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Perfect Morse Function on $SO(n)$

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Abstract. In this work, we define a Morse function on $SO(n)$ and show that this function is indeed a perfect Morse function.

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1. Introduction

The main point of Morse Theory, which was introduced in [6], is investigating the relation between shape of a smooth manifold M and critical points of a specific real-valued function $f : M \to \mathbb{R}$, that is called *Morse function*. [5] and [4] are two of main sources about this subject, so mostly we will use their beautiful tools for defining a Morse function on $SO(n)$. Also, we will refer [2] to use homological properties and to determine the Poincaré polynomial of $SO(n)$. Perfect Morse functions are widely studied in [7], that is one of our inspiration to show that the function, we defined, is also perfect.

2. Preliminaries

In this section, we give some definitions and theorems which will be used in this paper.

Definition 1. *Let M be an n-dimensional smooth manifold and $f : M \to \mathbb{R}$ be a smooth function. A point $p_0 \in M$ is said to be a critical point of M if we have*

$$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0 \tag{1}$$

with respect to a coordinate system $\{x_1, x_2, \ldots, x_n\}$ around p_0.

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A point $c \in \mathbb{R}$ is said to be *a critical value of* $f : M \to \mathbb{R}$, if $f(p_0) = c$ for a critical point p_0 of f.

Definition 2. *Let p_0 be a critical point of the function $f : M \to \mathbb{R}$. The Hessian of f at he point p_0 is the $n \times n$ matrix*

$$H_f(p_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2}(p_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(p_0) & \cdots & \frac{\partial^2 f}{\partial {x_n}^2}(p_0) \end{bmatrix} \tag{2}$$

Since $\frac{\partial^2 f}{\partial x_i \partial x_j}(p_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(p_0)$, the Hessian of f is a symmetric matrix.

Let p_0 be a critical point of f and $c_0 \in \mathbb{R}$ such that $f(p_0) = c_0$. Then, c_0 is said to be *a critical value of* f. If p_0 is a regular point of f, then c_0 is said to be a *regular value of* f.

If a is a regular value of f, it can be shown that the set $f^{-1}(a) = \{p \in M | f(p) = a\}$ is an $n-1$ dimensional manifold [1].

Definition 3. *A critical point of a function $f : M \to \mathbb{R}$ is called "non-degenerate point of f" if $detH_f(p_0) \neq 0$. Otherwise, it is called "degenerate critical point".*

Lemma 1. *Let p_0 be a critical point of a smooth function*

$$f : M \to \mathbb{R}, (U, \varphi = (x_1, \ldots, x_n)), (V, \psi = (X_1, \ldots, X_n))$$

be two charts of p_0, and $H_f(p_0), \mathscr{H}_f(p_0)$ be the Hessians of f at p_0, using the charts (U, φ), (V, ψ) respectively. Then the following holds:

$$\mathscr{H}_f(p_0) = J(p_0)^t H_f(p_0) J(p_0) \tag{3}$$

where $J(p_0)$ is the Jacobian matrix for the given coordinate transformation, defined by

$$J(p_0) = \begin{bmatrix} \frac{\partial x_1}{\partial X_1}(p_0) & \cdots & \frac{\partial x_1}{\partial X_n}(p_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial X_1}(p_0) & \cdots & \frac{\partial x_n}{\partial X_n}(p_0) \end{bmatrix} \tag{4}$$

and the matrix $J(p_0)^t$ is the transpose of $J(p_0)$.

For a critical point p_0, non-degeneracy does not depend on the choice of charts around p_0. The same argument is also true for degenerate critical points. In fact we have

$$\mathscr{H}_f(p_0) = J(p_0)^t H_f(p_0) J(p_0)$$

by the previous lemma, and hence

$$det\mathscr{H}_f(p_0) = detJ(p_0)^t detH_f(p_0) detJ(p_0) \tag{5}$$

by using determinant function on both sides. On the other hand, the determinant of the Jacobian matrix is non-zero. So the statement "$det\mathscr{H}_f(p_0) \neq 0$" and "$detH_f(p_0) \neq 0$" are equivalent. In other words,

$$det\mathscr{H}_f(p_0) \neq 0 \Leftrightarrow detH_f(p_0) \neq 0.$$

Now a function $f : M \to \mathbb{R}$ is called a *Morse function* if any critical point of f is non-degenerate. From now on, we only consider a Morse function f.

