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# Weighted Opial–type inequalities for fractional integral and differential operators involving generalized Mittag–Leffler functions

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**Abstract.** In this paper, by using Hölder integral inequality we give generalization of wighted Opial–type inequalities by using generalized fractional integral and differential operators involving generalized Mittag–Leffler functions.

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**Key Words and Phrases**: Opial–type inequality, fractional integral, fractional derivative, Mittag–Leffler function

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## 1. Introduction and preliminaries

In 1960 Opial established the following integral inequality [17].

> Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then
> $$\int_0^h |x(t)x'(t)|dt \le \frac{h}{4}\int_0^h \left(x'(t)\right)^2 dt, \tag{1}$$
> where constant $\frac{h}{4}$ is the best possible.

Opial's inequality [3, 4, 5, 6, 7, 14] is studied extensively by many researchers. It recognizes as a fundamental result in the theory of differential and difference equations and other areas of mathematics, and has attracted a great deal of attention in the literature

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(see, for instance, [1, 2]). In [3, 4, 5, 6, 7] Opial–type integral inequalities were considered for different kinds of fractional derivative and fractional integral operators for example Riemann-Liouville, Caputo, Canvati etc were established. Our paper is motivated by the work of Koliha and Pecaric [14] on Opial inequalities for fractional differential operators and presents a class of very general weighted Opial type inequalities using integral and differential operators in fractional calculus involving generalized Mittag-Leffler functions.

The following hypotheses are assumed throughout this section: Let $I$ be a closed interval in $\mathbb{R}, a$ is a fixed point in $I$, let $\Phi$ be a continuous function nonnegative on $I \times I$, and let $y, h \in C(I)$. We assume that the following condition involving $\Phi, h$ and $y$ is satisfied:

$$|y(x)| \leq \left| \int_a^x \Phi(x,t) |h(t)| \, dt \right|, \quad x \in I. \tag{2}$$

Koliha and Pecaric in [14] proved the following weighted Opial type inequalities by application of Hölder integral inequality.

**Theorem 1.** *Assume that (2) holds. Let $x \in I$, let $\alpha, \beta > 0$, $r > \max(1, \alpha)$, and let $U, V \in C(I)$ be such that $U(s) \geq 0$ and $V(s) > 0$ for all $s \in I$. Then*

$$\left| \int_a^x U(s) |y(s)|^\beta |h(s)|^\alpha \, ds \right| \leq C(x) \left| \int_a^x V(s) |h(s)|^r \, ds \right|^{(\alpha+\beta)/r} \tag{3}$$

*where*

$$C(x) = \left( \frac{\alpha}{\alpha+\beta} \right)^{\alpha/r} \left( \int_a^x \left( U^r(s) V^{-\alpha}(s) \right)^{1/(r-\alpha)} P(s)^{\beta(r-1)/(r-\alpha)} \, ds \right)^{(r-\alpha)/r}, \tag{4}$$

$$P(s) = \left| \int_a^s V(t)^{-1/(r-1)} \Phi(s,t)^{r/(r-1)} \, dt \right|. \tag{5}$$

**Theorem 2.** *Assume that (2) holds. Let $x \in I$, $\alpha, \beta > 0, r > \max(1, \alpha)$, and let $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$. Then*

$$\left| \int_a^x U(s) |y(s)|^\beta |h(s)|^\alpha \, ds \right| \leq \int_a^x U(\omega) \left| \int_a^\omega V(t) \Phi(\omega,t) \, dt \right|^{\frac{r-\alpha}{r}} d\omega \, \|V\|_\infty^\beta \, \|h\|_\infty^{\alpha+\beta}. \tag{6}$$

If the exponents $\alpha, \beta$ and $r$ in Theorem 2 are not necessarily positive, in this case the inequality (2) must be strengthened to equality

$$|y(s)| = \left| \int_a^s \Phi(s,t)\, |h(t)|\, dt \right|, \quad s \in I, \tag{7}$$

where $\Phi$ is a nonnegative continuous function on $I \times I$, and $y, h \in C(I)$.

**Theorem 3.** *Assume that (7) holds. Let $x \in I$, $U, V \in C(I)$ be such that $U(s) \geq 0$ and $V(s) > 0$ for all $s \in I$. Consider real numbers $\alpha, \beta, r$ and the following relations:*

*(i)* $r > 1,\ \beta > 0,\ 0 < \alpha < r$;
*(ii)* $r < \alpha < 0,\ \beta < 0$;
*(iii)* $-\alpha < \beta < 0,\ 0 < \alpha < r < 1$;
*(iv)* $\beta > 0, 0 < r < \min(\alpha, 1)$;
*(v)* $\alpha < 0 < r < 1, 0 < \beta < -\alpha$;
*(vi)* $\beta < 0,\ \alpha < 0,\ r > 1$;
*(vii)* $1 < r < \alpha,\ -\alpha < \beta < 0$;
*(viii)* $\beta > 0,\ r < 0 < \alpha$;
*(ix)* $\alpha < r < 0,\ 0 < \beta < -\alpha$.

*If one of the conditions (i)-(iii) is satisfied, then*

$$\left| \int_a^x U(s)\, |y(s)|^{\beta}\, |h(s)|^{\alpha}\, ds \right| \leq C(x) \left| \int_a^x V(s)\, |h(s)|^{r}\, ds \right|^{(\alpha+\beta)/r}. \tag{8}$$

*If one of the conditions (iv)-(ix) is satisfied, then*

$$\left| \int_a^x U(s)\, |y(s)|^{\beta}\, |h(s)|^{\alpha}\, ds \right| \geq C(x) \left| \int_a^x V(s)\, |h(s)|^{r}\, ds \right|^{(\alpha+\beta)/r}, \tag{9}$$

where $C(x)$ is defined by (4) and (5).

## 2. Fractional differential and integral operators involving Mittag-Leffler functions

Fractional calculus refers to integration and differentiation of fractional order. Several mathematicians contributed to this subject over the years. People like Liouville, Riemann, and Weyl made major contributions to the theory of fractional calculus. The story on the fractional calculus continued with contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov. For a historical survey the reader may see [12, 15, 16].

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators (see, [9, 11, 18, 20, 21] ).

Let $x > 0$. By $L^1(0,x)$ we denote the space of all Lebesgue integrable functions on the interval $(0,x)$. For any $f \in L^1(0,x)$ the *Riemann-Liouvill fractional integral* of $f$ of order $\nu$ is defined by

$$\left(I_{a+}^{\nu} f\right)(s) = \frac{1}{\Gamma(\nu)} \int_a^s (x-t)^{\nu-1} f(t)dt = (f * K_\nu)(s), \quad s \in [0,x], \quad \nu > 0, \tag{10}$$

where $K_\nu(s) = \frac{s^{\nu-1}}{\Gamma(\nu)}$. The integral on the right side of (10) exists for almost $s \in [0,x]$ and $I_{a+}^{\nu} f \in L^1(0,x)$.The *Riemann-Liouville fractional derivative of* $f \in L^1(0,x)$ of order $\nu$ is defined by

$$\left(D_{a+}^{\nu} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\nu} f\right)(x), (\nu > 0, n = [\nu] + 1) \tag{11}$$

