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Classical 2-Absorbing Submodules of Modules over Commutative Rings

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Abstract. In this article, all rings are commutative with nonzero identity. Let M be an R-module. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. We introduce the concept of "classical 2-absorbing submodules" as a generalization of "classical prime submodules". We say that a proper submodule N of M is a *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$.

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1. Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring and M be an R-module. A proper submodule N of M is said to be a *prime submodule*, if for each element $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{r \in R \mid rM \subseteq N\}$. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [9, 10] (see also, [11], in which, the notion of "weakly prime submodules" is investigated). For more information on weakly prime submodules, the reader is referred to [3, 4, 12].

Badawi gave a generalization of prime ideals in [5] and said such ideals 2-absorbing ideals. A proper ideal I of R is a *2-absorbing ideal of R* if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if

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whenever I_1, I_2, I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Anderson and Badawi [2] generalized the notion of 2-absorbing ideals to n-absorbing ideals. A proper ideal I of R is called an *n-absorbing* (resp. *a strongly n-absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp. $I_1 \ldots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. The reader is referred to [6–8] for more concepts related to 2-absorbing ideals. Yousefian Darani and Soheilnia in [13] extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a 2-*absorbing submodule of* M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Generally, a proper submodule N of M is called an *n-absorbing submodule* if whenever $a_1 \ldots a_n m \in N$ for $a_1, \ldots a_n \in R$ and $m \in M$, then either $a_1 \ldots a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose product with m is in N, see [14]. Several authors investigated properties of 2-absorbing submodules, for example [15].

In this paper we introduce the definition of classical 2-absorbing submodules. A proper submodule N of an R-module M is called *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$. Clearly, every classical prime submodule is a classical 2-absorbing submodule. We show that every Noetherian R-module M contains a finite number of minimal classical 2-absorbing submodules (Theorem 3). Further, we give the relationship between classical 2-absorbing submodules, classical prime submodules and 2-absorbing submodules (Proposition 2, Proposition 7). Moreover, we characterize classical 2-absorbing submodules in (Theorem 2, Theorem 4). In (Theorem 7, Theorem 8) we investigate classical 2-absorbing submodules of a finite direct product of modules.

2. Characterizations of Classical 2-Absorbing Submodules

First of all we give a module which has no classical 2-absorbing submodule.

Example 1. *Let p be a fixed prime integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then*

$$E(p) := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \text{ and } n \in \mathbb{N}_0 \right\}$$

is a nonzero submodule of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each $t \in \mathbb{N}_0$, set

$$G_t := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \right\}.$$

Notice that for each $t \in \mathbb{N}_0$, G_t is a submodule of $E(p)$ generated by $\frac{1}{p^t} + \mathbb{Z}$ for each $t \in \mathbb{N}_0$. Each proper submodule of $E(p)$ is equal to G_i for some $i \in \mathbb{N}_0$ (see, [17, Example 7.10]). However, no G_t is a classical 2-absorbing submodule of $E(p)$. Indeed, $\frac{1}{p^{t+3}} + \mathbb{Z} \in E(p)$. Then $p^3\left(\frac{1}{p^{t+3}} + \mathbb{Z}\right) = \frac{1}{p^t} + \mathbb{Z} \in G_t$ but $p^2\left(\frac{1}{p^{t+3}} + \mathbb{Z}\right) = \frac{1}{p^{t+1}} + \mathbb{Z} \notin G_t$.

Theorem 1. *Let $f : M \to M'$ be an epimorphism of R-modules.*

(i) If N' is a classical 2-absorbing submodule of M', then $f^{-1}(N')$ is a classical 2-absorbing submodule of M.

(ii) If N is a classical 2-absorbing submodule of M containing $Ker(f)$, then $f(N)$ is a classical 2-absorbing submodule of M'.

Proof. (i) Since f is epimorphism, $f^{-1}(N')$ is a proper submodule of M. Let $a, b, c \in R$ and $m \in M$ such that $abcm \in f^{-1}(N')$. Then $abcf(m) \in N'$. Hence $abf(m) \in N'$ or $acf(m) \in N'$ or $bcf(m) \in N'$, and thus $abm \in f^{-1}(N')$ or $acm \in f^{-1}(N')$ or $bcm \in f^{-1}(N')$. So, $f^{-1}(N')$ is a classical 2-absorbing submodule of M.

(ii) Let $a, b, c \in R$ and $m' \in M'$ be such that $abcm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abcm) \in f(N)$. Since $Ker(f) \subseteq N$, we have $abcm \in N$. It implies that $abm \in N$ or $acm \in N$ or $bcm \in N$. Hence $abm' \in f(N)$ or $acm' \in f(N)$ or $bcm' \in f(N)$. Consequently $f(N)$ is a classical 2-absorbing submodule of M'. □

As an immediate consequence of Theorem 1 we have the following corollary.