Now, we introduce Morse lemma on manifolds.

Theorem 1 (The Morse Lemma). *Let M be an n-dimensional smooth manifold and p_0 be a non-degenerate critical point of a Morse function $f : M \to \mathbb{R}$. Then, there exists a local coordinate system $(X_1, X_2, \ldots, X_n)$ around p_0 such that the coordinate representation of f has the following form:*

$$f = -X_1^2 - X_2^2 \ldots - X_\lambda^2 + X_{\lambda+1}^2 + \ldots + X_n^2 + c \tag{6}$$

where $c = f(p_0)$ and p_0 corresponds to the origin $(0, 0, \ldots, 0)$.

One may refer to [5] for the proof.

The number λ of minus signs in the equation (6) is the number of negative diagonal entries of the matrix $H_f(p_0)$ after diagonalization. By Sylvester's law, λ does not depend on how $H_f(p_0)$ is diagonalized. So, λ is determined by f and p_0. The number λ is called "the index of the non-degenerate critical point p_0". Obviously, λ is an integer between 0 and n. Note that,

(i) A non-degenerate critical point is isolated.

(ii) A Morse function on a compact manifold has only finitely many critical points [5].

3. A Morse Function on $SO(n)$

In this section, we will define a Morse function on $SO(n)$.

The set of all $n \times n$ orthogonal matrices, $O(n) = \{A = (a_{ij}) \in M_n(\mathbb{R}) : AA^t = I_n\}$ is a group with matrix multiplication. From the definition of $O(n)$,

$$detA = \pm 1, \text{ for any } A \in O(n).$$

An orthogonal matrix with determinant 1 is called *rotation matrix* and the set of this kind of matrices is also a group, called *special orthogonal group* and denoted by $SO(n)$. On the other hand, let $S_n(\mathbb{R})$ denote the set of symmetric $n \times n$ matrices. Since each symmetric matrix is uniquely determined by its entries on and above the main diagonal, that is a linear subspace of $M_n(\mathbb{R})$ of dimension $n(n+1)/2$.

Now we define a function $\varphi : GL_n(\mathbb{R}) \longrightarrow S_n(\mathbb{R})$ by

$$\varphi(A) := A^t A.$$

Then, the identity matrix I_n is a regular value of φ [3].

Let $C \in S_n(\mathbb{R})$ with entries c_i, with $0 \le c_1 < c_2 < \ldots < c_n$ fixed real numbers and $f_C : SO(n) \to \mathbb{R}$ be given by,

$$f_C(A) := < C, A >= c_1 x_{11} + c_2 x_{22} + \ldots + c_n x_{nn}, \tag{7}$$

where $A = (x_{ij}) \in SO(n)$.

Obviously, f_C is a smooth function. Now, we will determine its critical points.

Lemma 2. *The critical points of the function f_C defined above are:*

$$\begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \pm 1 \end{bmatrix} \tag{8}$$

Proof. Let A be a critical point of f_C. Then the derivative of f_C at A must be zero. Consider the matrix given by a rotation of first and second coordinate $B_{12}(\theta)$ defined by

$$B_{12}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 & \cdots & 0 \\ sin\theta & cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then, $AB_{12}(\theta) \in SO(n)$ and the matrix $B_{12}(\theta)$ forms a curve on $SO(n)$. Moreover, $B_{12}(\theta) = A$ for $\theta = 0$.

By the definition of f_C, and after computing the matrix product, we have

$$f_C(AB_{12}(\theta)) = c_1(x_{11}cos\theta + x_{12}sin\theta) + c_2(-x_{21}sin\theta + x_{22}cos\theta) + c_3 x_{33} + \ldots + c_n x_{nn}. \tag{9}$$

By differentiating f in the direction of the velocity vector $\frac{d}{d\theta}AB_{12}(\theta)|_{\theta=0}$ of the curve $AB_{12}(\theta)$ at A, we have

$$\frac{d}{d\theta} f_C(AB_{12}(\theta))|_{\theta=0} = c_1 x_{12} - c_2 x_{21} \tag{10}$$

and

$$\frac{d}{d\theta} f_C(B_{12}(\theta)A)|_{\theta=0} = -c_1 x_{21} + c_2 x_{12}. \tag{11}$$

However, by the assumption that A is a critical point of f_C, we require these derivatives to be zero. i.e.

