EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 2, 2016, 152-164
ISSN 1307-5543 – www.ejpam.com

New Generalized Classes of τ_ω

Otchana Thevar Ravi[1,*], Ilangovan Rajasekaran[1], Soundararajan Satheesh Kanna[2] and Malliharjunaiah Paranjothi[3]

[1] *Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai District, Tamil Nadu, India.*
[2] *Department of Mathematics, Research Scholar, Bharathidasan University, Tiruchirapalli, Tamil Nadu, India.*
[3] *Department of Mathematics, Sree Sowdambiga College of Engineering, Aruppukottai, Virudhunagar District, Tamil Nadu, India.*

Abstract. The purpose of this paper is to introduce a new class of sets called semi-ω-open which lies between the class of $\alpha-\omega$-open sets and the class of $\beta-\omega$-open sets and to investigate the basic properties of such sets. This apart, some new generalized classes of τ_ω are introduced and investigated on the line of research.

2010 Mathematics Subject Classifications: 54C05, 54C08, 54C10

Key Words and Phrases: ω-open set, $\alpha-\omega$-open set, pre-ω-open set, $\beta-\omega$-open set, $b-\omega$-open set, $\omega-t$-set, $\delta-\omega$-open set, semi$^\star-\omega$-closed set.

1. Introduction

In 1982, the notions of ω-closed sets and ω-open sets were introduced and studied by Hdeib [7]. In 2009, Noiri et al. [10] introduced some generalizations of ω-open sets and investigated some properties of the sets. Moreover, they used them to obtain decompositions of continuity.

In this paper, we introduce and investigate the new notion called semi-ω-open sets which is weaker than $\alpha-\omega$-open sets and stronger than $\beta-\omega$-open sets. Also we introduce and investigate some new generalized classes of τ_ω.

*Corresponding author.

Email addresses: siingam@yahoo.com (O. Ravi), rajasekarani@yahoo.com (I. Rajasekaran), satheesh_kanna@yahoo.co.in (S. Kanna) and jothimp123@gmail.com (M. Paranjothi)

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{Q}$, $\mathbb{Q}^{\star}$) denotes the set of all real numbers (resp. the set of all rational numbers, the set of all irrational numbers).

By a space (X,τ), we always mean a topological space (X,τ) with no separation properties assumed. If $H \subset X$, $cl(H)$ and $int(H)$ will, respectively, denote the closure and interior of H in (X,τ). τ_H denotes the relative topology on H and τ_u denotes the usual topology on $\mathbb{R}$.

Definition 1. *A subset H of a space (X,τ) is said to be semi-open [9] if $H \subset cl(int(H))$.*

Definition 2 ([11]). *Let H be a subset of a space (X,τ), a point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 3 ([7]). *A subset H of a space (X,τ) is called ω-closed if it contains all its condensation points.*

The complement of an ω-closed set is called ω-open.

It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X,τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 1 ([7]). *Let H be a subset of a space (X,τ). Then*

(i) H is ω-closed in X if and only if $H = cl_\omega(H)$.

(ii) $cl_\omega(X \backslash H) = X \backslash int_\omega(H)$.

(iii) $cl_\omega(H)$ is ω-closed in X.

(iv) $x \in cl_\omega(H)$ if and only if $H \cap G \neq \phi$ for each ω-open set G containing x.

(v) $cl_\omega(H) \subset cl(H)$.

(vi) $int(H) \subset int_\omega(H)$.

Remark 1. *For a subset of a space (X,τ), the following property holds:*
Every closed set is ω-closed but not conversely [2, 7].

Definition 4. *[1] A space (X,τ) is called anti-locally countable if each non-empty open set is uncountable.*

Lemma 2 ([8]). *Let (H,τ_H) be an anti-locally countable subspace of a space (X,τ). Then $cl(H) = cl_\omega(H)$.*

Lemma 3 ([6]). *If U is an open set, then $cl(U \cap H) = cl(U \cap cl(H))$ and hence $U \cap cl(H) \subset cl(U \cap H)$ for any subset H.*

Lemma 4 ([1, 4]). *If (X,τ) is an anti-locally countable space, then $int_\omega(H) = int(H)$ for every ω-closed set H of X and $cl_\omega(H) = cl(H)$ for every ω-open set H of X.*

Definition 5 ([10]). *A subset H of a space (X,τ) is called*

(i) $\alpha-\omega$*-open if* $H \subset int_{\omega}(cl(int_{\omega}(H)))$;

(ii) pre-ω*-open if* $H \subset int_{\omega}(cl(H))$;

(iii) $\beta-\omega$*-open if* $H \subset cl(int_{\omega}(cl(H)))$;

(iv) $b-\omega$*-open if* $H \subset int_{\omega}(cl(H)) \cup cl(int_{\omega}(H))$.

Definition 6 ([10]). *A subset H of a space (X,τ) is called an* $\omega-t$*-set if* $int(H) = int_{\omega}(cl(H))$.