Corollary 1. *Let M be an R-module and $L \subseteq N$ be submodules of M. Then N is a classical 2-absorbing submodule of M if and only if N/L is a classical 2-absorbing submodule of M/L.*

Proposition 1. *Let M be an R-module and N_1, N_2 be classical prime submodules of M. Then $N_1 \cap N_2$ is a classical 2-absorbing submodule of M.*

Proof. Let for some $a, b, c \in R$ and $m \in M$, $abcm \in N_1 \cap N_2$. Since N_1 is a classical prime submodule, then we may assume that $am \in N_1$. Likewise, assume that $bm \in N_2$. Hence $abm \in N_1 \cap N_2$ which implies $N_1 \cap N_2$ is a classical 2-absorbing submodule. □

Proposition 2. *Let N be a proper submodule of an R-module M.*

(i) If N is a 2-absorbing submodule of M, then N is a classical 2-absorbing submodule of M.

(ii) N is a classical prime submodule of M if and only if N is a 2-absorbing submodule of M and $(N :_R M)$ is a prime ideal of R.

Proof. (i) Assume that N is a 2-absorbing submodule of M. Let $a, b, c \in R$ and $m \in M$ such that $abcm \in N$. Therefore either $acm \in N$ or $bcm \in N$ or $ab \in (N : M)$. The first two cases lead us to the claim. In the third case we have that $abm \in N$. Consequently N is a classical 2-absorbing submodule.

(ii) It is evident that if N is classical prime, then it is 2-absorbing. Also, [3, Lemma 2.1] implies that $(N :_R M)$ is a prime ideal of R. Assume that N is a 2-absorbing submodule of M and $(N :_R M)$ is a prime ideal of R. Let $abm \in N$ for some $a, b \in R$ and $m \in M$ such that neither $am \in N$ nor $bm \in N$. Then $ab \in (N :_R M)$ and so either $a \in (N :_R M)$ or $b \in (N :_R M)$.This contradiction shows that N is classical prime. □

he following example shows that the converse of Proposition 2(i) is not true.

Example 2. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \bigoplus \mathbb{Z}_q \bigoplus \mathbb{Q}$ where p, q are two distinct prime integers. One can easily see that the zero submodule of M is a classical 2-absorbing submodule. Notice that $pq(1,1,0) = (0,0,0)$, but $p(1,1,0) \neq (0,0,0)$, $q(1,1,0) \neq (0,0,0)$ and $pq(1,1,1) \neq 0$. So the zero submodule of M is not 2-absorbing. Also, part (ii) of Proposition 2 shows that the zero submodule is not a classical prime submodule. Hence the two concepts of classical prime submodules and of classical 2-absorbing submodules are different in general.*

Let M be an R-module and N a submodule of M. For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_R a)$. It is easy to see that $(N :_M a)$ is a submodule of M containing N.

Theorem 2. *Let M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

(i) *N is classical 2-absorbing;*

(ii) *For every* $a, b, c \in R$, $(N :_M abc) = (N :_M ab) \cup (N :_M ac) \cup (N :_M bc)$;

(iii) *For every* $a, b \in R$ *and* $m \in M$ *with* $abm \notin N$, $(N :_R abm) = (N :_R am) \cup (N :_R bm)$;

(iv) *For every* $a, b \in R$ *and* $m \in M$ *with* $abm \notin N$, $(N :_R abm) = (N :_R am)$ *or* $(N :_R abm) = (N :_R bm)$;

(v) *For every* $a, b \in R$ *and every ideal* I *of* R *and* $m \in M$ *with* $abIm \subseteq N$, *either* $abm \in N$ *or* $aIm \subseteq N$ *or* $bIm \subseteq N$;

(vi) *For every* $a \in R$ *and every ideal* I *of* R *and* $m \in M$ *with* $aIm \nsubseteq N$, $(N :_R aIm) = (N :_R am)$ *or* $(N :_R aIm) = (N :_R Im)$;

(vii) *For every* $a \in R$ *and every ideals* I, J *of* R *and* $m \in M$ *with* $aIJm \subseteq N$, *either* $aIm \subseteq N$ *or* $aJm \subseteq N$ *or* $IJm \subseteq N$;

(viii) *For every ideals* I, J *of* R *and* $m \in M$ *with* $IJm \nsubseteq N$, $(N :_R IJm) = (N :_R Im)$ *or* $(N :_R IJm) = (N :_R Jm)$;

(ix) *For every ideals* I, J, K *of* R *and* $m \in M$ *with* $IJKm \subseteq N$, *either* $IJm \subseteq N$ *or* $IKm \subseteq N$ *or* $JKm \subseteq N$;

(x) *For every* $m \in M \backslash N$, $(N :_R m)$ *is a 2-absorbing ideal of R.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N is a classical 2-absorbing submodule of M. Let $m \in \left(N :_M abc\right)$. Then $abcm \in N$. Hence $abm \in N$ or $acm \in N$ or $bcm \in N$. Therefore $m \in \left(N :_M ab\right)$ or $m \in \left(N :_M ac\right)$ or $m \in \left(N :_M bc\right)$. Consequently,

$$\left(N :_M abc\right) = \left(N :_M ab\right) \cup \left(N :_M ac\right) \cup \left(N :_M bc\right).$$

$(ii) \Rightarrow (iii)$ Let $abm \notin N$ for some $a, b \in R$ and $m \in M$. Assume that $x \in (N :_R abm)$. Then $abxm \in N$, and so $m \in (N :_M abx)$. Since $abm \notin N$, $m \notin (N :_M ab)$. Thus by part (i), $m \in (N :_M ax)$ or $m \in (N :_M bx)$, whence $x \in (N :_R am)$ or $x \in (N :_R bm)$. Therefore $(N :_R abm) = (N :_R am) \cup (N :_R bm)$.
$(iii) \Rightarrow (iv)$ By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.
$(iv) \Rightarrow (v)$ Let for some $a, b \in R$, an ideal I of R and $m \in M$, $abIm \subseteq N$. Hence $I \subseteq (N :_R abm)$. If $abm \in N$, then we are done. Assume that $abm \notin N$. Therefore by part (iv) we have that $I \subseteq (N :_R am)$ or $I \subseteq (N :_R bm)$, i.e., $aIm \subseteq N$ or $bIm \subseteq N$.
$(v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii) \Rightarrow (ix)$ Have proofs similar to that of the previous implications.

