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$\mathscr{C}$-cycle Compatible Splitting Signed Graphs $\mathfrak{S}(S)$ and $\Gamma(S)$

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Abstract. A *signed graph* (or, in short, *sigraph*) $S = (S^u, \sigma)$ consists of an underlying graph $S^u := G = (V, E)$ and a function $\sigma : E(S^u) \longrightarrow \{+, -\}$, called the signature of S. A *marking* of S is a function $\mu : V(S) \longrightarrow \{+, -\}$. The *canonical marking* of a signed graph S, denoted μ_σ, is given as

$$\mu_\sigma(v) := \prod_{vw \in E(S)} \sigma(vw).$$

The *splitting signed graph* $\mathfrak{S}(S)$ of a signed graph S is formed as follows:

- Take a copy of S and for each vertex v of S, take a new vertex v'. Join v' to all vertices $u \in N(v)$ by negative edge, if $\mu_\sigma(u) = \mu_\sigma(v) = -$ in S and by positive edge otherwise.

The *splitting signed graph* $\Gamma(S)$ of a signed graph S is formed as follows:

- Take a copy of S and for each vertex v of S, take a new vertex v'. Join v' to all vertices $u \in N(v)$ and assign $\sigma(uv)$ as its sign. Here, $N(v)$ is the set of all adjacent vertices to v.

A signed graph is called *canonically consistent (or $\mathscr{C}$-consistent)* if its every cycle contains even number of negative vertices with respect to its canonical marking. A marked signed graph S is called *cycle-compatible* if for every cycle Z in S, the product of signs of its vertices equals the product of signs of its edges. A signed graph S is *$\mathscr{C}$-cycle compatible* if for every cycle Z in S,

$$\prod_{e \in E(Z)} \sigma(e) = \prod_{v \in V(Z)} \mu_\sigma(v).$$

In this paper, we establish a structural characterization of signed graph S for which $\mathfrak{S}(S)$ and $\Gamma(S)$ are isomorphic and $\mathscr{C}$-cycle compatible.

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1. Introduction

A *graph* is an ordered pair $G = (V, E)$, where $V = V(G)$ is a set of *vertices or points* of G and $E = E(G)$ is a collection of pairs of vertices of G, called *edges or lines* of G. For graph theoretical terminology, we refer to [2]. All graphs considered in the paper are finite, simple and connected.

A *signed graph* is an ordered pair $S = (S^u, \sigma)$, where $S^u := G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E(S^u) \longrightarrow \{+, -\}$ is a function, called the *signature* of S. In other terms, we say that the edges are *signed* by σ. In a pictorial representation of a signed graph S, its positive edges are shown as bold line segments ('Jorden curves' drawn on the plane) and negative lines as broken line segments as shown in Figure 1. $E^{+}(S) = \{e \in E(S^u) : \sigma(e) = +\}$ and $E^{-}(S) = \{e \in E(S^u) : \sigma(e) = -\}$. The elements of $E^{+}(S)$ ($E^{-}(S)$) are called *positive* (*negative*) *edges* of S and the set $E(S) = E^{+}(S) \cup E^{-}(S)$ is called the edge set of S.

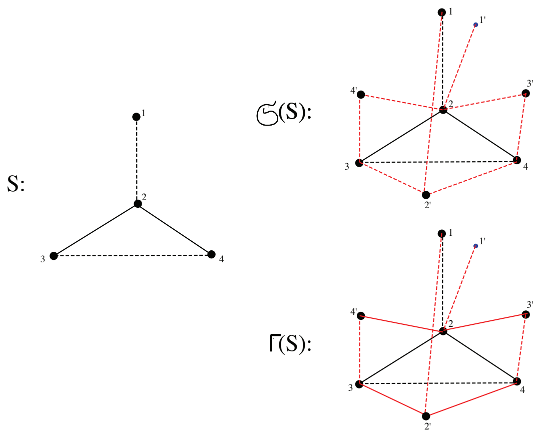

Figure 1: A signed graph S and its splitting signed graphs $\mathfrak{S}(S)$ and $\Gamma(S)$

A signed graph in which all the edges are positive, is called *all-positive signed graph* (*all-negative signed graph* is defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. By $d(v)$, we denote degree of $v \in V(S)$, $d(v) = d^+(v) + d^-(v)$, here $d^+(v)$ ($d^-(v)$) denotes the positive (negative) degree of v.