Definition 7. *A space (X,τ) is called submaximal [5] if every dense subset is open.*

Definition 8. *A subset H of a space (X,τ) is called ω-dense [3] if* $cl_{\omega}(H) = X$.

3. Properties of Semi-ω-Open Sets

Definition 9. *A subset H of a space (X,τ) is said to be*

(i) semi-ω*-open if* $H \subset cl(int_{\omega}(H))$.

(ii) semi-ω*-closed if* $int(cl_{\omega}(H)) \subset H$.

The complement of semi-ω-open set is called semi-ω-closed.

Example 1. *Let* $X = \{a,b,c\}$ *with the topology* $\tau = \{\phi, X, \{a\}, \{a,b\}\}$. *Then* $\{a\}$ *is semi-ω-open.*

Example 2. *Let* $X = \mathbb{R}$ *with the usual topology* τ_u. *Let* $H = (0,1) \cap \mathbb{Q}$. *Then H is not semi-$\omega$-open, since* $cl(int_{\omega}(H)) = cl(\phi) = \phi$.

Proposition 1. *In a space (X,τ), every semi-open subset is semi-ω-open.*

Proof. Let H be semi-open in (X,τ). Then $H \subset cl(int(H)) \subset cl(int_{\omega}(H))$. This proves that H is semi-ω-open. □

Remark 2. *The converse of Proposition 1 is not true.*

Example 3. *Let* $X = \mathbb{R}$ *with the usual topology* τ_u. *Then* $H = \mathbb{Q}^{\star}$ *is semi-ω-open for* $cl(int_{\omega}(H)) = cl(H) = \mathbb{R}$ *and* $H \subset cl(int_{\omega}(H))$. *But H is not semi-open for* $cl(int(H)) = cl(\phi) = \phi$ *and* $H \nsubseteq cl(int(H))$.

From the above Example, we observe that the converse fails in an anti-locally countable space also.

Theorem 1. *In an anti-locally countable space, an ω-closed and a semi-ω-open subset is semi-open.*

Proof. Let (X, τ) be an anti-locally countable space and H be an ω-closed and a semi-ω-open subset.

Since H is semi-ω-open, $H \subset cl(int_\omega(H))$. Since (X, τ) is anti-locally countable and H is ω-closed, $int_\omega(H) = int(H)$ by Lemma 4. Hence $H \subset cl(int_\omega(H)) = cl(int(H))$ and thus H is semi-open. □

Theorem 2. *For a subset of space (X, τ), the following properties hold:*

(i) Every ω-open set is semi-ω-open.

(ii) Every $\alpha - \omega$-open set is semi-ω-open.

(iii) Every semi-ω-open set is $\beta - \omega$-open.

(iv) Every semi-ω-open set is $b - \omega$-open.

Proof. (i). If H is an ω-open set, then $H = int_\omega(H) \subset cl(int_\omega(H))$. Therefore H is semi-$\omega$-open.

(ii). If H is an $\alpha - \omega$-open set, then $H \subset int_\omega(cl(int_\omega(H))) \subset cl(int_\omega(H))$. Therefore H is semi-$\omega$-open.

(iii). If H is an semi-ω-open set, then $H \subset cl(int_\omega(H)) \subset cl(int_\omega(cl(H)))$. Therefore H is $\beta - \omega$-open.

(iv). If H is an semi-ω-open set, then $H \subset cl(int_\omega(H)) \subset int_\omega(cl(H)) \cup cl(int_\omega(H))$. Therefore H is $b - \omega$-open. □

The following Examples support that the separate converses of Theorem 2 are not true in general.

Example 4. *Let $X = \mathbb{R}$ with the usual topology τ_u.*

(i) Let $H = (0, 1]$. Then H is semi-ω-open set but not ω-open, since $H = (0, 1] \neq (0, 1) = int_\omega(H)$.

(ii) Let $H = (0, 1]$. Then H is semi-ω-open set but not $\alpha - \omega$-open, since $int_\omega(cl(int_\omega(H))) = int_\omega(cl(0, 1)) = int_\omega([0, 1]) = (0, 1)$.

(iii) Let $H = [0, 1] \cap \mathbb{Q}$. Then H is $\beta - \omega$-open set but not semi-ω-open, since $cl(int_\omega(H)) = cl(\phi) = \phi$.

(iv) Let $H = \mathbb{Q}$. Then H is $b - \omega$-open set but not semi-ω-open, since $cl(int_\omega(H)) = cl(\phi) = \phi$.

Theorem 3. *Let H be a subset of a space (X, τ). Then H is $\alpha - \omega$-open if and only if it is semi-ω-open and pre-ω-open.*

Proof. Let H be an $\alpha - \omega$-open. Then $H \subset int_\omega(cl(int_\omega(H)))$. It implies that $H \subset int_\omega(cl(int_\omega(H))) \subset cl(int_\omega(H))$ and $H \subset int_\omega(cl(int_\omega(H))) \subset int_\omega(cl(H))$. Thus H is semi-$\omega$-open and pre-$\omega$-open.

