:E\to E_i$ and $\eta_i:K\to K_i$ be functions for each $i\in\Gamma$. The initial (L,M)-fuzzy (E,K)-soft uniform structure on X with respect to $(X,(\varphi_{\psi,\eta})_i,(X_i,\mathscr{U}^i),\Gamma)$ is the coarsest (L,M)-fuzzy (E,K)-soft uniform structure on X for which all $i\in\Gamma$, $(\varphi_{\psi,\eta})_i$ are uniformly continuous.*

From Theorem 8 and Definition 6, we have the following theorem:

Theorem 9. *The category $\mathbf{HFSU}(L,M)$ of (L,M)-fuzzy (E,K)-soft uniform spaces and uniformly continuous functions is a topological category over the category $\mathbf{SET}^3$ with respect to the usual forgetful functor $V:\mathbf{HFSU}(L,M)\to\mathbf{SET}^3$ which is defined by $V(X,\mathscr{U})=(X,E,K)$ and $V(\varphi_{\psi,\eta})=(\varphi,\psi,\eta)$.*

Definition 7. *Let $X=\Pi_{i\in\Gamma}X_i$, $E=\Pi_{i\in\Gamma}E_i$ and $K=\Pi_{i\in\Gamma}K_i$ be the product sets and $\{(X_i,\mathscr{U}^i)\}_{i\in\Gamma}$ be a family of (L,M)-fuzzy (E_i,K_i)-soft uniform spaces, for each $i\in\Gamma$. The initial (L,M)-fuzzy (E,K)-soft uniformity structure $\mathscr{U}$ on X with respect to the family $\{(p_{q,r})_i:X\to(X_i,\mathscr{U}^i)\}_{i\in\Gamma}$ of all projection functions is called the product of (L,M)-fuzzy (E_i,K_i)-soft uniformity $\{\mathscr{U}^i\}_{i\in\Gamma}$. The pair $(X,\mathscr{U})$ is called the product (L,M)-fuzzy (E,K)-soft uniform space.*

6. Conclusion

Since uniformity plays an important role in classical topology and fuzzy topology, a great number of interesting works has been done on the uniformity theory for classical sets and fuzzy sets. So, we found it reasonable to investigate Hutton uniformity in the context of fuzzy soft sets. For this reason, we defined fuzzy soft remote neighborhood system and used this to investigate the relation between fuzzy soft cotopology and fuzzy soft (quasi-)uniformity. We proved the existence of the initial structure of fuzzy soft uniformities. Therefore we defined the product fuzzy soft uniformity. Also, we showed that **HFSU**(L,M) is a topological category over $\mathbf{SET}^3$.

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