EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 8, No. 4, 2015, 458-461
ISSN 1307-5543 – www.ejpam.com

A Note on Prüfer $\star$-multiplication Domains

Olivier A. Heubo-Kwegna

Department of Mathematical Sciences, Saginaw Valley State University, University Center MI 48710, USA

Abstract. In this note, we prove that for an arbitrary star operation $\star$ on a domain R, the domain R is a Prüfer $\star$-multiplication domain if every 2-generated ideal of R is $\star_f$-invertible. Some characterizations of Prüfer-$\star$ multiplication domains are therefore obtained.

2010 Mathematics Subject Classifications: 13A15, 13A18, 16W50

Key Words and Phrases: Star operation; $\star$-ideal; Prüfer $\star$-multiplication domain

1. Introduction

Throughout this note R denotes an integral domain with quotient field K. Let $\mathscr{F}(R)$ be the set of all nonzero fractional ideals of R and $f(R)$ be the set of all nonzero finitely generated fractional ideals of R.

A *star operation* on R is a mapping $A \to A^\star$ of $\mathscr{F}(R)$ into $\mathscr{F}(R)$ such that for all $A, B \in \mathscr{F}(R)$ and for all $a \in K \setminus \{0\}$,

(i) $(a)^\star = (a)$ and $(aA)^\star = aA^\star$;

(ii) $A \subseteq A^\star$ and $A \subseteq B \Rightarrow A^\star \subseteq B^\star$, and

(iii) $A^{\star\star} := (A^\star)^\star = A^\star$.

For an overview of star operations, the reader may refer to [5, Sections 32 and 34]. Given a star operation $\star$ on R, one can construct a new star operation $\star_f$ as follows: for each $A \in F(R)$, $A^{\star_f} = \cup\{B^\star | B \subseteq A \text{ and } B \in f(R)\}$. A star operation is said to be of *finite type* if $\star_f = \star$. Since $(\star_f)_f = \star_f$, $\star_f$ is a finite type star operation for any given star operation $\star$ on R. Note that $d_f = d$, where d is the identity star operation and if $\star$ is the v-operation we denote $v_f := t$ and call it the t-operation. A nonzero ideal A of R is a $\star$-*ideal* if $A^\star = A$. Similarly, we call a $\star$-ideal of R a $\star$-*prime ideal* of R if it is also a prime ideal. We call a maximal element in the set of all proper $\star$-ideals of R a $\star$-*maximal* ideal of R. We denote $\text{Spec}^\star(R)$ the set of all $\star$-prime ideals

Email address: oheubokw@svsu.edu

of R and $\mathrm{Max}^\star(R)$ the set of all $\star$-maximal ideals of R. An $A\in\mathscr{F}(R)$ is said to be $\star$-invertible if $(AA^{-1})^\star = R$, whereas a domain R is a *Prüfer $\star$-multiplication domain* (in short, P$\star$MD) if every finitely generated ideal A of R is $\star_f$-invertible, i.e., $(AA^{-1})^{\star_f} = R$ for any $A\in f(R)$. Thus a Prüfer domain is a PdMD and PvMD is often called a Prüfer multiplication domain.

Many authors have previously produced several characterizations of Prüfer-$\star$ multiplication domains (for instance see [1–3, 6]). The aim of this note is to provide some new characterizations of Prüfer-$\star$ multiplication domains. We precisely show that a domain R is a P$\star$MD if and only if each 2-generated ideal of R is $\star_f$-invertible. Note that this result is a generalization of the fact that a domain is Prüfer if and only if each 2-generated ideal is invertible [9, page 7]. We also show that a domain R is a P$\star$MD if and only if $(a)\cap(b)$ is $\star_f$-invertible for all $a,b\in R\setminus\{0\}$. The latest result has also been shown in the v-domain context [8] and in the PvMD context [7].

2. Main Results

We start this section with the recollection of some facts about star operations. Let $\star$ be a star operation on R. Recall that $\star$ is *stable* if $(A\cap B)^\star = A^\star\cap B^\star$ for all $A,B\in\mathscr{F}(R)$. Now define $\widetilde{\star}$ by $A^{\widetilde{\star}} := \cap\{AR_M | M\in \mathrm{Max}^{\star_f}(R)\}$, for all $A\in\mathscr{F}(R)$. Then it is well known that $\widetilde{\star}$ is a stable star operation on R of finite type called the *stable star operation of finite type associated to* $\star$. It is not hard to see that $\mathrm{Max}^{\widetilde{\star}}(R)$=$\mathrm{Max}^{\star_f}(R)$[4, Corollary 3.5(2)]. From the latest fact, it then follows that an ideal A is $\widetilde{\star}$-invertible if and only if it is $\star_f$-invertible (in fact, if a star operation $\star$ is of finite type, then $(AA^{-1})^\star = R$ if and only if $AA^{-1}\nsubseteq M$ for all $M\in\mathrm{Max}^\star(R)$). From this observation it then follows that P$\star$MD, P$\star_f$MD, and P$\widetilde{\star}$MD coincide.

Lemma 1. *Let A be a finitely generated ideal of R and $\star$ a star operation on R. If A is $\star_f$-invertible, then AR_M is principal for every $M\in \mathrm{Max}^{\star_f}(R)$.*

Proof. Suppose that A is $\star_f$-invertible. From the above observation, it follows that A is $\widetilde{\star}$-invertible, i.e., $(AA^{-1})^{\widetilde{\star}} = R$. We have, for each maximal $\star_f$-ideal M,

$$R_M = (AA^{-1})^{\widetilde{\star}}R_M = \bigcap\{(AA^{-1})R_N | N\in\mathrm{Max}^{\star_f}(R)\}R_M = (AA^{-1})R_M$$

[4, Lemma 2.4.(1)]. Thus AR_M is invertible and therefore principal. □

Theorem 1. *Let R be an integral domain and let $\star$ be a star operation on R. Then the following statements are equivalent for an integral domain R.*

(i) R_M is a valuation domain for all $M\in \mathrm{Max}^{\star_f}(R)$.