Conversely, let H be semi-ω-open and pre-ω-open. Then we have $H \subset cl(int_\omega(H))$ and $H \subset int_\omega(cl(H))$. Hence $H \subset int_\omega(cl(H)) \subset int_\omega(cl(int_\omega(H)))$ which implies that H is $\alpha - \omega$-open. □

Remark 3. *The concepts of semi-ω-openness and pre-ω-openness are independent.*

Example 5. *Let $X = \mathbb{R}$ with the usual topology τ_u. The interval $H = (0,1]$ is semi-ω-open but not pre-ω-open, since $int_\omega(cl(H)) = int_\omega([0,1]) = (0,1)$.*

Example 6. *Let $X = \mathbb{R}$ with the usual topology τ_u. Let $H = \mathbb{Q}$. Then H is pre-ω-open but not semi-ω-open, since $cl(int_\omega(H)) = cl(\phi) = \phi$.*

Proposition 2. *The intersection of a semi-ω-open set and an open set is semi-ω-open.*

Proof. Let H be a semi-ω-open and U be an open set in X. Then $H \subset cl(int_\omega(H))$ and $int(U) = U$. By Lemma 3, we have

$$\begin{aligned} U \cap H \subset & U \cap cl(int_\omega(H)) \subset cl(U \cap int_\omega(H)) \\ =& cl(int(U) \cap int_\omega(H)) \subset cl(int_\omega(U) \cap int_\omega(H)) \\ =& cl(int_\omega(U \cap H)). \end{aligned}$$

Therefore $U \cap H$ is semi-ω-open. □

Remark 4. *The intersection of two semi-ω-open sets need not be semi-ω-open. This can be seen from the following Example.*

Example 7. *Let $X = \mathbb{R}$ with the usual topology τ_u. Let $A = (0,1]$ and $B = [1,2)$, then A and B are semi-ω-open, but $A \cap B = \{1\}$ which is not semi-ω-open, since $cl(int_\omega(A \cap B)) = cl(\phi) = \phi$.*

Theorem 4. *Let H be a subset of a space (X, τ). If H is both closed and $\beta - \omega$-open, then H is semi-ω-open.*

Proof. Since H is a $\beta - \omega$-open set, $H \subset cl(int_\omega(cl(H))) = cl(int_\omega(H))$, H being closed. Therefore H is semi-ω-open. □

Theorem 5. *Let H be a subset of a space (X, τ). If H is both $\beta - \omega$-open and $\omega - t$-set, then H is semi-ω-open.*

Proof. Since H is a $\omega - t$-set, $int(H) = int_\omega(cl(H))$. Since H is $\beta - \omega$-open also,

$$H \subset cl(int_\omega(cl(H))) \subset cl(int(H)) \subset cl(int_\omega(H)).$$

Therefore H is semi-ω-open. □

Theorem 6. *Let H be a subset of a space (X, τ). If H is both $b - \omega$-open and $\omega - t$-set, then H is semi-ω-open.*

Proof. Since H is $\omega - t$-set, $int_\omega(cl(H)) = int(H) \subset int_\omega(H)$. Since H is $b - \omega$-open also, $H \subset int_\omega(cl(H)) \cup cl(int_\omega(H)) \subset int_\omega(H) \cup cl(int_\omega(H)) = cl(int_\omega(H))$. Therefore H is semi-$\omega$-open. □

Proposition 3. *Let H be a subset of a space (X,τ). Then H is semi-ω-open if and only if $cl(H) = cl(int_\omega(H))$.*

Proof. Let H be semi-ω-open. Then $H \subset cl(int_\omega(H))$ and $cl(H) \subset cl(int_\omega(H))$. But always $cl(int_\omega(H)) \subset cl(H)$. Thus, we obtain that $cl(H) = cl(int_\omega(H))$.

Conversely, let the condition hold. We have $H \subset cl(H) = cl(int_\omega(H))$, by the given condition. Thus $H \subset cl(int_\omega(H))$ and hence H is semi-ω-open. □

Proposition 4. *Let $H \subset (X,\tau)$ be a $b-\omega$-open set such that $cl(H) = \phi$. Then H is semi-ω-open.*

Theorem 7. *For a subset H of a submaximal space (X,τ), the following properties are equivalent.*

(i) H is semi-ω-open,

(ii) H is $\beta-\omega$-open.

Proof. $(i) \Rightarrow (ii)$: It follows from the fact that every semi-ω-open set is $\beta-\omega$-open.

$(ii) \Rightarrow (i)$: Let H be a $\beta-\omega$-open set in X. Then $H \subset cl(int_\omega(cl(H)))$ and $cl(H) \subset cl(int_\omega(cl(H)))$. Thus, $cl(H)$ is semi-ω-open. Put $A = cl(H)$ and $K = H \cup (X \backslash cl(H))$. We have $H = cl(H) \cap K$ and $cl(K) = X$. This implies that $H = A \cap K$, where A is semi-ω-open and K is dense. Since X is submaximal, then K is open. By Proposition 2, $H = A \cap K$ is semi-ω-open. □

Theorem 8. *A subset H of a space (X,τ) is semi-ω-open if and only if there exists $U \in \tau_\omega$ such that $U \subset H \subset cl(U)$.*

Proof. Let H be semi-ω-open. Then $H \subset cl(int_\omega(H))$. Take $int_\omega(H) = U$. Then, we have $U \subset H \subset cl(U)$.