By $C^m[0,x]$ we denote the space of all functions which have continuous derivatives up to order $m$, and $AC[0,x]$ is the space of all absolutely continuous functions on $[0,x]$. By $AC^m[0,x]$ we denote the space of all functions $f \in C^m[0,x]$ with $f^{(m-1)} \in AC[0,x]$. By $L^\infty(0,x)$ we denote the space of all measurable functions essentially bounden on $[0,x]$. Let $\mu > 0$, $m = [\mu] + 1$, $f \in AC^m[a,b]$. The Caputo derivative of order $\mu > 0$ is defined as

$$\left({}^C D_{a+}^{\mu} f\right)(x) = \left(I_{a+}^{m-\mu} \frac{d^m}{dx^m} f\right)(x) \tag{12}$$

$$= \frac{1}{\Gamma(m-\mu)} \int_a^x (x-s)^{m-\mu-1} \frac{d^m}{ds^m} f(s)\, ds.$$

**Definition 1.** *[21] Let $f \in L^1[a,b]$, $f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0 < \mu < 1$ and type $0 \le \nu \le 1$ with respect to $x \in [a,b]$ is defined by*

$$\left(D_{a+}^{\mu,\nu} f\right)(x) = \left(I_{a+}^{\nu(1-\mu)} \frac{d}{dx} \left(I_{a+}^{(1-\nu)(1-\mu)} f\right)\right)(x) \tag{13}$$

*whenever the right hand side exists.*

This generalization gives the classical Riemann-Liouville fractional differentiation operator if $\nu = 0$. For $\nu = 1$ it gives the fractional differential operator introduced by Caputo. We denote it by $D_{a+}^{\mu,1} f = {}^C D_{a+}^{\mu} f$.

Several authors (see, [9, 20]) called (13) the Hilfer fractional derivative. Applications of $D_{a+}^{\mu,\nu}$ are given in [9, 20, 21, 23].

The purpose of this paper is to give weighted Opial type integral inequalities involving different kinds of fractional differential operators. For $0 < \mu < 1$ and $0 < \nu \le 1$, the Hilfer fractional differentiation operator $D_{a+}^{\mu,\nu}$ can be rewritten in the form

$$\left(D_{a+}^{\mu,\nu} f\right)(x) = \left(I_{a+}^{\nu(1-\mu)} \left(D_{a+}^{\mu+\nu-\mu\nu} f\right)\right)(x) \tag{14}$$

$$= \frac{1}{\Gamma(\nu(1-\mu))} \int_{a+}^x (x-\tau)^{\nu(1-\mu)-1} \left(D_{a+}^{\mu+\nu-\mu\nu} f\right)(\tau)\, d\tau.$$

Definition of this generalized fractional integral operator containing Mittag–Leffler function is as follows.

**Definition 2.** *(Prabhakar [18]) Let $\mu, \nu, \gamma$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operator $\epsilon^{\gamma}_{\mu,\nu,\omega,a+}$ for a real-valued continuous function $f$ is defined by:*

$$(\epsilon^{\gamma}_{\mu,\nu,\omega,a+}f)(x) = \int_{a+}^{x} (x-t)^{\nu-1} E^{\gamma}_{\mu,\nu}(\omega(x-t)^{\mu}) f(t)dt, \tag{15}$$

*where the function $E^{\gamma}_{\mu,\nu}$ is generalized Mittag–Leffler function defined as*

$$E^{\gamma}_{\mu,\nu}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\Gamma(\mu n+\nu)} t^n, \tag{16}$$

*and $(\gamma)_n$ is the Pochhammer symbol: $(\gamma)_n = \gamma(\gamma+1)...(\gamma+n-1)$, $(\gamma)_0 = 1$.*

The integral operaor $\epsilon^{\gamma}_{\mu,\nu,\omega,a^+}$ is bounded in the space $C(I)$ with a finite norm $\|f\|_C = \max_{x\in I} |f(x)|$, and there exists a positive constant $M > 0$, such that (see [11])

$$\left\| \epsilon^{\gamma}_{\mu,\nu,\omega,a^+} f \right\|_C \leq M \|f\|_C \ .$$

For $\omega = 0$ in (15), integral operator $\epsilon^{\gamma}_{\mu,\nu,\omega,a^+}$ would correspond essentially to the Riemann-Liouville fractional integral operator $I^{\nu}_{a+}f$.

Let $e^{\gamma}_{\mu,\nu}(t,\omega) = t^{\nu-1} E^{\gamma}_{\mu,\nu}(-\omega t^{\mu})$. In [22] Tomovski et al. proved the following uniform estimate for the function $e^{\gamma}_{\mu,\nu}(\omega, t)$ :

**Lemma 1.** *If $\mu \in (0,1)$, $\gamma, \omega > 0$, $\mu\gamma > \nu - 1 > 0$, then the following uniform bound holds true*

$$\left| e^{\gamma}_{\mu,\nu}(t,\omega) \right| \leq \frac{\Gamma\left(\gamma - \frac{\nu-1}{\mu}\right)\Gamma\left(\frac{\nu-1}{\mu}\right)}{\pi\mu\omega^{\frac{\nu-1}{\mu}}\Gamma(\gamma)\left[\cos\left(\frac{\pi\mu}{2}\right)\right]^{\gamma-\frac{\nu-1}{\mu}}},\ t > 0. \tag{17}$$

**Lemma 2.** *[22] If $\mu \in (0,1)$, $\gamma, \omega > 0, \nu \geq \mu\gamma$, then $e^{\gamma}_{\mu,\nu}(t,\omega) > 0$, for all $t > 0$.*

We define a variant of Sobolev space:

$$W^{m,1}[a,b] = \left\{ f \in L^1[a,b] : \frac{d^m}{dt^m} f \in L^1[a,b] \right\}. \tag{18}$$

**Definition 3.** *(Prabhakar derivative [9]) Let $f \in L^1[0,b]$, $0 < t < b \leq \infty$, $\mu, \nu, \gamma > 0$, and $f * e^{-\gamma}_{\mu,m-\nu,\omega} \in W^{m,1}[0,b]$, $m = [\nu]$. Then the Prabhakar derivative is defined by following relation*

$$\left(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+} f\right)(t) = \frac{d^m}{dt^m} \epsilon^{-\gamma}_{\mu,m-\nu,\omega,0+} f(t). \tag{19}$$

**Definition 4.** *(Caputo-Prabhakar derivative [9]) Let* $f \in L^1[0,b]$, $0 < t < b \leq \infty$, $\mu, \nu, \gamma > 0$ , $m = [\nu]$. *Then the Caputo-Prabhakar derivative for* $f \in AC^m[0,b]$ *is defined by following relation*

$$\left({}^C D^{\gamma}_{\mu,\nu,\omega,0+} f\right)(t) = \epsilon^{-\gamma}_{\mu,m-\nu,\omega,0+} \frac{d^m}{dt^m} f(t) \tag{20}$$

$$= \left(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+} f\right)(t) - \sum_{k=0}^{m-1} t^{k-\mu} E^{-\gamma}_{\mu,k-\nu+1}(\omega t^{\mu}) f^{(k)}(0+).$$

**Remark 1.** *Let* $\mu, \nu, \gamma > 0$ *and* $f \in AC^m[0,b]$ , $0 < t < b \leq \infty$, *then*

$$\left({}^C D^{\gamma}_{\mu,\nu,\omega,0+} f\right)(t) = \mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+} \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0+)\right). \tag{21}$$

*Moreover, if* $f^{(k)}(0+) = 0$, $k = 0, 1, 2, ...m-1$, *then*

$$\left({}^C D^{\gamma}_{\mu,\nu,\omega,0+} f\right)(t) = \left(\mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+} f\right)(t).$$