A *marking* of S is a function $\mu : V(S) \longrightarrow \{+, -\}$. Sampathkumar in [4] introduced the idea of marking derived from the signs of edges incident to vertices, given as

$$\mu_\sigma(v) := \prod_{vw \in E(S)} \sigma(vw).$$

This marking is called canonical marking. Clearly, $\mu_\sigma(v) = +$ if $d^-(v)$ is even and $\mu_\sigma(v) = -$ if $d^-(v)$ is odd. Thus, in canonical marking of a signed graph, we assign + sign to a vertex if its negative degree is even and - sign if its negative degree is odd. In this paper, a vertex v of $d^-(v) =$ even (odd) is called *positive* (*negative*) *vertex.*

Signed graphs S_1 and S_2 are called *isomorphic*, written as $S_1 \cong S_2$, if there is a graph isomorphism $f : S_1^u \to S_2^u$ that preserves edge signs.

A cycle in a signed graph is said to be *positive (negative) cycle* if the product of the signs of its edges is positive (negative), i.e., its an even (odd) number of edges are negative. A signed graph is said to be *balanced* if every cycle in it is positive (*see* [3]).

A cycle in a marked signed graph is said to be *consistent* if its an even number of vertices are negative and a marked signed graph is called consistent if its all cycles are consistent (*see* [6]). Similarly, a cycle in a signed graph is said to be *canonically consistent (or $\mathscr{C}$-consistent)* if for every cycle in S, the product of signs of its vertices with respect to canonical marking, is positive, i.e., its an even number of vertices are negative and a signed graph is called $\mathscr{C}$-consistent if its all cycles are $\mathscr{C}$-consistent.

A marked signed graph S is called *cycle-compatible* if for every cycle Z in S, the product of signs of its vertices equals the product of signs of its edges. A signed graph S is called *canonically cycle* (or *$\mathscr{C}$-cycle*) *compatible* if for every cycle Z in S,

$$\prod_{e \in E(Z)} \sigma(e) = \prod_{v \in V(Z)} \mu_\sigma(v).$$

A signed graph $S = (S^u, \sigma)$ is said to be *sign-compatible* [6] if it has a vertex marking μ such that every edge $e = uv$ has $\sigma(e) = -$ if and only if $\mu(u) = \mu(v) = -$. If the canonical marking μ_σ has this property, then S is said to be *canonically sign-compatible (or $\mathscr{C}$-sign-compatible)*.

2. Splitting Signed Graphs

Sampathkumar and Walikar introduced the concept of *splitting graph* of a graph in [5]. The splitting graph of a graph G, denoted here $\mathfrak{S}(G)$, is formed as follows:

Take a copy of G and for each vertex v of G, take a new vertex v'. Join v' to all adjacent vertices of v.

There are two notions of *splitting signed graphs* of a signed graph $S = (S^u, \sigma)$ in the literature, viz., $\mathfrak{S}(S)$ and $\Gamma(S)$, both of which have $\mathfrak{S}(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $\mathfrak{S}(S^u)$ differ. An edge uv' in $\mathfrak{S}(S)$ is negative whenever u and v are negative vertices of S and an edge uv' in $\Gamma(S)$ is negative whenever uv is a negative edge of S as reported in [1] and [7] respectively.

A signed graph S is called a $\mathfrak{S}$-splitting (Γ-splitting) signed graph if there exists a signed graph T such that S is isomorphic to $\mathfrak{S}(T)$ ($\Gamma(T)$).