Conversely, let $U \subset H \subset cl(U)$ for some $U \in \tau_\omega$. Since $U \subset H$, we have $U \subset int_\omega(H)$ and hence $cl(U) \subset cl(int_\omega(H))$. Thus we obtain $H \subset cl(int_\omega(H))$ and H is semi-ω-open. □

Corollary 1. *If A is a semi-ω-open set in a space (X,τ) and $A \subset B \subset cl(A)$, then B is semi-ω-open in X.*

Proof. Since A is semi-ω-open, $A \subset cl(int_\omega(A)) \subset cl(int_\omega(B))$ for $A \subset B$. So $cl(A) \subset cl(int_\omega(B))$. Since $B \subset cl(A), B \subset cl(int_\omega(B))$. Thus B is semi-$\omega$-open. □

4. Properties of $\delta-\omega$-Open Sets

Definition 10. *A subset H of a space (X,τ) is said to be*

(i) $\delta-\omega$-open if $int_\omega(cl(H)) \subset cl(int_\omega(H))$.

(ii) $\delta-\omega$-closed if $int(cl_\omega(H)) \subset cl_\omega(int(H))$.

The complement of $\delta-\omega$-open set is called $\delta-\omega$-closed.

Example 8. *Let $X = \mathbb{R}$ with the usual topology τ_u. Let $H = \mathbb{Q}$. Then H is not $\delta - \omega$-open, since $int_\omega(cl(\mathbb{Q})) = int_\omega(\mathbb{R}) = \mathbb{R}$ and $cl(int_\omega(\mathbb{Q})) = cl(\phi) = \phi$.*

Example 9. *Let $X = \mathbb{R}$ with the usual topology τ_u. Let $H = (0, 1]$. Then H is $\delta - \omega$-open, since $int_\omega(cl((0, 1])) = int_\omega([0, 1]) = (0, 1)$ and $cl(int_\omega(H)) = cl(0, 1) = [0, 1]$.*

Proposition 5. *For a subset of a space (X, τ), the following properties hold:*

(i) Every $\alpha - \omega$-open set is $\delta - \omega$-open.

(ii) Every $\omega - t$-set is $\delta - \omega$-open.

Proof. (i) Since H is an $\alpha - \omega$-open set, $H \subset int_\omega(cl(int_\omega(H))) \subset cl(int_\omega(H))$. Then we obtain $cl(H) \subset cl(int_\omega(H))$ and $int_\omega(cl(H)) \subset cl(H) \subset cl(int_\omega(H))$. Therefore H is $\delta - \omega$-open.

(ii) Since H is an $\omega - t$-set, $int_\omega(cl(H)) = int(H) \subset H$. Then we obtain

$$int_\omega(cl(H)) \subset int_\omega(H) \subset cl(int_\omega(H)).$$

Therefore H is $\delta - \omega$-open. □

Example 10. *Let $X = \mathbb{R}$ with the usual topology τ_u.*

(i) Let $H = (0, 1]$. Then H is $\delta - \omega$-open but not $\alpha - \omega$-open, since $int_\omega(cl(H)) = (0, 1)$ and $cl(int_\omega(H)) = [0, 1]$.

(ii) Let $H = \mathbb{Q}^\star$. Then H is $\delta - \omega$-open but not $\omega - t$-set, since $int(\mathbb{Q}^\star) = \phi$, $int_\omega(cl(\mathbb{Q}^\star)) = \mathbb{R}$ and $cl(int_\omega(\mathbb{Q}^\star)) = cl(\mathbb{Q}^\star) = \mathbb{R}$.

Definition 11. *A subset H of a space (X, τ) is said to be $\beta - \omega$-closed if $int(cl_\omega(int(H))) \subset H$. The complement of $\beta - \omega$-open set is called $\beta - \omega$-closed.*

Proposition 6. *Let H be a subset of a space (X, τ). Then H is $\beta - \omega$-closed if and only if $int(cl_\omega(int(H))) = int(H)$.*

Proof. Since H is $\beta - \omega$-closed set, $int(cl_\omega(int(H))) \subset H$ and then we obtain $int(cl_\omega(int(H))) \subset int(H)$. But $int(H) \subset int(cl_\omega(int(H)))$. Thus we have $int(H) = int(cl_\omega(int(H)))$.

Conversely, let the condition hold. We have $int(cl_\omega(int(H))) = int(H) \subset H$. Therefore H is $\beta - \omega$-closed. □

Theorem 9. *For a subset H of a space (X, τ), the following properties are equivalent:*

(i) H is semi-ω-closed.

(ii) H is $\beta - \omega$-closed and $\delta - \omega$-closed.