**Definition 5.** *(Hilfer-Prabhakar derivative [9]). Let* $\mu \in (0,1)$, $\nu \in [0,1]$, *and let* $f \in L^1[a,b]$, $0 < t < b \leq \infty$, $f * e^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega} \in AC^1[0,b]$. *The Hilfer-Prabhakar derivative is defined by*

$$\left(\mathrm{D}^{\gamma,\,\mu,\,\nu}_{\rho,\,\omega,\,0+} f\right)(t) = \left(\epsilon^{-\gamma\nu}_{\rho,\,\nu(1-\mu),\,\omega,\,0+} \frac{d}{dt}\left(\epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+} f\right)\right)(t), \tag{22}$$

*where* $\gamma, \omega \in \mathbb{R}$, $\rho > 0$, *and* $\epsilon^{0}_{\rho,\,0,\,\omega,\,0+} f = f$. *Moreover,*

$$\left(\mathrm{D}^{\gamma,\,\mu}_{\rho,\,\omega,\,0+} f\right)(t) = \left(\mathrm{D}^{\gamma,\,\mu,\,1}_{\rho,\,\omega,\,0+} f\right)(t) = \left(\epsilon^{-\gamma}_{\rho,\,1-\mu,\,\omega,\,0+} \frac{d}{dt} f\right)(t). \tag{23}$$

## 3. Main Results

Our first main result is given in the following theorem. Namely, we present Opial type inequalities for Hilfer fractional operator (13).

**Theorem 4.** *Let* $x > 0$, $\alpha, \beta > 0$, $\mu \in (0,1)$, $\nu \in (0,1]$ *and* $U, V \in C(I)$ *be such that* $U(s) \geq 0$, $V(s) > 0$ *for all* $s \in I$. *Then let* $f \in L(0,x)$ *have an integrable fractional derivative* $D^{\mu+\nu-\mu\nu}_{0+} f \in L^{\infty}(0,x)$.

*(i) If* $r > \max\left\{1, \alpha, (\nu(1-\mu))^{-1}\right\}$, *then*

$$\int_0^x U(s) \left|\left(D^{\mu,\nu}_{0+} f\right)(s)\right|^{\beta} \left|\left(D^{\mu+\nu-\mu\nu}_{0+} f\right)(s)\right|^{\alpha} ds \leq \Omega(x) \times$$

$$\left(\int_0^x V(s)\left|\left(D_{0+}^{\mu+\nu-\mu\nu}f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}}, \quad (24)$$

*where*

$$\Omega(x)=\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}}\left(\int_0^x \left(U^r(s)V^{-\alpha}(s)\right)^{\frac{1}{r-\alpha}}(\Delta(s))^{\frac{\beta(r-1)}{r-\alpha}}ds\right)^{\frac{r-\alpha}{r}}, \quad (25)$$

$$\Delta(s)=\int_0^s (V(t))^{-\frac{1}{r-1}}\left[\frac{1}{\Gamma(\nu(1-\mu))}(s-t)^{\nu(1-\mu)-1}\right]^{\frac{r}{r-1}}dt. \quad (26)$$

*(ii) If* $0<r<\min\left\{\alpha,1,(\nu(1-\mu))^{-1}\right\}$, *then*

$$\int_0^x U(s)\left|\left(D_{0+}^{\mu,\nu}f\right)(s)\right|^{\beta}\left|\left(D_{0+}^{\mu+\nu-\mu\nu}f\right)(s)\right|^{\alpha}ds \geq \Omega(x)\times$$

$$\left(\int_0^x V(s)\left|\left(D_{0+}^{\mu+\nu-\mu\nu}f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}}, \quad (27)$$

*where* $\Omega(x)$ *and* $\Delta(s)$ *are given by (25) and (26).*

*Proof.* *According to (14),*

$$\left(D_{0+}^{\mu,\nu}f\right)(s)=\frac{1}{\Gamma(\nu(1-\mu))}\int_0^s (s-\tau)^{\nu(1-\mu)-1}\left(D_{0+}^{\mu+\nu-\mu\nu}f\right)(\tau)\,d\tau, \quad s\in[0,x]. \quad (28)$$

*Setting*

$$y(s)=\left(D_{0+}^{\mu,\nu}f\right)(s), \quad h(s)=\left(D_{0+}^{\mu+\nu-\mu\nu}f\right)(s), \quad \Phi(s,t)=\frac{(s-t)^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))},$$

*we observe that condition (2) is satisfied with* $a=0$ *and* $I=[0,x]$ :

$$|y(s)|\leq\int_0^s \Phi(s,t)|h(t)|\,dt, \quad 0\leq s\leq x.$$

*The rest of the proof of (i) is the same as Theorem 4.2 of [14].*

*For* $\nu=1$, *we obtain the following Opial type inequalities for Caputo fractional derivative, defined by (12).*

**Corollary 1.** *Let* $x>0$, $\alpha,\beta>0$, $\mu\in(0,1)$ *and* $U,V\in C(I)$ *be such that* $U(s)\geq 0$, $V(s)>0$ *for all* $s\in I$ *and let* $f\in AC^1(0,x)$.

*(i) If* $r > \max\left\{1, \alpha, (1-\mu)^{-1}\right\}$, *then*

$$\int_0^x U(s) \left|\left({}^C D_{0+}^{\mu} f\right)(s)\right|^{\beta} \left|\frac{d}{ds} f(s)\right|^{\alpha} ds \le \Omega(x) \left(\int_0^x V(s) \left|\frac{d}{ds} f(s)\right|^{r} ds\right)^{\frac{\alpha+\beta}{r}}, \tag{29}$$

*where*

$$\Omega(x) = \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} \left(\int_0^x \left(U^r(s) V^{-\alpha}(s)\right)^{\frac{1}{r-\alpha}} (\Delta(s))^{\frac{\beta(r-1)}{r-\alpha}} ds\right)^{\frac{r-\alpha}{r}}, \tag{30}$$

$$\Delta(s) = \int_0^s (V(t))^{-\frac{1}{r-1}} \left[\frac{1}{\Gamma(1-\mu)(s-t)^{\mu}}\right]^{\frac{r}{r-1}} dt. \tag{31}$$

*(ii) If* $0 < r < \min\left\{\alpha, 1, (1-\mu)^{-1}\right\}$, *then*

$$\int_0^x U(s) \left|\left({}^C D_{0+}^{\mu} f\right)(s)\right|^{\beta} \left|\frac{d}{ds} f(s)\right|^{\alpha} ds \ge \Omega(x) \left(\int_0^x V(s) \left|\frac{d}{ds} f(s)\right|^{r} ds\right)^{\frac{\alpha+\beta}{r}}, \tag{32}$$

*where* $\Omega(x)$ *and* $\Delta(s)$ *are given by (30) and (31).*

**Corollary 2.** *Let* $x > 0$, $\alpha, \beta > 0$, $\mu \in (0,1)$, $\nu \in (0,1]$ *and let* $f \in L(0,x)$ *have an integrable fractional derivative* $D_{0+}^{\mu+\nu-\mu\nu} f \in L^{\infty}(0,x)$.