$(ix) \Rightarrow (i)$ Is trivial.
$(ix) \Leftrightarrow (x)$ Straightforward. □

Corollary 2. *Let R be a ring and I be a proper ideal of R.*

(i) $_RI$ *is a classical 2-absorbing submodule of R if and only if I is a 2-absorbing ideal of R.*

(ii) Every proper ideal of R is 2-absorbing if and only if for every R-module M and every proper submodule N of M, N is a classical 2-absorbing submodule of M.

Proof. (i) Let I be a classical 2-absorbing submodule of R. Then by Theorem 2, $(I :_R 1) = I$ is a 2-absorbing ideal of R. For the converse see part (i) of Proposition 2.

(ii) Assume that every proper ideal of R is 2-absorbing. Let N be a proper submodule of an R-module M. Since for every $m \in M\backslash N$, $(N :_R m)$ is a proper ideal of R, then it is a 2-absorbing ideal of R. Hence by Theorem 2, N is a classical 2-absorbing submodule of M. We have the converse immediately by part (i). □

Proposition 3. *Let M be an R-module and* $\{K_i \mid i \in I\}$ *be a chain of classical 2-absorbing submodules of M. Then* $\cap_{i\in I}K_i$ *is a classical 2-absorbing submodule of M.*

Proof. Suppose that $abcm \in \cap_{i\in I}K_i$ for some $a, b, c \in R$ and $m \in M$. Assume that $abm \notin \cap_{i\in I}K_i$ and $acm \notin \cap_{i\in I}K_i$. Then there are $t,\ l \in I$ where $abm \notin K_t$ and $acm \notin K_l$. Hence, for every $K_s \subseteq K_t$ and every $K_d \subseteq K_l$ we have that $abm \notin K_s$ and $acm \notin K_d$. Thus, for every submodule K_h such that $K_h \subseteq K_t$ and $K_h \subseteq K_l$ we get $bcm \in K_h$. Hence $bcm \in \cap_{i\in I}K_i$. □

A classical 2-absorbing submodule of M is called *minimal*, if for any classical 2-absorbing submodule K of M such that $K \subseteq N$, then $K = N$. Let L be a classical 2-absorbing submodule of M. Set

$$\Gamma = \{K \mid K \text{ is a classical 2-absorbing submodule of } M \text{ and } K \subseteq L\}.$$

If $\{K_i : i \in I\}$ is any chain in Γ, then $\cap_{i\in I}K_i$ is in Γ, by Proposition 3. By Zorn's Lemma, Γ contains a minimal member which is clearly a minimal classical 2-absorbing submodule of M. Thus, every classical 2 -absorbing submodule of M contains a minimal classical 2-absorbing submodule of M. If M is a finitely generated, then it is clear that M contains a minimal classical 2-absorbing submodule.

Theorem 3. *Let M be a Noetherian R-module. Then M contains a finite number of minimal classical* 2*-absorbing submodules.*

Proof. Suppose that the result is false. Let Γ denote the collection of proper submodules N of M such that the module M/N has an infinite number of minimal classical 2-absorbing submodules. Since $0 \in \Gamma$ we get $\Gamma \neq \emptyset$. Therefore Γ has a maximal member T, since M is a Noetherian R-module. It is clear that T is not a classical 2-absorbing submodule. Therefore, there exists an element $m \in M\backslash T$ and ideals I, J, K in R such that $IJKm \subseteq T$ but $IJm \nsubseteq T$, $IKm \nsubseteq T$ and $JKm \nsubseteq T$. The maximality of T implies that $M/(T + IJm)$, $M/(T + IKm)$

and $M/(T+JKm)$ have only finitely many minimal classical 2-absorbing submodules. Suppose P/T be a minimal classical 2-absorbing submodule of M/T. So $IJKm \subseteq T \subseteq P$, which implies that $IJm \subseteq P$ or $IKm \subseteq P$ or $JKm \subseteq P$. Thus $P/(T+IJm)$ is a minimal classical 2-absorbing submodule of $M/(T+IJm)$ or $P/(T+IKm)$ is a minimal classical 2-absorbing submodule of $M/(T+IKm)$ or $P/(T+JKm)$ is a minimal classical 2-absorbing submodule of $M/(T+JKm)$. Thus, there are only a finite number of possibilities for the submodule P. This is a contradiction. □

We recall from [5] that if I is a 2-absorbing ideal of a ring R, then either $\sqrt{I} = P$ where P is a prime ideal of R or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are the only distinct minimal prime ideals of I.