Proof. $(i) \Rightarrow (ii)$: Let H be semi-ω-closed. By Theorem 2(iii), H is $\beta-\omega$-closed. Since H is semi-ω-closed, $int(cl_{\omega}(H)) \subset H$ and $int(cl_{\omega}(H)) \subset int(H)$. It gives that $cl_{\omega}(int(cl_{\omega}(H))) \subset cl_{\omega}(int(H))$. Thus $int(cl_{\omega}(H)) \subset cl_{\omega}(int(cl_{\omega}(H))) \subset cl_{\omega}(int(H))$ and so H is $\delta-\omega$-closed.

$(ii) \Rightarrow (i)$: Since H is $\delta-\omega$-closed, $int(cl_{\omega}(H)) \subset cl_{\omega}(int(H))$ and $int(cl_{\omega}(H)) \subset int(cl_{\omega}(int(H)))$. Since H is $\beta-\omega$-closed, $int(cl_{\omega}(int(H))) \subset H$. Then $int(cl_{\omega}(H)) \subset H$ and so H is semi-ω-closed. □

Remark 5. *The concepts of $\beta-\omega$-closedness and $\delta-\omega$-closedness are independent.*

Example 11.

(i) Let $X=\mathbb{R}$ with the topology $\tau=\{\phi, X, \mathbb{Q}^{\star}\}$. Let $H=\mathbb{Q}^{\star}$. Then H is $\delta-\omega$-closed but not $\beta-\omega$-closed, since $\mathbb{Q}$ is not $\beta-\omega$-open.

(ii) Let $X=\mathbb{R}$ with the usual topology τ_u. Let $H=\mathbb{Q}^{\star}$. Then H is $\beta-\omega$-closed but not $\delta-\omega$-closed, since $\mathbb{Q}$ is not $\delta-\omega$-open.

Theorem 10. *Let (X,τ) be a space. Then a subset of X is $\alpha-\omega$-open if and only if it is both $\delta-\omega$-open and pre-ω-open.*

Proof. **Necessity:** Let H be an $\alpha-\omega$-open set. Then $H \subset int_{\omega}(cl(int_{\omega}(H)))$. It implies that $cl(H) \subset cl(int_{\omega}(H))$ and $int_{\omega}(cl(H)) \subset int_{\omega}(cl(int_{\omega}(H))) \subset cl(int_{\omega}(H))$. Hence, H is a $\delta-\omega$-open set. On the other hand, since H is an $\alpha-\omega$-open set, H is a pre-ω-open set.

Sufficiency: Let H be both $\delta-\omega$-open and pre-ω-open. Since H is $\delta-\omega$-open, we have $int_{\omega}(cl(H)) \subset cl(int_{\omega}(H))$ and hence $int_{\omega}(cl(H)) \subset int_{\omega}(cl(int_{\omega}(H)))$. Since H is pre-$\omega$-open, we have $H \subset int_{\omega}(cl(H))$. Therefore we obtain that $H \subset int_{\omega}(cl(int_{\omega}(H)))$ which proves that H is an $\alpha-\omega$-open set. □

Remark 6. *The concepts of $\delta-\omega$-openness and pre-ω-openness are independent.*

Example 12. *Let $X=\mathbb{R}$ with the usual topology τ_u.*

(i) $H=(0,1]$ is $\delta-\omega$-open but not pre-ω-open.

(ii) $H=\mathbb{Q}$ is pre-ω-open but not $\delta-\omega$-open.

Proposition 7. *Let A and B be subsets of a space (X,τ). If $A \subset B \subset cl(A)$ and A is $\delta-\omega$-open in X, then B is $\delta-\omega$-open in X.*

Proof. Suppose that $A \subset B \subset cl(A)$ and A is $\delta-\omega$-open in X. Then, we have $int_{\omega}(cl(A)) \subset cl(int_{\omega}(A))$. Since $A \subset B, cl(int_{\omega}(A)) \subset cl(int_{\omega}(B))$ and $int_{\omega}(cl(A)) \subset cl(int_{\omega}(B))$. Since $B \subset cl(A)$, we have $cl(B) \subset cl(cl(A)) = cl(A)$ and $int_{\omega}(cl(B)) \subset int_{\omega}(cl(A))$. Therefore we obtain that $int_{\omega}(cl(B)) \subset cl(int_{\omega}(B))$. This shows that B is a $\delta-\omega$-open set. □

Corollary 2. *Let (X,τ) be a space. If $A \subset X$ is $\delta-\omega$-open and dense in (X,τ), then every subset of X containing A is $\delta-\omega$-open.*

Proof. It is obvious by Proposition 7. □

5. Properties of Semi $^\star - \omega$-Open Sets

Definition 12. *A subset H of a space (X,τ) is said to be*

(i) semi$^\star - \omega$-open if $H \subset cl_\omega(int(H))$.

(ii) semi$^\star - \omega$-closed if $int_\omega(cl(H)) \subset H$.

The complement of a semi$^\star - \omega$-open set is called semi$^\star - \omega$-closed.