*(i) If* $r > \max\left\{1, \alpha, (\nu(1-\mu))^{-1}\right\}$, *then*

$$\int_0^x \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^{\beta} \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^{\alpha} ds \le \Omega_1 x^{\beta\nu(1-\mu)-\frac{\alpha+1}{r}+1} \times \left(\int_0^x \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^{r} ds\right)^{\frac{\alpha+\beta}{r}}, \tag{33}$$

*where*

$$\Omega_1 = \frac{\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \left(\frac{r-1}{\nu(1-\mu)r-1}\right)^{\frac{\beta(r-1)}{r}}}{\left[\frac{\beta[\nu(1-\mu)r-1]}{r-\alpha}+1\right]^{\frac{r-\alpha}{r}}}. \tag{34}$$

*(ii) If* $0 < r < \min\left\{\alpha, 1, (\nu(1-\mu))^{-1}\right\}$, *then*

$$\int_0^x \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^\beta \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^\alpha ds \geq \Omega_1 x^{\beta\nu(1-\mu)-\frac{\alpha+1}{r}+1} \times$$

$$\left(\int_0^x \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}}, \quad (35)$$

*where* $\Gamma$ *is the Euler Gamma function and* $\Omega_1$ *is given by (34).*

*Proof. By Theorem 4,*

$$\int_0^x \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^\beta \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^\alpha ds \leq \Omega(x) \left(\int_0^x \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}}, \quad (36)$$

*where*

$$\Omega(x) = \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} \left(\int_0^x \left(\int_0^s \left[\frac{(s-t)^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))}\right]^{\frac{r}{r-1}} dt\right)^{\frac{\beta(r-1)}{r-\alpha}} ds\right)^{\frac{r-\alpha}{r}}$$

$$= \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \left(\frac{r-1}{\nu(1-\mu)r-1}\right)^{\frac{\beta(r-1)}{r}} \left(\int_0^x s^{\frac{\beta[\nu(1-\mu)r-1]}{r-\alpha}} ds\right)^{\frac{r-\alpha}{r}}$$

$$= \frac{\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \left(\frac{r-1}{\nu(1-\mu)r-1}\right)^{\frac{\beta(r-1)}{r}}}{\left[\frac{\beta[\nu(1-\mu)r-1]}{r-\alpha}+1\right]^{\frac{r-\alpha}{r}}} x^{\beta\nu(1-\mu)-\frac{\alpha+1}{r}+1}.$$

**Corollary 3.** *Let* $x>0$, $\alpha,\beta>0$, $p>q>0$, $\mu\in(0,1)$, $\nu\in(0,1]$. *Then let* $f\in L(0,x)$ *have an integrable fractional derivative* $D_{0+}^{\mu+\nu-\mu\nu} f \in L^\infty(0,x)$.

*(i) If* $r > \max\left\{1, \alpha, 1+q, (\nu(1-\mu))^{-1}\right\}$, *then*

$$\int_0^x s^p \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^\beta \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^\alpha ds \leq \Omega_2 x^{\beta\nu(1-\mu)+p-\frac{(\beta+\alpha)(q+1)}{r}+1} \quad (37)$$

$$\times \left(\int_0^x s^q \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}},$$

*where*

$$\Omega_2 = \frac{\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \left(B\left(\frac{r-1-q}{r-1}, \frac{\nu(1-\mu)r-1}{r-1}\right)\right)^{\frac{\beta(r-1)}{r}}}{\left[\frac{\beta[\nu(1-\mu)r-1-q]+pr-q\alpha}{r-\alpha}+1\right]^{\frac{r-\alpha}{r}}}. \quad (38)$$

*(ii) If* $0 < r < \min\left\{\alpha, 1, 1+q, (\nu(1-\mu))^{-1}\right\}$, *then*

$$\int_0^x s^p \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^\beta \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^\alpha ds \geq \Omega_2 x^{\beta\nu(1-\mu)+p-\frac{(\alpha+\beta)(q+1)}{r}+1} \times \left(\int_0^x s^q \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}}, \tag{39}$$

*where* $\Gamma$ *and* $B$ *are the Euler Gamma and Beta functions and* $\Omega_2$ *is given by (38).*

*Proof.* By Theorem 4,

$$\int_0^x s^p \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^\beta \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^\alpha ds \leq \Omega(x) \left(\int_0^x s^q \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^r ds\right)^{\frac{\alpha+\beta}{r}}, \tag{40}$$

where

$$\begin{aligned}
\Omega(x) &= \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} \\
&\quad \times \left(\int_0^x s^{\frac{pr-q\alpha}{r-\alpha}} \left(\int_0^s (t^q)^{-\frac{1}{r-1}} \left[\frac{(s-t)^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))}\right]^{\frac{r}{r-1}} dt\right)^{\frac{\beta(r-1)}{r-\alpha}} ds\right)^{\frac{r-\alpha}{r}} \\
&= \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \\
&\quad \times \left(\int_0^x s^{\frac{pr-q\alpha}{r-\alpha}} \left(\int_0^s t^{\frac{r-1-q}{r-1}-1} (s-t)^{\frac{\nu(1-\mu)r-1}{r-1}-1} dt\right)^{\frac{\beta(r-1)}{r-\alpha}} ds\right)^{\frac{r-\alpha}{r}} \\
&= \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \left(B\left(\frac{r-1-q}{r-1}, \frac{\nu(1-\mu)r-1}{r-1}\right)\right)^{\frac{\beta(r-1)}{r}} \\
&\quad \times \left(\int_0^x s^{\frac{\beta[\nu(1-\mu)r-1-q]+pr-q\alpha}{r-\alpha}} ds\right)^{\frac{r-\alpha}{r}} \\
&= \frac{\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} (\Gamma(\nu(1-\mu)))^{-\beta} \left(B\left(\frac{r-1-q}{r-1}, \frac{\nu(1-\mu)r-1}{r-1}\right)\right)^{\frac{\beta(r-1)}{r}}}{\left[\frac{\beta[\nu(1-\mu)r-1-q]+pr-q\alpha}{r-\alpha}+1\right]^{\frac{r-\alpha}{r}}} \\
&\quad \times x^{\beta\nu(1-\mu)+p-\frac{(\beta+\alpha)(q+1)}{r}+1}.
\end{aligned}$$

**Corollary 4.** *Let* $x>0,\ \alpha,\beta>0,\ \mu\in(0,1)$ *and let* $f\in AC^1(0,x)$.

*(i) If* $r>\max\left\{1,\alpha,(1-\mu)^{-1}\right\}$, *then*

$$\int_0^x \left|\left({}^C D_{0+}^{\mu} f\right)(s)\right|^{\beta}\left|\frac{d}{ds}f(s)\right|^{\alpha} ds \le \Omega_3 x^{\beta(1-\mu)-\frac{\alpha+1}{r}+1}\left(\int_0^x \left|\frac{d}{ds}f(s)\right|^{r} ds\right)^{\frac{\alpha+\beta}{r}}, \tag{41}$$

*where*

$$\Omega_3=\frac{\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}}\left(\Gamma(1-\mu)\right)^{-\beta}\left(\frac{r-1}{(1-\mu)r-1}\right)^{\frac{\beta(r-1)}{r}}}{\left[\frac{\beta[(1-\mu)r-1]}{r-\alpha}+1\right]^{\frac{r-\alpha}{r}}}. \tag{42}$$

*(ii) If* $0<r<\min\left\{\alpha,1,(1-\mu)^{-1}\right\}$, *then*

$$\int_0^x \left|\left({}^C D_{0+}^{\mu} f\right)(s)\right|^{\beta}\left|\frac{d}{ds}f(s)\right|^{\alpha} ds \ge \Omega_3 x^{\beta(1-\mu)-\frac{\alpha+1}{r}+1}\left(\int_0^x \left|\frac{d}{ds}f(s)\right|^{r} ds\right)^{\frac{\alpha+\beta}{r}}, \tag{43}$$

*where* $\Gamma$ *is the Euler Gamma function and* $\Omega_3$ *is given by (42).*