Theorem 1 (Acharya *et al.*[1]). *Following statements hold:*

(i) If $v \in V(S)$ is a positive vertex then $v, v' \in V(\mathfrak{S}(S))$ are positive.

(ii) If $v \in V(S)$ is a negative vertex having an even (odd) number of negative vertices in its neighbourhood then $v \in V(\mathfrak{S}(S))$ is negative (positive) vertex and v' is of opposite sign to v.

Here v' is the vertex as defined above.

Theorem 2 (Acharya *et al.* [1]). *$\mathfrak{S}(S)$ is balanced if and only if the following conditions hold in S:*

(i) S is balanced and;

(ii) S does not contain a homogeneous path P_3 of marking +, -, - and the marking of a heterogeneous path P_3 is +, -, - only.

Lemma 1 (Sinha *et al.* [7]). *The following statements hold in $\Gamma(S)$:*

(i) If $v \in V(S)$ is any vertex then $v \in V(\Gamma(S))$ is positive.

(ii) If $v \in V(S)$ is a negative vertex then $v' \in V(\Gamma(S))$ is negative.

Theorem 3 (Sinha *et al.* [7]). *The splitting signed graph $\Gamma(S)$ of a signed graph S is balanced if and only if S is balanced.*

Figure 1 illustrates a signed graph S and its splitting signed graphs $\mathfrak{S}(S)$ and $\Gamma(S)$.

3. Main Results

Theorem 4. *For a signed graph S, $\mathfrak{S}(S) \cong \Gamma(S)$ if and only if S is any one of the following:*

(i) All-positive or;

(ii) All-negative in which degree of each vertex is odd or;

(iii) Heterogeneous in which end vertices of every negative (positive) edge are (are not) negative.

Proof. Necessity: Let, for a signed graph S, $\mathfrak{S}(S) \cong \Gamma(S)$. Since S is a subsignedgraph of $\mathfrak{S}(S)$ and $\Gamma(S)$, we concentrate our attention only on the sign of edge uv' in $\mathfrak{S}(S)$ and $\Gamma(S)$. By the definition of $\mathfrak{S}(S)$, $uv' \in E^-(\mathfrak{S}(S))$ if and only if $u, v \in V(S)$ are negative and by the definition of $\Gamma(S)$, $uv' \in E^-(\Gamma(S))$ if and only if $uv \in E^-(S)$. Therefore, we have following three possible cases:

Case I: If $\mathfrak{S}(S) \cong \Gamma(S)$ and both $\mathfrak{S}(S)$ and $\Gamma(S)$ are all-positive then no edge of S will be negative. Hence, **(i)** follows.

Case II: If $\mathfrak{S}(S) \cong \Gamma(S)$ and both are all-negative then every edge and every vertex of S will be negative. Hence, **(ii)** follows.

Case III: If $\mathfrak{S}(S) \cong \Gamma(S)$ and both are heterogeneous then S will be heterogeneous and edge uv' in both $\mathfrak{S}(S)$ and $\Gamma(S)$ must be of the same sign. This implies that end vertices of every negative (positive) edge of S are (are not) negative. Hence, **(iii)** follows.

Thus, the necessity follows.

Sufficiency: Suppose S is any one of the following:

(i) All-positive or;

(ii) All-negative in which degree of each vertex is odd or;

(iii) Heterogeneous in which end vertices of every negative (positive) edge are (are not) negative.

then by the definitions $\mathfrak{S}$ - and Γ- splitting signed graphs, we obtain following results:

Case I: If S is all-positive then $\mathfrak{S}(S)$ and $\Gamma(S)$ will be all-positive and $\mathfrak{S}(S) \cong \Gamma(S)$.

Case II: If S is all-negative in which degree of each vertex is odd then $\mathfrak{S}(S)$ and $\Gamma(S)$ will be all-negative and $\mathfrak{S}(S) \cong \Gamma(S)$.