Example 13. *Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a,b\}\}$.*

(i) Let $H = \{a\}$. Then H is semi$^\star - \omega$-open, since $int(H) = \{a\}$ and $cl_\omega(int(H)) = \{a\}$.

(ii) Let $H = \{c\}$. Then H is not semi$^\star - \omega$-open, since $int(H) = \phi$ and $cl_\omega(int(H)) = \phi$.

Proposition 8. *For a subset of a space (X,τ), every semi$^\star - \omega$-open set is semi-ω-open.*

Proof. If H is semi$^\star - \omega$-open set, then $H \subset cl_\omega(int(H)) \subset cl(int_\omega(H))$. Therefore H is semi-$\omega$-open. □

Example 14. *Let $X = \mathbb{R}$ with the usual topology τ_u. Let $H = \mathbb{Q}^\star$. Then H is semi-ω-open but not semi$^\star - \omega$-open, since $cl(int_\omega(H)) = cl(\mathbb{Q}^\star) = \mathbb{R}$ and $cl_\omega(int(H)) = cl_\omega(\phi) = \phi$.*

Proposition 9. *A subset H of a space (X,τ) is semi$^\star - \omega$-open if and only if $cl_\omega(H) = cl_\omega(int(H))$.*

Proof. If H is semi$^\star - \omega$-open set, then $H \subset cl_\omega(int(H))$ and $cl_\omega(H) \subset cl_\omega(int(H))$. But $cl_\omega(int(H)) \subset cl_\omega(H)$. Hence $cl_\omega(H) = cl_\omega(int(H))$.

Conversely, let the condition hold. We have $H \subset cl_\omega(H)$ and $cl_\omega(H) = cl_\omega(int(H))$. Therefore H is semi$^\star - \omega$-open. □

Definition 13. *A subset H of a space (X,τ) is said to be $\omega^\star - t$-set if $int_\omega(cl(H)) = int_\omega(H)$.*

Example 15. *Let $X = \mathbb{R}$ with the usual topology τ_u.*

(i) Let $H = (0,1]$. Then H is a $\omega^\star - t$-set.

(ii) Let $H = \mathbb{Q}^\star$. Then H is not a $\omega^\star - t$-set.

Proposition 10. *In a space (X,τ), every closed set is a $\omega^\star - t$-set.*

Proof. Let H be a closed set. Then $H = cl(H)$ and we have $int_\omega(cl(H)) = int_\omega(H)$ which proves that H is a $\omega^\star - t$-set. □

The converse of Proposition 10 is not true as can be seen from the following Example.

Example 16. *Let $X = \mathbb{R}$ with the usual topology τ_u. Let $H = (0,1]$. Then H is $\omega^\star - t$-set but not closed.*

Proposition 11. *In a space (X,τ), every $\omega - t$-set is a $\omega^\star - t$-set.*

Proof. If H is a $\omega - t$-set, then $int_{\omega}(cl(H)) = int(H) \subset int_{\omega}(H) \subset int_{\omega}(cl(H))$. Thus we have $int_{\omega}(cl(H)) = int_{\omega}(H)$ and hence H is a $\omega^{\star} - t$-set. □

Example 17. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then $H = \{c\}$ is a $\omega^{\star} - t$-set but not a $\omega - t$-set. Since $int_{\omega}(H) = H$, $int(H) = \phi$ and $int_{\omega}(cl(H)) = int_{\omega}(H) = H$, we have $int_{\omega}(cl(H)) = int_{\omega}(H)$ and $int_{\omega}(cl(H)) \neq int(H)$. This proves that H is a $\omega^{\star} - t$-set but not a $\omega - t$-set.*

Theorem 11. *A subset H of a space (X, τ) is semi$^{\star} - \omega$-closed if and only if H is a $\omega^{\star} - t$-set.*

Proof. Let H be a semi$^{\star} - \omega$-closed set in X. Then $X \backslash H$ is semi$^{\star} - \omega$-open. By Proposition 9, we have $cl_{\omega}(X \backslash H) = cl_{\omega}(int(X \backslash H))$. It follows that

$$X \backslash int_{\omega}(H) = cl_{\omega}(X \backslash cl(H)) = X \backslash int_{\omega}(cl(H)).$$

Thus, $int_{\omega}(cl(H)) = int_{\omega}(H)$ and hence H is a $\omega^{\star} - t$-set in X.

Conversely, let H be a $\omega^{\star} - t$-set. Then $int_{\omega}(cl(H)) = int_{\omega}(H) \subset H$. Therefore H is semi$^{\star} - \omega$-closed. □

Proposition 12. *If A and B are $\omega^{\star} - t$-sets of a space (X, τ), then $A \cap B$ is a $\omega^{\star} - t$-set.*

Proof. Let A and B be $\omega^{\star} - t$-sets. Then we have

$$\begin{aligned} int_{\omega}(A \cap B) &\subset int_{\omega}(cl(A \cap B)) \subset int_{\omega}(cl(A) \cap cl(B)) \\ &= int_{\omega}(cl(A)) \cap int_{\omega}(cl(B)) = int_{\omega}(A) \cap int_{\omega}(B) = int_{\omega}(A \cap B). \end{aligned}$$

Then $int_{\omega}(A \cap B) = int_{\omega}(cl(A \cap B))$ and hence $A \cap B$ is an $\omega^{\star} - t$-set. □

Definition 14. *A subset H of a space (X, τ) is said to be semi-ω-regular if H is semi-ω-open and a $\omega^{\star} - t$-set.*

Example 18. *Let $X = \mathbb{R}$ with the usual topology τ_u.*

(i) Let $H = (0, 1]$. Then H is semi-ω-regular.