Corollary 3. *Let N be a classical 2-absorbing submodule of an R-module M. Suppose that $m \in M \backslash N$ and $\sqrt{(N :_R m)} = P$ where P is a prime ideal of R and $(N :_R m) \neq P$. Then for each $x \in \sqrt{(N :_R m)} \backslash (N :_R m)$, $(N :_R xm)$ is a prime ideal of R containing P. Furthermore, either $(N :_R xm) \subseteq (N :_R ym)$ or $(N :_R ym) \subseteq (N :_R xm)$ for every $x, y \in \sqrt{(N :_R m)} \backslash (N :_R m)$.*

Proof. By Theorem 2 and [5, Theorem 2.5]. □

Corollary 4. *Let N be a classical 2-absorbing submodule of an R-module M. Suppose that $m \in M \backslash N$ and $\sqrt{(N :_R m)} = P_1 \cap P_2$ where P_1 and P_2 are the only nonzero distinct prime ideals of R that are minimal over $(N :_R m)$. Then for each $x \in \sqrt{(N :_R m)} \backslash (N :_R m)$, $(N :_R xm)$ is a prime ideal of R containing P_1 and P_2. Furthermore, either $(N :_R xm) \subseteq (N :_R ym)$ or $(N :_R ym) \subseteq (N :_R xm)$ for every $x, y \in \sqrt{(N :_R m)} \backslash (N :_R m)$.*

Proof. By Theorem 2 and [5, Theorem 2.6]. □

An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of N and K is independent of presentations of N and K.

Proposition 4. *Let M be a multiplication R-module and N be a proper submodule of M. The following conditions are equivalent:*

(i) N is a classical 2-absorbing submodule of M;

(ii) If $N_1N_2N_3m \subseteq N$ for some submodules N_1, N_2, N_3 of M and $m \in M$, then either $N_1N_2m \subseteq N$ or $N_1N_3m \subseteq N$ or $N_2N_3m \subseteq N$.

Proof. $(i) \Rightarrow (ii)$ Let $N_1N_2N_3m \subseteq N$ for some submodules N_1, N_2, N_3 of M and $m \in M$. Since M is multiplication, there are ideals I_1, I_2, I_3 of R such that $N_1 = I_1M$, $N_2 = I_2M$ and $N_3 = I_3M$. Therefore $I_1I_2I_3m \subseteq N$, and so either $I_1I_2m \subseteq N$ or $I_1I_3m \subseteq N$ or $I_2I_3m \subseteq N$. Hence $N_1N_2m \subseteq N$ or $N_1N_3m \subseteq N$ or $N_2N_3m \subseteq N$.
$(ii) \Rightarrow (i)$ Suppose that $I_1I_2I_3m \subseteq N$ for some ideals I_1, I_2, I_3 of R and some $m \in M$. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 = I_3M$ in part (ii). □

In [16], Quartararo *et al.* said that a commutative ring R is a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them. They show that every Bézout ring is a u-ring. Moreover, they proved that every Prüfer domain is a u-domain. Also, any ring which contains an infinite field as a subring is a u-ring, [17, Exercise 3.63].

Theorem 4. *Let R be a um-ring, M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

(i) N is classical 2-absorbing;

(ii) For every $a,b,c \in R$, $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M ac)$ or $(N :_M abc) = (N :_M bc)$;

(iii) For every $a,b,c \in R$ and every submodule L of M, $abcL \subseteq N$ implies that $abL \subseteq N$ or $acL \subseteq N$ or $bcL \subseteq N$;

(iv) For every $a,b \in R$ and every submodule L of M with $abL \nsubseteq N$, $(N :_R abL) = (N :_R aL)$ or $(N :_R abL) = (N :_R bL)$;

(v) For every $a,b \in R$, every ideal I of R and every submodule L of M, $abIL \subseteq N$ implies that $abL \subseteq N$ or $aIL \subseteq N$ or $bIL \subseteq N$;

(vi) For every $a \in R$, every ideal I of R and every submodule L of M with $aIL \nsubseteq N$, $(N :_R aIL) = (N :_R aL)$ or $(N :_R aIL) = (N :_R IL)$;

(vii) For every $a \in R$, every ideals I, J of R and every submodule L of M, $aIJL \subseteq N$ implies that $aIL \subseteq N$ or $aJL \subseteq N$ or $IJL \subseteq N$;

(viii) For every ideals I, J of R and every submodule L of M with $IJL \nsubseteq N$, $(N :_R IJL) = (N :_R IL)$ or $(N :_R IJL) = (N :_R JL)$;

(ix) For every ideals I, J, K of R and every submodule L of M, $IJKL \subseteq N$ implies that $IJL \subseteq N$ or $IKL \subseteq N$ or $JKL \subseteq N$;

(x) For every submodule L of M not contained in N, $(N :_R L)$ is a 2-absorbing ideal of R.