(ii) R is a P$\star$MD.

(iii) Every nonzero fractional finitely generated ideal of R is $\star_f$-invertible.

(iv) Every nonzero fractional 2-generated ideal is $\star_f$-invertible.

Proof. For $(i) \Longleftrightarrow (ii)$ (see [1, Corollary 1.2]). $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are clear. So it remains to prove that $(iv) \Rightarrow (i)$. Let $x, y \in R$, note that if P is a prime ideal of R, we have $xR_P + yR_P = (a, b)R_P$ for some $a, b \in R$. But if P is a $\star_f$-maximal ideal of R then, by Lemma 1, $(a, b)R_P$ is principal, that is, R_P is a valuation domain. □

Corollary 1. *A domain R is a P⋆MD if and only if $(a) \cap (b)$ is $\star_f$-invertible for all $a, b \in R \setminus \{0\}$.*

Proof. Note that we have $(ab)^{-1}[(a) \cap (b)] = (a, b)^{-1}$. So $(ab)^{-1}[(a) \cap (b)](a, b) = (a, b)^{-1}(a, b)$ and $\left((ab)^{-1}[(a) \cap (b)](a, b)\right)^{\star_f} = \left((a, b)^{-1}(a, b)\right)^{\star_f}$. Thus if $a, b \in R \setminus \{0\}$, $(a) \cap (b)$ is $\star_f$ invertible if and only if (a, b) is $\star_f$-invertible. Hence R is a P⋆MD if and only if $(a) \cap (b)$ is $\star_f$-invertible for all $a, b \in R \setminus \{0\}$ by Theorem 1(iv). □

Recall that a $\star$-ideal A of R is of *finite type* if $A = (a_1, \ldots, a_n)^\star$ for some $(0) \neq (a_1, \ldots, a_n) \subseteq A$. Note that if $\star = \star_f$, then $A^\star$ is of finite type if and only if $A^\star = (a_1, \ldots, a_n)^\star$ for some $(0) \neq (a_1, \ldots, a_n) \subseteq A$. If $\star$ is a star operation of finite type, then a $\star$-invertible ideal is of finite type. Also note that from [5, Proposition 32.2(b)] and the fact that $(z)^\star = (z)$ for any $z \in K$, we have $((a) \cap (b))^\star = (a) \cap (b)$ for any star operation $\star$ on R. Thus $(a) \cap (b)$ is a $\star$-ideal of R for all $a, b \in R \setminus \{0\}$.

Corollary 2. *Let R be an integral domain such that $\left((ab)^{-1}[(a) \cap (b)](a, b)\right)^\star = R$. Then R is a P⋆MD if and only if $(a) \cap (b)$ is of finite type.*

Proof. Suppose that R is a P⋆MD. Then $(a) \cap (b)$ is $\star_f$-invertible by Corollary 1. So $(a) \cap (b)$ is of finite type following the above discussion. Conversely if we assume that $(a) \cap (b)$ is of finite type, then from $\left((ab)^{-1}[(a) \cap (b)](a, b)\right)^\star = R$, it follows that $(a) \cap (b)$ is $\star_f$-invertible and hence R is a P⋆MD by Corollary 1. □

Remark 1. *Note that the preceding theorem and corollaries give new characterizations of Prüfer $\star$-multiplication domains which generalize some of the classical characterizations of Prüfer v-multiplication domains (see [7, Lemma 1.7, Corollary 1.8, and Corollary 1.9]).*

ACKNOWLEDGEMENTS The author wishes to express his gratitude to Bruce Olberding for bringing up the problem treated in this paper following some discussions on Prüfer-⋆ multiplication domains.

References

[1] D. D. Anderson, D. F. Anderson, M. Fontana, and M. Zafrullah. *On v-Domains and Star Operations*, Communications in Algebra, 2, 141-145. 2008.

[2] D. F. Anderson, M. Fontana, and M. Zafrullah. *Some remarks on Prüfer ⋆-multiplication domains and class groups*, Journal of Algebra, 319, 272-295. 2008.

[3] G. W. Chang. *Prüfer ∗-Multiplication Domains, Nagata rings, and Kroneker function rings*, Journal of Algebra, 319(1), 309-319. 2008.

[4] M. Fontana and K.A. Loper. Nagata rings. *Kronecker function rings and related smistar operations*, Communications in Algebra, 31, 4775-4805. 2003.

[5] R. Gilmer. *Multiplicative ideal theory*, Corrected reprint of the 1972 edition. Queen's Papers in Pure and Applied Mathematics, 90. Queen's University, Kingston, ON, 1992.

[6] E. G. Houston, S. B. Malik, and J.L. Mott. *Characterization of $\star$-Multiplication Domains*, Canadian Mathematics Bulletin, 27, 48-52. 1984.

[7] S. Malik, J. Mott, and M. Zafrullah. *On t-invertibility*, Communications in Algebra, 16, 149-170. 1988.

[8] J. Mott, B. Nashier, and M. Zafrullah. *Contents of polynomials and invertibility*, Communications in Algebra, 18(5), 1569-1583. 1990.

[9] H. Prüfer. *Untersuchen über Teilbarkeitseigenshafen in Körper*, Journal of Reine Angew of Mathematics, 168, 1-36. 1932.