(ii) Let $H = \mathbb{R} \backslash \mathbb{Q}$. Then H is not semi-$\omega$-regular, since H is not $\omega^{\star} - t$-set.

Theorem 12. *Let H be a subset of a space (X, τ). Then H is semi-ω-regular if and only if H is both $\beta - \omega$-open and semi$^{\star} - \omega$-closed.*

Proof. If H is semi-ω-regular, then H is both semi-ω-open and a $\omega^{\star} - t$-set. Since every semi-ω-open set is $\beta - \omega$-open, H is both $\beta - \omega$-open and a $\omega^{\star} - t$-set. By Theorem 11, we obtain the result.

Conversely, let H be semi$^{\star} - \omega$-closed and $\beta - \omega$-open. Since H is a semi$^{\star} - \omega$-closed, by Theorem 11 H is a $\omega^{\star} - t$-set. Since H is $\beta - \omega$-open, $H \subset cl(int_{\omega}(cl(H))) = cl(int_{\omega}(H))$. Therefore H is semi-$\omega$-open. Since H is both semi-ω-open and a $\omega^{\star} - t$-set, H is semi-ω-regular. □

Remark 7. *The concepts of $\beta-\omega$-openness and semi$^\star-\omega$-closedness are independent.*

Example 19. *(i) Let $X=\mathbb{R}$ with the topology $\tau=\{\phi, X, \mathbb{Q}^\star\}$. Then $H=\mathbb{Q}$ is semi$^\star-\omega$-closed but not $\beta-\omega$-open. Since $int_\omega(cl(H))=int_\omega(H)=\phi\subset H$, H is semi$^\star-\omega$-closed. Again since $H\not\subseteq cl(int_\omega(cl(H)))=\phi$, H is not $\beta-\omega$-open.*

(ii) Let $X=\mathbb{R}$ with the usual topology τ_u. Let $H=\mathbb{Q}$. Then H is $\beta-\omega$-open but not semi$^\star-\omega$-closed, since $int_\omega(cl(H))=int_\omega(\mathbb{R})=\mathbb{R}$.

6. Properties of $\omega-\mathscr{R}$-Closed Sets

Definition 15. *A subset H of a space (X,τ) is called $\omega-\mathscr{R}$-closed if $H=cl(int_\omega(H))$.*

Theorem 13. *Let (X,τ) be a space and H a subset of X. Then the following properties are equivalent.*

(i) $H\neq\phi$ is $\omega-\mathscr{R}$-closed.

(ii) There exists a non-empty ω-open set G such that $G\subset H=cl(G)$.

(iii) There exists a non-empty ω-open set G such that $H=G\cup(cl(G)-G)$.

Proof. $(i)\Rightarrow(ii)$: Suppose $H\neq\phi$ is an $\omega-\mathscr{R}$-closed set. Then $H=cl(int_\omega(H))$. Let $G=int_\omega(H)$. G is the required ω-open set such that $G\subset H=cl(G)$.

$(ii)\Rightarrow(iii)$: Since $H=cl(G)=G\cup(cl(G)-G)$ where G is a nonempty ω-open set, (iii) follows.

$(iii)\Rightarrow(i)$: $H=G\cup(cl(G)-G)$ implies that $H=cl(G)=cl(int_\omega(G))\subset cl(int_\omega(H))$, since G is ω-open and $G\subset H$. Again $int_\omega(H)\subset H$ implies that $cl(int_\omega(H))\subset cl(H)=cl(G)=H$. Therefore $H=cl(int_\omega(H))$ which implies that H is $\omega-\mathscr{R}$-closed. □

Theorem 14. *Let H be a subset of a space (X,τ). If H is $\beta-\omega$-open, then cl(H) is $\omega-\mathscr{R}$-closed.*

Proof. Suppose H is $\beta-\omega$-open. Then $H\subset cl(int_\omega(cl(H)))$ and so $cl(H)\subset cl(int_\omega(cl(H)))\subset cl(H)$ which implies that $cl(H)=cl(int_\omega(cl(H)))$. Therefore $cl(H)$ is $\omega-\mathscr{R}$-closed. □

Theorem 15. *Let H be a subset of a space (X,τ). Then the following properties are equivalent.*

(i) H is $\omega-\mathscr{R}$-closed.

(ii) H is semi-ω-open and closed.

(iii) H is $\beta-\omega$-open and closed.