Proof. Similar to the proof of Theorem 2. □

Proposition 5. *Let R be a um-ring and N be a proper submodule of an R-module M. Then N is a classical 2-absorbing submodule of M if and only if N is a 3-absorbing submodule of M and $(N :_R M)$ is a 2-absorbing ideal of R.*

Proof. It is trivial that if N is classical 2-absorbing, then it is 3-absorbing. Also, Theorem 4 implies that $(N :_R M)$ is a 2-absorbing ideal of R. Now, assume that N is a 3-absorbing submodule of M and $(N :_R M)$ is a 2-absorbing ideal of R. Let $a_1a_2a_3m \in N$ for some $a_1,a_2,a_3 \in R$ and $m \in M$ such that neither $a_1a_2m \in N$ nor $a_1a_3m \in N$ nor $a_2a_3m \in N$. Then $a_1a_2a_3 \in (N :_R M)$

and so either $a_1a_2 \in (N :_R M)$ or $a_1a_3 \in (N :_R M)$ or $a_2a_3 \in (N :_R M)$. This contradiction shows that N is classical 2-absorbing. □

Proposition 6. *Let M be an R-module and N be a classical 2-absorbing submodule of M. The following conditions hold:*

(i) For every $a, b, c \in R$ and $m \in M$, $(N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)$;

(ii) If R is a u-ring, then for every $a, b, c \in R$ and $m \in M$, $(N :_R abcm) = (N :_R abm)$ or $(N :_R abcm) = (N :_R acm)$ or $(N :_R abcm) = (N :_R bcm)$.

Proof. (i) Let $a, b, c \in R$ and $m \in M$. Suppose that $r \in (N :_R abcm)$. Then $abc(rm) \in N$. So, either $ab(rm) \in N$ or $ac(rm) \in N$ or $bc(rm) \in N$. Therefore, either $r \in (N :_R abm)$ or $r \in (N :_R acm)$ or $r \in (N :_R bcm)$. Consequently
$(N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)$.
(ii) Use part (i). □

Proposition 7. *Let R be a um-ring, M be a multiplication R-module and N be a proper submodule of M. The following conditions are equivalent:*

(i) N is a classical 2-absorbing submodule of M;

(ii) If $N_1N_2N_3N_4 \subseteq N$ for some submodules N_1, N_2, N_3, N_4 of M, then either $N_1N_2N_4 \subseteq N$ or $N_1N_3N_4 \subseteq N$ or $N_2N_3N_4 \subseteq N$;

(iii) If $N_1N_2N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M, then either $N_1N_2 \subseteq N$ or $N_1N_3 \subseteq N$ or $N_2N_3 \subseteq N$;

(iv) N is a 2-absorbing submodule of M;

(v) $(N :_R M)$ is a 2-absorbing ideal of R.

Proof. $(i) \Rightarrow (ii)$ Let $N_1N_2N_3N_4 \subseteq N$ for some submodules N_1, N_2, N_3, N_4 of M. Since M is multiplication, there are ideals I_1, I_2, I_3 of R such that $N_1 = I_1M$, $N_2 = I_2M$ and $N_3 = I_3M$. Therefore $I_1I_2I_3N_4 \subseteq N$, and so $I_1I_2N_4 \subseteq N$ or $I_1I_3N_4 \subseteq N$ or $I_2I_3N_4 \subseteq N$. Thus by Theorem 4, either $N_1N_2N_4 \subseteq N$ or $N_1N_3N_4 \subseteq N$ or $N_2N_3N_4 \subseteq N$.
$(ii) \Rightarrow (iii)$ Is easy.
$(iii) \Rightarrow (iv)$ Suppose that $I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 = K$ in part (iii).
$(iv) \Rightarrow (i)$ By part (i) of Proposition 2.
$(iv) \Rightarrow (v)$ By [15, Theorem 2.3].
$(v) \Rightarrow (iv)$ Let $I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M. Since M is multiplication, then there is an ideal I_3 of R such that $K = I_3M$. Hence $I_1I_2I_3 \subseteq (N :_R M)$ which implies that either $I_1I_2 \subseteq (N :_R M)$ or $I_1I_3 \subseteq (N :_R M)$ or $I_2I_3 \subseteq (N :_R M)$. If $I_1I_2 \subseteq (N :_R M)$, then we are done. So, suppose that $I_1I_3 \subseteq (N :_R M)$. Thus $I_1I_3M = I_1K \subseteq N$. Similarly if $I_2I_3 \subseteq (N :_R M)$, then we have $I_2K \subseteq N$. □

Definition 1. *Let R be a um-ring, M be an R-module and S be a subset of $M\setminus\{0\}$. If for all ideals I, J, Q of R and all submodules K, L of M, $(K+IJL)\cap S\neq\emptyset$ and $(K+IQL)\cap S\neq\emptyset$ and $(K+JQL)\cap S\neq\emptyset$ implies $(K+IJQL)\cap S\neq\emptyset$, then the subset S is called classical 2-absorbing m-closed.*

Proposition 8. *Let R be a um-ring, M be R-module and N a submodule of M. Then N is a classical 2-absorbing submodule if and only if $M\setminus N$ is a classical 2-absorbing m-closed.*