**Example 1.** *If we put* $\alpha=1,\beta=1,\ r=2,\ \mu,\nu\in(0,1),\ 2\nu(1-\mu)\ge 1,\ \gamma,\omega>0,\nu\ge\mu\gamma$, $U(s)=1,\ V(s)=\frac{1}{e_{\mu,\nu}^{\gamma}(s,\omega)}$ *in Theorem 4, by using the integral formula (see [18])*

$$\int_0^x (x-t)^{\nu-1}E_{\mu,\nu}^{\gamma}\left(\omega(x-t)^{\mu}\right)t^{\delta-1}dt=\Gamma(\delta)\,x^{\nu+\delta-1}E_{\mu,\nu+\delta}^{\gamma}\left(\omega x^{\mu}\right)$$

*we obtain*

$$\begin{aligned}
\Delta(s) &= \int_0^s e_{\mu,\nu}^{\gamma}(t,\omega)\left[\frac{1}{\Gamma(\nu(1-\mu))}(s-t)^{\nu(1-\mu)-1}\right]^2 dt\\
&= \frac{1}{\Gamma^2(\nu(1-\mu))}\int_0^s t^{\nu-1}E_{\mu,\nu}^{\gamma}\left(-\omega t^{\mu}\right)(s-t)^{2[\nu(1-\mu)-1]}dt\\
&= \frac{1}{\Gamma^2(\nu(1-\mu))}\int_0^s (s-t)^{\nu-1}E_{\mu,\nu}^{\gamma}\left(-\omega(s-t)^{\mu}\right)t^{2[\nu(1-\mu)-1]}dt\\
&= \frac{\Gamma(2\nu(1-\mu)-1)}{\Gamma^2(\nu(1-\mu))}s^{\nu+2[\nu(1-\mu)-1]}E_{\mu,\nu+2\nu(1-\mu)-1}^{\gamma}\left(-\omega s^{\mu}\right)\\
&= \frac{1}{\Gamma(2\nu(1-\mu)+1)}e_{\mu,\nu+2\nu(1-\mu)-1}^{\gamma}(s,\omega).
\end{aligned}$$

*Hence,*

$$\Omega(x)=\frac{\sqrt{2}}{2\Gamma(2\nu(1-\mu)+1)}\left(\int_0^x e^{\gamma}_{\mu,\nu}(s,\omega)\,e^{\gamma}_{\mu,\nu+2\nu(1-\mu)-1}(s,\omega)\,ds\right)^{1/2}. \tag{44}$$

*By Lemma 2.2, we obtain that* $e^{\gamma}_{\mu,\nu}(s,\omega)\,e^{\gamma}_{\mu,\nu+2\nu(1-\mu)-1}(s,\omega)>0$, *for all* $s\in(0,x]$, *i.e.* $\Omega(x)>0$, *for all* $x>0$. *By Theorem 4, we obtain,*

$$\int_0^x \left|\left(D^{\mu,\nu}_{0+}f\right)(s)\right|\left|\left(D^{\mu+\nu-\mu\nu}_{0+}f\right)(s)\right|ds\le\Omega(x)\left(\int_0^x\frac{\left|\left(D^{\mu+\nu-\mu\nu}_{0+}f\right)(s)\right|^2}{e^{\gamma}_{\mu,\nu}(s,\omega)}ds\right), \tag{45}$$

*where* $\Omega(x)$ *is given by (44).*

**Theorem 5.** *Let* $x>0$, $x\in I$, $\alpha,\beta>0, r>\max(1,\alpha)$, $\mu\in(0,1)$, $\nu\in(0,1]$ *and let* $U,V\in C(I)$ *be such that* $U(s)\ge 0$, $V(s)>0$ *for all* $s\in I$. *If* $f\in L(0,x)$ *have an integrable fractional derivative* $D^{\mu+\nu-\mu\nu}_{0+}f\in L^{\infty}(0,x)$, *then*

$$\left|\int_0^x U(s)\left|\left(D^{\mu,\nu}_{0+}f\right)(s)\right|^{\beta}\left|\left(D^{\mu+\nu-\mu\nu}_{0+}f\right)(s)\right|^{\alpha}ds\right|$$
$$\le\left(\frac{1}{\Gamma(\nu(1-\mu))}\right)^{\frac{r-\alpha}{r}}\int_0^x U(\lambda)\left|\int_0^{\lambda}V(t)(\lambda-t)^{\mu+\nu-\mu\nu}dt\right|^{\frac{r-\alpha}{r}}d\lambda\times$$
$$\|V\|^{\beta}_{\infty}\left\|\left(D^{\mu+\nu-\mu\nu}_{0+}f\right)\right\|^{\alpha+\beta}_{\infty}.$$

If we take $U=V=1$ in Theorem 5, since

$$\int_0^x\left|\int_0^{\lambda}(\lambda-t)^{\mu+\nu-\mu\nu}dt\right|^{\frac{r-\alpha}{r}}d\lambda$$
$$=\frac{1}{(\mu+\nu-\mu\nu+1)^{\frac{r-\alpha}{r}}}\int_0^x\lambda^{(\mu+\nu-\mu\nu+1)\frac{r-\alpha}{r}}d\lambda$$
$$=\frac{x^{(\mu+\nu-\mu\nu+1)\frac{r-\alpha}{r}+1}}{(\mu+\nu-\mu\nu+1)^{\frac{r-\alpha}{r}}\left[(\mu+\nu-\mu\nu+1)^{\frac{r-\alpha}{r}}+1\right]},$$

we get the following special inequality of Theorem 5.

**Corollary 5.** *Let* $x>0$, $\alpha,\beta>0, r>\max(1,\alpha)$, $\mu\in(0,1)$, $\nu\in(0,1]$. *If* $f\in L(0,x)$ *have an integrable fractional derivative* $D^{\mu+\nu-\mu\nu}_{0+}f\in L^{\infty}(0,x)$, *then*

$$\left|\int_0^x \left|\left(D_{0+}^{\mu,\nu} f\right)(s)\right|^{\beta} \left|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)(s)\right|^{\alpha} ds\right| \leq \left(\frac{1}{\Gamma\left(\nu\left(1-\mu\right)\right)\left(\mu+\nu-\mu\nu+1\right)}\right)^{\frac{r-\alpha}{r}} \tag{46}$$

$$\times \frac{x^{(\mu+\nu-\mu\nu+1)\frac{r-\alpha}{r}+1}}{\left[\left(\mu+\nu-\mu\nu+1\right)^{\frac{r-\alpha}{r}}+1\right]} \left\|\left(D_{0+}^{\mu+\nu-\mu\nu} f\right)\right\|_{\infty}^{\alpha+\beta}.$$

**Corollary 6.** *Let $x>0,\ x\in I,\ \alpha,\beta>0, r>\max(1,\alpha),\ \mu\in(0,1)$ and let $U,V\in C(I)$ be such that $U(s)\geq 0,\ V(s)>0$ for all $s\in I$. If $f\in L(0,x)$ have an integrable fractional derivative $f\in AC^1(0,x)$, then*

$$\left|\int_0^x U(s)\left|\left({}^C D_{0+}^{\mu} f\right)(s)\right|^{\beta} \left|\frac{d}{ds} f(s)\right|^{\alpha} ds\right| \tag{47}$$

$$\leq \left(\frac{1}{\Gamma(1-\mu)}\right)^{\frac{r-\alpha}{r}} \int_0^x U(\lambda)\left|\int_0^{\lambda} V(t)(\lambda-t)\,dt\right|^{\frac{r-\alpha}{r}} d\lambda \left\|V\right\|_{\infty}^{\beta} \left\|\left(\frac{d}{ds} f(s)\right)\right\|_{\infty}^{\alpha+\beta}.$$

*Moreover for $U(s)=V(s)=1$, there holds*

$$\left|\int_0^x \left|\left({}^C D_{0+}^{\mu} f\right)(s)\right|^{\beta} \left|\frac{d}{ds} f(s)\right|^{\alpha} ds\right| \leq \left(\frac{rx}{3r-2\alpha}\right)\left(\frac{x^2}{2\Gamma(1-\mu)}\right)^{\frac{r-\alpha}{r}} \left\|\left(\frac{d}{ds} f(s)\right)\right\|_{\infty}^{\alpha+\beta}. \tag{48}$$

We present some interesting Opial type inequalities regarding Prabhakar integral operator (15) and Riemamn-Liouville integral operator.