Proof. $(i) \Rightarrow (ii)$: If H is $\omega - \mathscr{R}$-closed, then $H = cl(int_\omega(H))$ and $cl(H) = cl(int_\omega(H))$. Since $H \subset cl(int_\omega(H))$, H is semi-$\omega$-open. Also, $H = cl(H)$ and so H is closed.

$(ii) \Rightarrow (iii)$: It follows from the fact that every semi-ω-open set is a $\beta - \omega$-open.

$(iii) \Rightarrow (i)$: Suppose H is $\beta - \omega$-open and closed. Then $H \subset cl(int_\omega(cl(H)))$ and $H = cl(H)$. Now $cl(int_\omega(H)) \subset cl(H) = H$. Also, $H \subset cl(int_\omega(H))$. Therefore $H = cl(int_\omega(H))$ which implies that H is $\omega - \mathscr{R}$-closed. □

Remark 8. *(i) The concepts of semi-ω-openness and closedness are independent.*

(ii) The concepts of $\beta - \omega$-openness and closedness are independent.

Example 20. *(i) Let $X = \mathbb{R}$ with the usual topology τ. Let $H = (0,1]$. Then H is semi-ω-open but not closed.*

(ii) Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$. Let $H = \mathbb{Q}$. Then H is closed but not semi-ω-open.

Example 21. *(i) Let $X = \mathbb{R}$ with the usual topology τ_u. Let $H = (0,1]$. Then H is $\beta - \omega$-open but not closed.*

(ii) Let $X = \mathbb{R}$ with the topology $\tau_u = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$. Let $H = \mathbb{Q}$. Then H is closed but not $\beta - \omega$-open.

7. Further Properties

Definition 16. *A space (X, τ) is called ω-submaximal if every ω-dense subset of X is ω-open.*

Proposition 13. *Every submaximal space is ω-submaximal.*

Proof. Let $H \subset X$ be ω-dense. Then $X = cl_\omega(H) \subset cl(H)$ and $X = cl(H)$. Thus H is dense in X. Since X is submaximal, H is open and hence ω-open in X. Therefore, X is ω-submaximal. □

Example 22. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{c\}, \{b, c\}\}$. Set $H = \{a, c\}$. Then $cl(H) = X$ and $H \notin \tau$. Hence X is not submaximal but it is ω-submaximal, since the only ω-dense set is X.*

Definition 17. *A subset H of a space (X, τ) is called ω-codense if $X \backslash H$ is ω-dense.*

Theorem 16. *For a space (X, τ), the following are equivalent.*

(i) X is ω-submaximal,

(ii) Every ω-codense subset H of X is ω-closed.

Proof. $(i) \Rightarrow (ii)$: Let H be a ω-codense subset of X. Then $X \backslash H$ is ω-dense and therefore $X \backslash H$ is ω-open, X being ω-submaximal by assumption. Thus H is ω-closed.

$(ii) \Rightarrow (i)$: Let H be a ω-dense subset of X. Then $X \backslash H$ is ω-codense in X and by assumption $X \backslash H$ is ω-closed. Hence H is ω-open and thus X is ω-submaximal. □

ACKNOWLEDGEMENTS The authors thank the referee(s) for his/her/their suggestions.

References

[1] S. Al-Ghour. *Certain covering properties related to paracompactness*, Ph.D Thesis, University of Jordan, Amman, 1999.

[2] A. Al-Omari and M. S. M. Noorani. *Regular generalized ω-closed sets*, International Journal of Mathematics and Mathematical Sciences, Article ID 16292, 11 pages. 2007. doi:10.1155/2007/1629

[3] A. Al-Omari and M.S.M. Noorani. *Contra- ω- continuous and Almost contra- ω- continuous*, International Journal of Mathematics and Mathematical Sciences, Article ID 40469, 13 pages. 2007. doi:10.1155/2007/40469

[4] K. Al-Zoubi and B. Al-Nashef. *The topology of ω-open sets*, Al-Manarch, 9(2), 169 - 179. 2003.

[5] J. Dontchev. *On submaximal spaces*, Tamkang Journal of Mathematics, 26, 253 - 260. 1995.

[6] R. Engelking. *General topology*, Heldermann Veriag Berlin, 2nd edition, 1989.

[7] H. Z. Hdeib. *ω-closed mappings*, Revista Colombiana De Mathematics, 16, 65 - 78. 1982.

[8] Khalid Y. Al-Zoubi. *On generalized ω-closed sets*, International Journal of Mathematics and Mathematical Sciences, 13, 2011 - 2021. 2005.

[9] N. Levine. *Semi-open sets and semi-continuity in topological spaces*, American Mathematical Monthly, 70, 36 - 41. 1963.

[10] T. Noiri, A. Al-Omari, and M. S. M. Noorani. *Weak forms of ω-open sets and decompositions of continuity*, European Journal of Pure and Applied Mathematics, 2, 73 - 84. 2009.

[11] S. Willard. *General Topology*, Addison-Wesley, Reading, Mass, USA, 1970.