Proof. Suppose that N is a classical 2-absorbing submodule of M and I, J, Q are ideals of R and K, L are submodules of M such that $(K+IJL)\cap S\neq\emptyset$ and $(K+IQL)\cap S\neq\emptyset$ and $(K+JQL)\cap S\neq\emptyset$ where $S=M\setminus N$. Assume that $(K+IJQL)\cap S=\emptyset$. Then $K+IJQL\subseteq N$ and so $K\subseteq N$ and $IJQL\subseteq N$. Since N is a classical 2-absorbing submodule, we get $IJL\subseteq N$ or $IQL\subseteq N$ or $JQL\subseteq N$. If $IJL\subseteq N$, then we get $(K+IJL)\cap S=\emptyset$, since $K\subseteq N$. This is a contradiction. By the other cases we get similar contradictions. Now for the converse suppose that $S=M\setminus N$ is a classical 2-absorbing m-closed and assume that $IJQL\subseteq N$ for some ideals I, J, Q of R and submodule L of M. Then we get for submodule $K=(0)$, $K+IJQL\subseteq N$. Thus $(K+IJQL)\cap S=\emptyset$. Since S is a classical 2-absorbing m-closed, $(K+IJL)\cap S=\emptyset$ or $(K+IQL)\cap S=\emptyset$ or $(K+JQL)\cap S=\emptyset$. Hence $IJL\subseteq N$ or $IQL\subseteq N$ or $JQL\subseteq N$. So N is a classical 2-absorbing submodule. □

Proposition 9. *Let R be a um-ring, M be an R-module, N a submodule of M and $S=M\setminus N$. The following conditions are equivalent:*

(i) N is a classical 2-absorbing submodule of M;

(ii) S is a classical 2-absorbing m-closed;

(iii) For every ideals I, J, Q of R and every submodule L of M, if $IJL\cap S\neq\emptyset$ and $IQL\cap S\neq\emptyset$ and $JQL\cap S\neq\emptyset$, then $IJQL\cap S\neq\emptyset$;

(iv) For every ideals I, J, Q of R and every $m\in M$, if $IJm\cap S\neq\emptyset$ and $IQm\cap S\neq\emptyset$ and $JQm\cap S\neq\emptyset$, then $IJQm\cap S\neq\emptyset$.

Proof. It follows from the previous Proposition, Theorem 2 and Theorem 4. □

Theorem 5. *Let R be a um-ring, M be an R-module and S be a classical 2-absorbing m-closed. Then the set of all submodules of M which are disjoint from S has at least one maximal element. Any such maximal element is a classical 2-absorbing submodule.*

Proof. Let $\Psi=\{N\mid N$ is a submodule of M and $N\cap S=\emptyset\}$. Then $(0)\in\Psi\neq\emptyset$. Since Ψ is partially ordered by using Zorn's Lemma we get at least a maximal element of Ψ, say P, with property $P\cap S=\emptyset$. Now we will show that P is classical 2-absorbing. Suppose that $IJQL\subseteq P$ for ideals I, J, Q of R and submodule L of M. Assume that $IJL\nsubseteq P$ or $IQL\nsubseteq P$ or $JQL\nsubseteq P$. Then by the maximality of P we get $(IJL+P)\cap S\neq\emptyset$ and $(IQL+P)\cap S\neq\emptyset$ and $(JQL+P)\cap S\neq\emptyset$. Since S is a classical 2-absorbing m-closed we have $(IJQL+P)\cap S\neq\emptyset$. Hence $P\cap S\neq\emptyset$, which is a contradiction. Thus P is a classical 2-absorbing submodule of M. □

Theorem 6. *Let R be a um-ring and M be an R-module.*

(i) If F is a flat R-module and N is a classical 2-absorbing submodule of M such that $F\otimes N\neq F\otimes M$, then $F\otimes N$ is a classical 2-absorbing submodule of $F\otimes M$.

(ii) Suppose that F is a faithfully flat R-module. Then N is a classical 2-absorbing submodule of M if and only if $F\otimes N$ is a classical 2-absorbing submodule of $F\otimes M$.

Proof. (i) Let $a,b,c\in R$. Then we get by Theorem 4, $\left(N:_M abc\right)=\left(N:_M ab\right)$ or $\left(N:_M abc\right)=\left(N:_M ac\right)$ or $\left(N:_M abc\right)=\left(N:_M bc\right)$. Assume that $\left(N:_M abc\right)=\left(N:_M ab\right)$. Then by [4, Lemma 3.2],

$$\left(F\otimes N:_{F\otimes M} abc\right)=F\otimes\left(N:_M abc\right)=F\otimes\left(N:_M ab\right)=\left(F\otimes N:_{F\otimes M} ab\right).$$

Again Theorem 4 implies that $F\otimes N$ is a classical 2-absorbing submodule of $F\otimes M$.

(ii) Let N be a classical 2-absorbing submodule of M and assume that $F\otimes N=F\otimes M$. Then $0\to F\otimes N\overset{\subseteq}{\to}F\otimes M\to 0$ is an exact sequence. Since F is a faithfully flat module, $0\to N\overset{\subseteq}{\to}M\to 0$ is an exact sequence. So $N=M$, which is a contradiction. So $F\otimes N\neq F\otimes M$. Then $F\otimes N$ is a classical 2-absorbing submodule by (1). Now for conversely, let $F\otimes N$ be a classical 2-absorbing submodule of $F\otimes M$. We have $F\otimes N\neq F\otimes M$ and so $N\neq M$. Let $a,b,c\in R$. Then $\left(F\otimes N:_{F\otimes M} abc\right)=\left(F\otimes N:_{F\otimes M} ab\right)$ or $\left(F\otimes N:_{F\otimes M} abc\right)=\left(F\otimes N:_{F\otimes M} ac\right)$ or $\left(F\otimes N:_{F\otimes M} abc\right)=\left(F\otimes N:_{F\otimes M} bc\right)$ by Theorem 4. Assume that $\left(F\otimes N:_{F\otimes M} abc\right)=\left(F\otimes N:_{F\otimes M} ab\right)$. Hence

$$F\otimes\left(N:_M ab\right)=\left(F\otimes N:_{F\otimes M} ab\right)=\left(F\otimes N:_{F\otimes M} abc\right)=F\otimes\left(N:_M abc\right).$$