Case III: If S is heterogeneous in which end vertices of every negative (positive) edge are (are not) negative then $\mathfrak{S}(S)$ and $\Gamma(S)$ will be heterogeneous as S be a subsignedgraph of $\mathfrak{S}(S)$ and $\Gamma(S)$ and edge uv' in both $\mathfrak{S}(S)$ and $\Gamma(S)$ will be of the same sign. Hence, $\mathfrak{S}(S) \cong \Gamma(S)$.

This completes the proof. □

Corollary 1. *For a signed graph S, $\mathfrak{S}(S) \cong \Gamma(S)$ if and only if S is $\mathscr{C}$-sign compatible.*

Theorem 5. $\mathfrak{S}(S)$ *is $\mathscr{C}$-cycle compatible if and only if the following conditions hold in S:*

(i) if Z is a positive (negative) cycle then an even (odd) number of negative vertices of cycle Z contain even numbers of negative vertices in their neighbourhoods and;

(ii) for a path $P_3 = (u, v, w)$, any one condition holds:

- *it is homogeneous of marking +, +, +;*
- *it is heterogeneous of marking +, -, +;*
- *it is homogeneous (heterogeneous) of marking -, +, + or -, -, + and $N(u)$ contains an odd (even) number of negative vertices;*
- *it is homogeneous (heterogeneous) of marking -, +, - and vertices u, w are (are not) of same parity (i.e., $N(u)$ and $N(w)$ contain even number of negative vertices or odd number of negative vertices);*
- *it is homogeneous (heterogeneous) of marking -, -, - and vertices u, w are not (are) of the same parity.*

Proof. Necessity: Let $\mathfrak{S}(S)$ be $\mathscr{C}$-cycle compatible. Therefore, every cycle in $\mathfrak{S}(S)$ is either positive and $\mathscr{C}$-consistent or negative and $\mathscr{C}$-inconsistent. By Theorem 1, every positive vertex of S is positive in $\mathfrak{S}(S)$ and every negative vertex of S having an even (odd) number of negative vertices in its neighbourhood is negative (positive) in $\mathfrak{S}(S)$. Since S is subsignedgraph of $\mathfrak{S}(S)$, if Z is a positive (negative) cycle of S then Z must be $\mathscr{C}$-consistent ($\mathscr{C}$-inconsistent) in $\mathfrak{S}(S)$, i.e., an even (odd) number of negative vertices of cycle Z must contain an even numbers of negative vertices in their neighbourhoods. Thus, **(i)** follows.

By the definition of $\mathfrak{S}(S)$, a path $P_3 = (u, v, w)$ of S induces a cycle $C_4 = (u, v, w, v')$ in $\mathfrak{S}(S)$. The marking of path $P_3 = (u, v, w)$ may be one of the following:

1. +, +, + 2. +, -, + 3. -, +, + 4. -, -, + 5. -, +, - 6. -, -, -

Hence, the following cases arise:

- if marking of path $P_3 = (u, v, w)$ is +, +, + then by Theorem 1, vertices u, v, w, v' have signs +, +, +, + respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathscr{C}$-consistent cycle C_4 in $\mathfrak{S}(S)$. By Theorem 2, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible, P_3 will be homogeneous.

- if marking of path $P_3 = (u, v, w)$ is +, -, + then by Theorem 1, vertices u, v, w, v' have signs +, -, +, + or +, +, +, - respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathscr{C}$-inconsistent cycle C_4 in $\mathfrak{S}(S)$. By Theorem 2, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible, P_3 will be heterogeneous.

- if marking of path $P_3 = (u, v, w)$ is -, +, + and $N(u)$ contains an odd (even) number of negative vertices then by Theorem 1, vertices u, v, w, v' have signs +, +, +, + (-, +, +, +) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathscr{C}$-consistent ($\mathscr{C}$-inconsistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 2, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible, for homogeneous (heterogeneous) P_3, $N(u)$ must contain an odd (even) number of negative vertices.