**Theorem 6.** *Let $x>0,\ x\in I,\ \alpha,\beta>0,\ r>\max(1,\alpha),\ \mu\in(0,1),\ \gamma,\omega>0,\ \mu\gamma>\nu-1>0$ and let $U,V\in C(I)$ be such that $U(s)\geq 0$ and $V(s)>0$ for all $s\in I$. If $h\in L(0,x)$, then*

$$\left|\int_0^x U(s)\left|(\epsilon_{\mu,\nu,\omega,0+}^{\gamma} h)(s)\right|^{\beta} \left|h(s)\right|^{\alpha} ds\right| \leq C(x)\left|\int_0^x V(s)\left|h(s)\right|^{r} ds\right|^{(\alpha+\beta)/r}, \tag{49}$$

*where*

$$C(x)=\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r}\left(\frac{\Gamma\left(\gamma-\frac{\nu-1}{\mu}\right)\Gamma\left(\frac{\nu-1}{\mu}\right)}{\pi\mu\omega^{\frac{\nu-1}{\mu}}\Gamma(\gamma)\left[\cos\left(\frac{\pi\mu}{2}\right)\right]^{\gamma-\frac{\nu-1}{\mu}}}\right)^{\beta} \tag{50}$$

$$\times\left(\int_0^x \left(U^r(s)V^{-\alpha}(s)\right)^{1/(r-\alpha)}\left(\int_0^s (V(t))^{-1/(r-1)}\,dt\right)^{\beta(r-1)/(r-\alpha)} ds\right)^{(r-\alpha)/r}.$$

**Corollary 7.** *Let $x > 0$ and $h \in L(0, x)$. If $\alpha, \beta > 0$, $r > \max(1, \alpha)$, $\mu \in (0, 1)$, $\gamma, \omega > 0$, $\mu\gamma > \nu - 1 > 0$, then*

$$\left| \int_0^x \left| (\epsilon^{\gamma}_{\mu,\nu,\omega,0+} h)(s) \right|^{\beta} |h(s)|^{\alpha} \, ds \right| \le C(x) \left| \int_0^x |h(s)|^r \, ds \right|^{(\alpha+\beta)/r}, \tag{51}$$

*where*

$$C(x) = \frac{\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r} \left( \frac{\Gamma\left(\gamma - \frac{\nu-1}{\mu}\right)\Gamma\left(\frac{\nu-1}{\mu}\right)}{\pi\mu\omega^{\frac{\nu-1}{\mu}}\Gamma(\gamma)\left[\cos\left(\frac{\pi\mu}{2}\right)\right]^{\gamma-\frac{\nu-1}{\mu}}} \right)^{\beta}}{\left(\frac{\beta(r-1)}{r-\alpha} + 1\right)^{\frac{r-\alpha}{r}}} x^{\frac{\beta(r-1)+r-\alpha}{r}}. \tag{52}$$

**Theorem 7.** *Let $x > 0$, $x \in I$, $\alpha, \beta > 0$, $r > \max(1, \alpha)$, $\mu$, $\nu$, $\gamma > 0$, $\omega \in \mathbb{R}$ and let $U, V, h \in C(I)$ be such that $U(s) \ge 0$, $V(s) > 0$ for all $s \in I$. Then the following inequality holds true:*

$$\left| \int_0^x U(s) \left| (\epsilon^{\gamma}_{\mu,\nu,\omega,0+} h)(s) \right|^{\beta} |h(s)|^{\alpha} \, ds \right| \tag{53}$$

$$\le \int_0^x U(\lambda) \left| \int_0^{\lambda} V(t) \, e^{\gamma}_{\mu,\nu}(\lambda - t, \omega) \, dt \right|^{\frac{r-\alpha}{r}} d\lambda \, \|V\|^{\beta}_{\infty} \, \|h\|^{\alpha+\beta}_{\infty}.$$

*If we take $U = V = 1$ and $\omega = 0$ in Theorem 7, since*

$$\int_0^x \left| \int_0^{\lambda} \left[ \frac{1}{\Gamma(\nu)} (\lambda - t)^{\nu-1} \right] dt \right|^{\frac{r-\alpha}{r}} d\lambda = \frac{rx}{r(\nu+1) - \nu\alpha} \left( \frac{x^{\nu}}{\Gamma(\nu+1)} \right)^{\frac{r-\alpha}{r}}$$

*we get the following Opial type inequality regarding Riemann-Liouville integral operator $I^{\nu}_{0+}h$:*

**Corollary 8.** *Let $x > 0$, $x \in I$, $\alpha, \beta > 0$, $r > \max(1, \alpha)$, $\nu > 0$. If $h \in L(0, x)$ and $h \in C(I)$, then*

$$\int_0^x \left| (I^{\nu}_{0+} h)(s) \right|^{\beta} |h(s)|^{\alpha} \, ds \le \frac{rx}{r(\nu+1) - \nu\alpha} \left( \frac{x^{\nu}}{\Gamma(\nu+1)} \right)^{\frac{r-\alpha}{r}} \|h\|^{\alpha+\beta}_{\infty}. \tag{54}$$

We present some Opial type inequalities for Prabhakar operator (15) and Caputo-Prabhakar derivative (12).

**Theorem 8.** *Let $x > 0$, $x \in I$, $f \in L(0, x)$, and let $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$, $\alpha, \beta > 0$, $\mu, \nu, \gamma > 0, m = [\nu]$, and $f \in AC^m[0, x]$.*
*(i) If $r > \max(1, \alpha)$, then*

$$\int_0^x U(s) \left| \left( {}^C D^{\gamma}_{\mu,\nu,\omega,0+} f \right)(s) \right|^{\beta} \left| f^{(m)}(s) \right|^{\alpha} ds \leq \Omega_4(x) \left( \int_0^x V(s) \left| f^{(m)}(s) \right|^r ds \right)^{\frac{\alpha+\beta}{r}}, \quad (55)$$

*where*

$$\Omega_4(x) = \left( \frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{r}} \left( \int_0^x \left( U^r(s) V^{-\alpha}(s) \right)^{\frac{1}{r-\alpha}} (\Delta(s))^{\frac{\beta(r-1)}{r-\alpha}} ds \right)^{\frac{r-\alpha}{r}}, \quad (56)$$

$$\Delta(s) = \int_0^s (V(t))^{-\frac{1}{r-1}} \left[ e^{\gamma}_{\mu,m-\nu}(s-t,\omega) \right]^{\frac{r}{r-1}} dt. \quad (57)$$

*(ii) If $r < \max(1, \alpha)$, then*

$$\int_0^x U(s) \left| \left( {}^C D^{\gamma}_{\mu,\nu,\omega,0+} f \right)(s) \right|^{\beta} \left| f^{(m)}(s) \right|^{\alpha} ds \geq \Omega_4(x) \left( \int_0^x V(s) \left| f^{(m)}(s) \right|^r ds \right)^{\frac{\alpha+\beta}{r}}, \quad (58)$$