So $0\to F\otimes\left(N:_M ab\right)\overset{\subseteq}{\to}F\otimes\left(N:_M abc\right)\to 0$ is an exact sequence. Since F is a faithfully flat module, $0\to\left(N:_M ab\right)\overset{\subseteq}{\to}\left(N:_M abc\right)\to 0$ is an exact sequence which implies that $\left(N:_M ab\right)=\left(N:_M abc\right)$. Consequently N is a classical 2-absorbing submodule of M by Theorem 4. □

Corollary 5. *Let R be a um-ring, M be an R-module and X be an indeterminate. If N is a classical 2-absorbing submodule of M, then $N[X]$ is a classical 2-absorbing submodule of $M[X]$.*

Proof. Assume that N is a classical 2-absorbing submodule of M. Notice that $R[X]$ is a flat R-module. So by Theorem 6, $R[X]\otimes N\simeq N[X]$ is a classical 2-absorbing submodule of $R[X]\otimes M\simeq M[X]$. □

For an R-module M, the set of zero-divisors of M is denoted by $Z_R(M)$.

Proposition 10. *Let M be an R-module, N be a submodule and S be a multiplicative subset of R.*

(i) If N is a classical 2-absorbing submodule of M such that $\left(N:_R M\right)\cap S=\emptyset$, then $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$.

(ii) If $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$ such that $Z_R(M/N)\cap S=\emptyset$, then N is a classical 2-absorbing submodule of M.

Proof. (i) Let N be a classical 2-absorbing submodule of M and $\left(N:_R M\right)\cap S=\emptyset$. Suppose that $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{a_3}{s_3}\frac{m}{s_4}\in S^{-1}N$. Then there exist $n\in N$ and $s\in S$ such that $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{a_3}{s_3}\frac{m}{s_4}=\frac{n}{s}$. Therefore there exists an $s'\in S$ such that $s'sa_1a_2a_3m=s's_1s_2s_3s_4n\in N$. So $a_1a_2a_3(s^*m)\in N$ for $s^*=s's$. Since N is a classical 2-absorbing submodule we get $a_1a_2(s^*m)\in N$ or $a_1a_3(s^*m)\in N$ or $a_2a_3(s^*m)\in N$. Thus $\frac{a_1a_2m}{s_1s_2s_4}=\frac{a_1a_2(s^*m)}{s_1s_2s_4s^*}\in S^{-1}N$ or $\frac{a_1a_3m}{s_1s_3s_4}\in S^{-1}N$ or $\frac{a_2a_3m}{s_2s_3s_4}\in S^{-1}N$.

(ii) Assume that $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$ and $Z_R(M/N)\cap S=\emptyset$. Let $a,b,c\in R$ and $m\in M$ such that $abcm\in N$. Then $\frac{a}{1}\frac{b}{1}\frac{c}{1}\frac{m}{1}\in S^{-1}N$. Therefore $\frac{a}{1}\frac{b}{1}\frac{m}{1}\in S^{-1}N$ or $\frac{a}{1}\frac{c}{1}\frac{m}{1}\in S^{-1}N$ or $\frac{b}{1}\frac{c}{1}\frac{m}{1}\in S^{-1}N$. We may assume that $\frac{a}{1}\frac{b}{1}\frac{m}{1}\in S^{-1}N$. So there exists $u\in S$ such that $uabm\in N$. But $Z_R(M/N)\cap S=\emptyset$, whence $abm\in N$. Consequently N is a classical 2-absorbing submodule of M. □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i=1,2$. Let $R=R_1\times R_2$. Then $M=M_1\times M_2$ is an R-module and each submodule of M is in the form of $N=N_1\times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 7. *Let $R=R_1\times R_2$ be a decomposable ring and $M=M_1\times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N=N_1\times N_2$ is a proper submodule of M. Then the following conditions are equivalent:*

(i) N is a classical 2-absorbing submodule of M;

(ii) Either $N_1=M_1$ and N_2 is a classical 2-absorbing submodule of M_2 or $N_2=M_2$ and N_1 is a classical 2-absorbing submodule of M_1 or N_1, N_2 are classical prime submodules of M_1, M_2, respectively.