  Similarly, if marking of path $P_3 = (u, v, w)$ is -, -, + and $N(u)$ contains an even (odd) number of negative vertices then by Theorem 1, vertices u, v, w, v' have signs -, -, +, + or -, +, +, - (+, -, +, + or +, +, +, -) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathscr{C}$-consistent ($\mathscr{C}$-inconsistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 2, this cycle C_4 is positive (negative) if and only if P_3 is heterogeneous (homogeneous). Since $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible, for heterogeneous (homogeneous) P_3, $N(u)$ will contain an even (odd) number of negative vertices.

- if marking of path $P_3 = (u, v, w)$ is -, +, - and vertices u and w are (are not) of the same parity, i.e., $N(u)$ and $N(w)$ contain even number of negative vertices or odd number of negative vertices, then by Theorem 1, vertices u, v, w, v' have signs -, +, -, + or +, +, +, + (-, +, +, + or +, +, -, +) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathscr{C}$-consistent ($\mathscr{C}$-inconsistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 2, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible, for homogeneous (heterogeneous) P_3, vertices u and w will (will not) be of the same parity;

- if marking of path $P_3 = (u, v, w)$ is -, -, - and vertices u and w are (are not) of the same parity, i.e., $N(u)$ and $N(w)$ contain even number of negative vertices or odd number of negative vertices, then by Theorem 1, vertices u, v, w, v' have signs -, -, -, +; -, +, -, - or +, -, +, +; +, +, +, - (-, -, +, +; -, +, +, - or +, -, -, +; +, +, -, -) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathscr{C}$-inconsistent ($\mathscr{C}$-consistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 2, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible, for homogeneous (heterogeneous) P_3, vertices u and w will not (will) be of the same parity.

Thus, the necessity follows.

Sufficiency: A cycle in $\mathfrak{S}(S)$ is induced due to a cycle or a path P_3 or their combinations in S. If conditions hold then it can be easily seen that every cycle in $\mathfrak{S}(S)$ is positive and $\mathscr{C}$-consistent or negative and $\mathscr{C}$-inconsistent, i.e, $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible. This completes the proof. □

Signed graph S shown in Figure 2 does not satisfy conditions **(i)** and **(ii)** of Theorem 5, $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle incompatible.

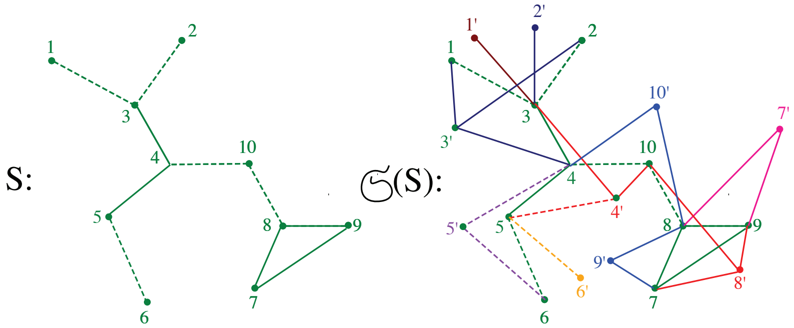

Figure 2: A signed graph S and its $\mathscr{C}$-cycle incompatible $\mathfrak{S}(S)$

Signed graph S shown in Figure 3 satisfies conditions **(i)** and **(ii)** of Theorem 5, $\mathfrak{S}(S)$ is $\mathscr{C}$-cycle compatible.