*where $\Omega_4(x)$ is given by (56) and (57).*

**Corollary 9.** *Let $x > 0$ and $f \in L(0, x)$, and let $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$, $\alpha, \beta > 0$, $\mu, \nu, \gamma > 0$, $\omega \in \mathbb{R}$ and $f * e^{-\gamma}_{\mu,m-\nu,\omega} \in W^{m,1}(0, x)$, $m = [\nu]$, $f \in AC^m[0, x]$, $f^{(k)}(0+) = 0$, $k = 0, 1, 2, ...m - 1$.*

*(i) If $r > \max(1, \alpha)$, then*

$$\int_0^x U(s) \left| \left( \mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+} f \right)(s) \right|^{\beta} \left| f^{(m)}(s) \right|^{\alpha} ds \leq \Omega_4(x) \left( \int_0^x V(s) \left| f^{(m)}(s) \right|^r ds \right)^{\frac{\alpha+\beta}{r}}. \quad (59)$$

*(ii) If $r < \max(1, \alpha)$, then*

$$\int_0^x U(s) \left| \left( \mathbf{D}^{\gamma}_{\mu,\nu,\omega,0+} f \right)(s) \right|^{\beta} \left| f^{(m)}(s) \right|^{\alpha} ds \geq \Omega_4(x) \left( \int_0^x V(s) \left| f^{(m)}(s) \right|^r ds \right)^{\frac{\alpha+\beta}{r}}, \quad (60)$$

*where $\Omega_4(x)$ is given by (56) and (57).*

**Theorem 9.** *Let $x > 0$, $h \in L(0, x)$, $x \in I$, $\alpha, \beta > 0, r > \max(1, \alpha)$, $\mu, \nu, \gamma > 0$ and let $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$. If $m = [\nu]$, $h \in AC^m[0, x]$, then*

$$\left| \int_0^x U(s) \left| \left( {}^C D^{\gamma}_{\mu,\nu,\omega,0+} h \right)(s) \right|^{\beta} \left| h^{(m)}(s) \right|^{\alpha} ds \right| \quad (61)$$

$$\leq \int_0^x U(\lambda) \left| \int_0^\lambda V(t)\, e^{-\gamma}_{\mu,m-\nu}(\lambda - t, \omega)\, dt \right|^{\frac{r-\alpha}{r}} d\lambda \, \|V\|_\infty^\beta \left\| h^{(m)} \right\|_\infty^{\alpha+\beta}.$$

**Theorem 10.** *Let $x > 0$, $h \in L(0,x)$, $x \in I$, and let $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$, $\alpha, \beta > 0, r > \max(1, \alpha)$, $\mu, \nu, \gamma > 0$. If $h * e^{-\gamma}_{\mu,m-\nu,\omega} \in W^{m,1}(0,x)$, $m = [\nu]$, $h \in AC^m[0,x]$, $h^{(k)}(0+) = 0$, $k = 0, 1, 2, ..., m-1$, then*

$$\left| \int_0^x U(s) \left| \left( \mathbf{D}^\gamma_{\mu,\nu,\omega,0+} h \right)(s) \right|^\beta \left| h^{(m)}(s) \right|^\alpha ds \right| \tag{62}$$

$$\leq \int_0^x U(\lambda) \left| \int_0^\lambda V(t)\, e^{-\gamma}_{\mu,m-\nu}(\lambda - t, \omega)\, dt \right|^{\frac{r-\alpha}{r}} d\lambda \, \|V\|_\infty^\beta \left\| h^{(m)} \right\|_\infty^{\alpha+\beta}.$$

**Theorem 11.** *Let $x > 0$, $\alpha, \beta, \rho > 0$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\gamma, \omega \in \mathbb{R}$ and $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$. Also let $f \in L(0,x)$ and $f * e^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega} \in AC^1(0,x)$.*

*(i) If $r > \max\{1, \alpha\}$, then*

$$\int_0^x U(s) \left| \left( \mathrm{D}^{\gamma,\,\mu,\,\nu}_{\rho,\,\omega,\,0+} f \right)(s) \right|^\beta \left| \frac{d}{ds} \left( \epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+} f \right)(s) \right|^\alpha ds \tag{63}$$

$$\leq \Omega_5(x) \left( \int_0^x V(s) \left| \frac{d}{ds} \left( \epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+} f \right)(s) \right|^r ds \right)^{\frac{\alpha+\beta}{r}},$$

*where*

$$\Omega_5(x) = \left( \frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{r}} \left( \int_0^x \left( U^r(s)\, V^{-\alpha}(s) \right)^{\frac{1}{r-\alpha}} (\Delta(s))^{\frac{\beta(r-1)}{r-\alpha}} ds \right)^{\frac{r-\alpha}{r}}, \tag{64}$$

$$\Delta(s) = \int_0^s (V(t))^{-\frac{1}{r-1}} \left[ e^{-\gamma\nu}_{\rho,\,\nu(1-\mu)}(s-t, \omega) \right]^{\frac{r}{r-1}} dt. \tag{65}$$

*(ii) If $0 < r < \min\{\alpha, 1\}$, then*

$$\int_0^x U(s) \left| \left( \mathrm{D}^{\gamma,\,\mu,\,\nu}_{\rho,\,\omega,\,0+} f \right)(s) \right|^\beta \left| \frac{d}{ds} \left( \epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+} f \right)(s) \right|^\alpha ds \tag{66}$$

$$\geq \Omega_5(x) \left( \int_0^x V(s) \left| \frac{d}{ds} \left( \epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+} f \right)(s) \right|^r ds \right)^{\frac{\alpha+\beta}{r}},$$

*where $\Omega_5(x)$ and $\Delta(s)$ are given by (64) and (65).*

**Corollary 10.** *Let* $x>0,\ f\in W^{1,1}(0,x),\ \alpha,\beta,\rho>0,\ \mu\in(0,1),\ \gamma,\omega\in\mathbb{R}$ *and* $U,V\in C(I)$ *be such that* $U(s)\geq 0,\ V(s)>0$ *for all* $s\in I$.

*(i) If* $r>\max\{1,\alpha\}$, *then*

$$\int_0^x U(s)\left|\left(\mathrm{D}^{\gamma,\,\mu}_{\rho,\,\omega,\,0+}f\right)(s)\right|^{\beta}\left|\frac{d}{ds}f(s)\right|^{\alpha}ds\leq\Omega_6(x)\left(\int_0^x V(s)\left|\frac{d}{ds}f(s)\right|^{r}ds\right)^{\frac{\alpha+\beta}{r}},\tag{67}$$

*where*

$$\Omega_6(x)=\left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}}\left(\int_0^x\left(U^r(s)V^{-\alpha}(s)\right)^{\frac{1}{r-\alpha}}(\Delta(s))^{\frac{\beta(r-1)}{r-\alpha}}ds\right)^{\frac{r-\alpha}{r}},\tag{68}$$

$$\Delta(s)=\int_0^s(V(t))^{-\frac{1}{r-1}}\left[e^{-\gamma}_{\rho,\,1-\mu}(s-t,\omega)\right]^{\frac{r}{r-1}}dt.\tag{69}$$

*(ii) If* $0<r<\min\{\alpha,1\}$, *then*

$$\int_0^x U(s)\left|\left(\mathrm{D}^{\gamma,\,\mu}_{\rho,\,\omega,\,0+}f\right)(s)\right|^{\beta}\left|\frac{d}{ds}f(s)\right|^{\alpha}ds\geq\Omega_6(x)\left(\int_0^x V(s)\left|\frac{d}{ds}f(s)\right|^{r}ds\right)^{\frac{\alpha+\beta}{r}},\tag{70}$$

*where* $\Omega_6(x)$ *and* $\Delta(s)$ *are given by (68) and (69).*

Finally, we present Opial type inequalities regarding Hilfer-Prabhakar operator.