Proof. $(i)\Rightarrow(ii)$ Suppose that N is a classical 2-absorbing submodule of M such that $N_2=M_2$. From our hypothesis, N is proper, so $N_1\neq M_1$. Set $M'=\frac{M}{\{0\}\times M_2}$. Hence $N'=\frac{N}{\{0\}\times M_2}$ is a classical 2-absorbing submodule of M' by Corollary 1. Also observe that $M'\cong M_1$ and $N'\cong N_1$. Thus N_1 is a classical 2-absorbing submodule of M_1. Suppose that $N_1\neq M_1$ and $N_2\neq M_2$. We show that N_1 is a classical prime submodule of M_1. Since $N_2\neq M_2$, there exists $m_2\in M_2\backslash N_2$. Let $abm_1\in N_1$ for some $a,b\in R_1$ and $m_1\in M_1$. Thus

$$(a,1)(b,1)(1,0)(m_1,m_2)=(abm_1,0)\in N=N_1\times N_2.$$

So either $(a,1)(1,0)(m_1,m_2)=(am_1,0)\in N$ or $(b,1)(1,0)(m_1,m_2)=(bm_1,0)\in N$. Hence either $am_1\in N_1$ or $bm_1\in N_1$ which shows that N_1 is a classical prime submodule of M_1. Similarly we can show that N_2 is a classical prime submodule of M_2.

$(ii)\Rightarrow(i)$ Suppose that $N=N_1\times M_2$ where N_1 is a classical 2-absorbing (resp. classical prime) submodule of M_1. Then it is clear that N is a classical 2-absorbing (resp. classical prime) submodule of M. Now, assume that $N=N_1\times N_2$ where N_1 and N_2 are classical prime submodules of M_1 and M_2, respectively. Hence $(N_1\times M_2)\cap(M_1\times N_2)=N_1\times N_2=N$ is a classical 2-absorbing submodule of M, by Proposition 1. □

Lemma 1. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable ring and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R-module where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. A proper submodule N of M is a classical prime submodule of M if and only if $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, \ldots, n\}$, N_k is a classical prime submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, \ldots, n\} \backslash \{k\}$.*

Proof. ($\Rightarrow$) Let N be a classical prime submodule of M. We know $N = \times_{i=1}^{n} N_i$ where for every $1 \leq i \leq n$, N_i is a submodule of M_i, respectively. Assume that N_r is a proper submodule of M_r and N_s is a proper submodule of M_s for some $1 \leq r < s \leq n$. So, there are $m_r \in M_r \backslash N_r$ and $m_s \in M_s \backslash N_s$. Since

$$\begin{aligned}(0, \ldots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \ldots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \ldots, 0) \\ = (0, \ldots, 0) \in N,\end{aligned}$$

then either

$$\begin{aligned}(0, \ldots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \ldots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \ldots, 0) \\ = (0, \ldots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \ldots, 0) \in N\end{aligned}$$

or

$$\begin{aligned}(0, \ldots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \ldots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \ldots, 0) \\ = (0, \ldots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \ldots, 0) \in N,\end{aligned}$$

which is a contradiction. Hence exactly one of the N_i's is proper, say N_k. Now, we show that N_k is a classical prime submodule of M_k. Let $abm_k \in N_k$ for some $a, b \in R_k$ and $m_k \in M_k$. Therefore

$$\begin{aligned}(0, \ldots, 0, \overbrace{a}^{k\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{b}^{k\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{m_k}^{k\text{-th}}, 0, \ldots, 0) \\ = (0, \ldots, 0, \overbrace{abm_k}^{k\text{-th}}, 0, \ldots, 0) \in N,\end{aligned}$$

and so

$$(0, \ldots, 0, \overbrace{a}^{k\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{m_k}^{k\text{-th}}, 0, \ldots, 0) = (0, \ldots, 0, \overbrace{am_k}^{k\text{-th}}, 0, \ldots, 0) \in N$$

or

$$(0, \ldots, 0, \overbrace{b}^{k\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{m_k}^{k\text{-th}}, 0, \ldots, 0) = (0, \ldots, 0, \overbrace{bm_k}^{k\text{-th}}, 0, \ldots, 0) \in N.$$

Thus $am_k \in N_k$ or $bm_k \in N_k$ which implies that N_k is a classical prime submodule of M_k.
($\Leftarrow$) Is easy. □

Theorem 8. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposable ring and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R-module where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. For a proper submodule N of M the following conditions are equivalent:*

(i) N is a classical 2-absorbing submodule of M;

(ii) Either $N = \times_{t=1}^{n} N_t$ such that for some $k \in \{1, 2, \ldots, n\}$, N_k is a classical 2-absorbing submodule of M_k, and $N_t = M_t$ for every $t \in \{1, 2, \ldots, n\} \backslash \{k\}$ or $N = \times_{t=1}^{n} N_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$, N_k is a classical prime submodule of M_k, N_m is a classical prime submodule of M_m, and $N_t = M_t$ for every $t \in \{1, 2, \ldots, n\} \backslash \{k, m\}$.

Proof. We argue induction on n. For $n = 2$ the result holds by Theorem 7. Then let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \cdots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 7, N is a classical 2-absorbing submodule of M if and only if either $N = L \times M_n$ for some classical 2-absorbing submodule L of K or $N = K \times L_n$ for some classical 2-absorbing submodule L_n of M_n or $N = L \times L_n$ for some classical prime submodule L of K and some classical prime submodule L_n of M_n. Notice that by Lemma 1, a proper submodule L of K is a classical prime submodule of K if and only if $L = \times_{t=1}^{n-1} N_t$ such that for some $k \in \{1, 2, \ldots, n-1\}$, N_k is a classical prime submodule of M_k, and $N_t = M_t$ for every $t \in \{1, 2, \ldots, n-1\} \backslash \{k\}$. Consequently we reach the claim. □

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