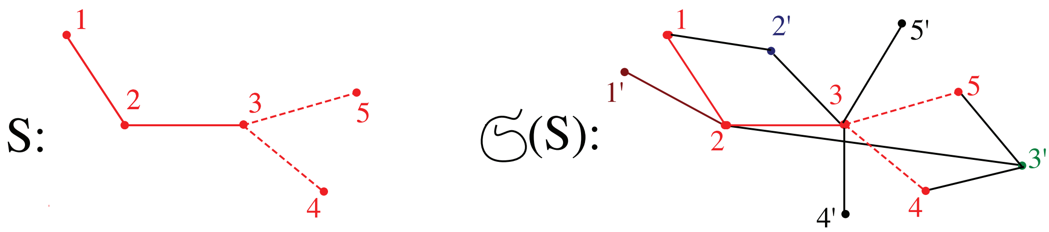

Figure 3: A signed graph S and its $\mathscr{C}$-cycle compatible $\mathfrak{S}(S)$

Theorem 6. *For a signed graph S, $\Gamma(S)$ is $\mathscr{C}$-cycle compatible if and only if the following conditions hold in S:*

(i) S is balanced;

(ii) each non-pendant vertex of S is positive.

Proof. Necessity: Let $\Gamma(S)$ be $\mathscr{C}$-cycle-compatible, i.e., every cycle in $\Gamma(S)$ is either positive and $\mathscr{C}$-consistent or negative and $\mathscr{C}$-inconsistent. By Lemma 1, every vertex of S is a positive vertex of $\Gamma(S)$. Hence, every cycle Z of $\Gamma(S)$ that is due to a cycle Z of S is $\mathscr{C}$-consistent. Since $\Gamma(S)$ is $\mathscr{C}$-cycle-compatible, this cycle Z of S must be positive. Therefore, S will be balanced. Thus, **(i)** follows.

By the definition of $\Gamma(S)$, a path $P_3 = (u, v, w)$ of S induces a positive cycle $C_4 = (u, v, w, v')$ in $\Gamma(S)$ and by Lemma 1, vertices u, v, w, v' have signs +, +, +, + or +, +, +, - in $\Gamma(S)$ if v is a positive (negative) vertex of S. Thus, this cycle C_4 is $\mathscr{C}$-consistent if $v \in V(S)$ is a positive vertex. Since $\Gamma(S)$ is $\mathscr{C}$-cycle compatible and cycle C_4 is positive, C_4 must be $\mathscr{C}$-consistent. Hence, every non-pendant vertex of S will be positive. Thus, the necessity follows.

Sufficiency: A cycle in $\Gamma(S)$ is induced due to a cycle or a path P_3 or their combinations in S. If conditions hold then it can be easily seen that every cycle in $\Gamma(S)$ is positive and $\mathscr{C}$-consistent, i.e, $\Gamma(S)$ is $\mathscr{C}$-cycle-compatible. This completes the proof. □

Signed graph S shown in Figure 4 satisfies conditions **(i)** and **(ii)** of Theorem 6, $\Gamma(S)$ is $\mathscr{C}$-cycle compatible.

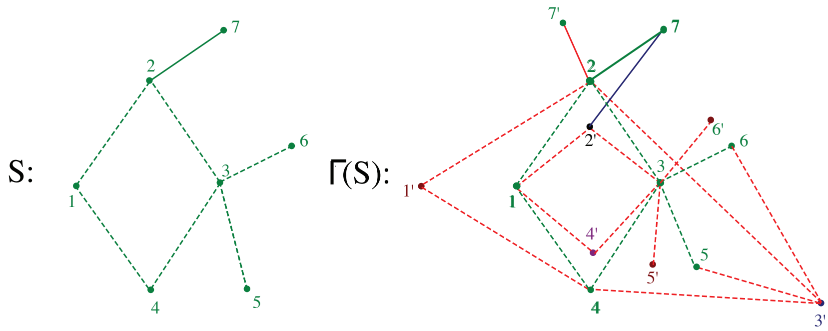

Figure 4: A signed graph S and its $\mathscr{C}$-cycle compatible $\Gamma(S)$

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