**Theorem 12.** *Let* $x>0,\ \alpha,\beta,\rho>0,\ \mu\in(0,1),\nu\in[0,1],\ \gamma,\omega\in\mathbb{R}$ *and* $U,V\in C(I)$ *be such that* $U(s)\geq 0,\ V(s)>0$ *for all* $s\in I$ *and* $f\in L(0,x)$, $f*e^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega}\in AC^1(0,x)$. *If* $r>\max\{1,\alpha\}$, *then*

$$\left|\int_0^x U(s)\left|\left(\mathrm{D}^{\gamma,\,\mu,\,\nu}_{\rho,\,\omega,\,0+}f\right)(s)\right|^{\beta}\left|\frac{d}{ds}\left(\epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+}f\right)(s)\right|^{\alpha}ds\right|$$
$$\leq\int_0^x U(\lambda)\left|\int_0^\lambda V(t)\,e^{-\gamma\nu}_{\rho,\,\nu(1-\mu)}(\lambda-t,\omega)\,dt\right|^{\frac{r-\alpha}{r}}d\lambda\times$$
$$\|V\|^{\beta}_{\infty}\left\|\frac{d}{ds}\left(\epsilon^{-\gamma(1-\nu)}_{\rho,\,(1-\nu)(1-\mu),\,\omega,\,0+}f\right)(s)\right\|^{\alpha+\beta}_{\infty}.$$

**Corollary 11.** *Let* $x>0,\ \alpha,\beta,\rho>0,\ \mu\in(0,1),\ \gamma,\omega\in\mathbb{R}$ *and* $U,V\in C(I)$ *be such that* $U(s)\geq 0,\ V(s)>0$ *for all* $s\in I$ *and* $f\in L(0,x)$, $f\in AC^1(0,x)$. *If* $r>\max\{1,\alpha\}$, *then*

$$\left|\int_0^x U(s)\left|\left(\mathrm{D}^{\gamma,\,\mu}_{\rho,\,\omega,\,0+}f\right)(s)\right|^{\beta}\left|\frac{d}{ds}f(s)\right|^{\alpha}ds\right|\tag{71}$$

$$\leq \int_0^x U(\lambda) \left| \int_0^\lambda V(t)\, e^{-\gamma}_{\rho,\,(1-\mu)}(\lambda - t, \omega)\, dt \right|^{\frac{r-\alpha}{r}} d\lambda \, \|V\|_\infty^\beta \left\| \frac{d}{ds} f(s) \right\|_\infty^{\alpha+\beta}.$$

## 4. Further Generalizations

In this section we give Opial–type integral inequalities for fractional integral operator containing more generalized Mittag–Leffler function in the kernel [19].

**Definition 6.** *Let $\mu, \nu, k, l, \gamma$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operator containing Mittag–Leffler function $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ for a real valued continuous function $f$ is defined by:*

$$(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+} f)(x) = \int_a^x (x-t)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-t)^\mu) f(t)dt, \tag{72}$$

*where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is generalized Mittag–Leffler function defined as*

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\mu n + \nu)} \frac{t^n}{(\delta)_{ln}}. \tag{73}$$

If $\delta = l = 1$ in (72), then integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to an integral operator containing generalized Mittag–Leffler function $E^{\gamma,1,k}_{\mu,\nu,1}$ introduced by Srivastava, and Tomovski in [20]. Along $\delta = l = 1$ in addition if $k = 1$ (72) reduces to an integral operator defined by Prabhakar in [18] containing Mittag-Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega = 0$ in (72), integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ would correspond essentially to the right-handed Riemann–Liouville fractional integral operators.

Here we present some general results involving generalized fractional integral operator, $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ containing more general form of Mittag–Leffler function $E^{\gamma,\delta,k}_{\mu,\nu,l}$.

**Theorem 13.** *Let $x > 0$, $\alpha, \beta, \mu, \nu, k, l, \gamma > 0$ with $k < l + \mu, \nu > 1$ and $r > \max\left\{1, \alpha, \frac{1}{\nu}\right\}$, also let $U, V \in C(I)$ be such that $U(s) \geq 0$, $V(s) > 0$ for all $s \in I$. Then for $\omega \in \mathbb{R}$ and $f \in L(0,x)$ we have*

$$\int_0^x U(s) \left| \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+} f\right)(s) \right|^\beta |f(s)|^\alpha ds \leq \Omega(x) \left( \int_0^x V(s) |f(s)|^r ds \right)^{\frac{\alpha+\beta}{r}}, \tag{74}$$

*where*

$$\Omega(x) = \left( \frac{\alpha}{\alpha+\beta} \right)^{\frac{\alpha}{r}} \left( \int_0^x \left( U^r(s) V^{-\alpha}(s) \right)^{\frac{1}{r-\alpha}} (\Delta(s))^{\frac{\beta(r-1)}{r-\alpha}} ds \right)^{\frac{r-\alpha}{r}}, \tag{75}$$

$$\Delta(s) = \int_0^s (V(t))^{-\frac{1}{r-1}} \left( E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-t)^\mu)(s-t)^{\nu-1} \right)^{\frac{r}{r-1}} dt. \tag{76}$$

*Proof.* According to (72),

$$(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f)(s) = \int_a^s (s-t)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-t)^{\mu})f(t)dt,$$

by setting

$$y(s) = (\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f)(s), \quad h(s) = f(s), \quad \Phi(s,t) = E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-t)^{\mu})(s-t)^{\nu-1},$$

we observe that condition (2) is satisfied with $a = 0$ and $I = [0, x]$ :

$$|y(s)| \le \int_0^s \Phi(s,t)|h(t)|dt, \quad 0 \le s \le x.$$

The rest of the proof of is the same as Theorem 4.2 of [14].

**Corollary 12.** *Let* $x > 0$, $\alpha, \beta, \mu, \nu, k, l, \gamma > 0$ *with* $k < l+\mu, \nu > 1$ *and* $r > \max\left\{1, \alpha, \frac{1}{\nu}\right\}$. *Then for* $\omega \in \mathbb{R}$ *and* $f \in L(0, x)$ *we have*

$$\int_0^x \left|\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f\right)(s)\right|^{\beta} |f(s)|^{\alpha} ds \le \left(\frac{\alpha}{\alpha+\beta}\right)^{\frac{\alpha}{r}} \times$$

$$\left(\int_0^x \left(\int_0^s \left(E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-t)^{\mu})(s-t)^{\nu-1}\right)^{\frac{r}{r-1}} dt\right)^{\frac{\beta(r-1)}{r-\alpha}} ds\right)^{\frac{(r-\alpha)}{r}} \left(\int_0^x |f(s)|^r ds\right)^{\frac{\alpha+\beta}{r}}. \tag{77}$$

**Remark 2.** *If* $\delta = l = 1$ *in above results, then we obtain results involving integral operator* $\epsilon^{\gamma,1,k}_{\mu,\nu,1,\omega,a^+}$ *containing generalized Mittag–Leffler function* $E^{\gamma,1,k}_{\mu,\nu,1}$ *introduced by Srivastava, and Tomovski in [20]. Along* $\delta = l = 1$ *in addition if* $k = 1$, *then we obtain results involving integral operator defined by Prabhakar in [18] containing Mittag-Leffler function* $E^{\gamma}_{\mu,\nu}$. *If* $\omega = 0$, *then we obtain results right-handed Riemann–Liouville fractional integral operator (see, [19]). Similar inequalities of Opial type can also be obtained for integral operators which contain multinomial Mittag–Leffler function [10] and Multiindex Mittag-Leffler function [13] in the kernel.*

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