$$c_1 x_{12} - c_2 x_{12} = 0$$
$$-c_1 x_{21} + c_2 x_{12} = 0$$

Solving this system for x_{12}, x_{21} gives $x_{12} = x_{21} = 0$. We can carry out the similar calculation for $B_{ij}(\theta)$ with $i < j$, where $B_{ij}(\theta)$ is with the entries: $(i,i) = cos\theta$, $(i,j) = -sin\theta$, $(j,i) = sin\theta$

and $(j, j) = cos\theta$. Thus, for the matrix A, $x_{ij} = 0$ whenever $i \neq j$. So, that is, a critical point of f_C is a diagonal matrix. On the other hand $A \in SO(n)$, so we have $AA^t = I_n$. So each entry on the main diagonal of A must be ± 1.

Conversely, let A be a matrix in the form (8). In order to check that A is a critical point, we need to compute the derivative of f_C. If we could find $n(n-1)/2$ curves C_i going through A with velocity vector at A and linearly independent from each other. Since the velocity vector of C_i at A plays a role of a local coordinate of A, we only need to check that the derivative of $f_C(C_i)$ vanishes to see that $Df(A) = 0$. Now, the claim is the curves C_i's are in fact $AB_{ij}\theta$'s defined above. Let $\epsilon_i = A_{ii}$ where A_{ii} is the i-th diagonal entry of $A(\epsilon_i = \pm 1)$. Then, the derivative of the matrix $AB_{ij}(\theta)$ at A is (we did for the case B_{12}, but it is same for other indices with $i < j$),

$$\frac{d}{d\theta} AB_{12}(\theta)|_{\theta=0} = \begin{bmatrix} 0 & -\epsilon_1 & 0 & \cdots & 0 \\ \epsilon_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

This matrix is regarded as a vector in $\mathbb{R}^{n^2}$. By considering all $1 \leq i \leq j \leq n$, these matrices (vectors) form a basis for the tangent space $T_A SO(n)$.

So, for a given matrix A in the form (8), it is easy to compute that, the derivative of f_C at A is zero. This means nothing but A is a critical point of f_C. □

After now, we know the coordinate system of $SO(n)$ and the critical points of the the given function f_C. It is straightforward to compute the Hessian of f_C at A. Suppose that A is a critical matrix with diagonal entries $A_{ii} = \epsilon_i = \pm 1$. Then, we want to compute

$$\frac{\partial^2}{\partial\theta\partial\varphi} f_C(AB_{\alpha\beta}(\theta)B_{\gamma\delta}(\varphi))|_{\theta=0,\varphi=0}.$$

Notice that is linear $AB_{\alpha\beta}(\theta)B_{\gamma\delta}(\varphi)$ is linear in θ and in φ, and f_C is a linear function. Thus, we can bring the derivative inside f_C. So,

$$\begin{aligned} \frac{\partial^2}{\partial\theta\partial\varphi} f_C(AB_{\alpha\beta}(\theta)B_{\gamma\delta}(\varphi))|_{\theta=0,\varphi=0} &= f_C(A\frac{d}{d\theta}B_{\alpha\beta}(\theta)|_{\theta=0}\frac{d}{d\varphi}B_{\gamma\delta}(\varphi)|_{\varphi=0} \\ &= \begin{cases} -c_\alpha\epsilon_\alpha - c_\beta\epsilon_\beta & \text{if } \alpha = \gamma, \beta = \delta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This calculation becomes easier if we consider the matrix multiplication $c_{ij} = \sum_k a_{ik}b_{kj}$. The calculation above shows that the Hessian matrix is diagonal. Since $c_\alpha \neq c_\beta$ for $\alpha \neq \beta$, the entries on the diagonal are non-zero. Therefore, A is a non-degenerate critical point of f_C, meaning that f_C is a Morse function on $SO(n)$.

Assume that the subscripts i of the diagonal entries ϵ_i of A, $1 \leq i \leq n$, with $\epsilon_i = 1$ are

$$i_1, i_2, \ldots, i_m$$

in ascending order. Then the index of the critical point A (the number of minus signs on the diagonal of Hessian) is

$$(i_1 - 1) + (i_2 - 1) + \ldots + (i_m - 1).$$

And the index is 0 if all ϵ_i's are -1. Also, the critical value at the critical point is

$$2(c_{i1} + c_{i2} + \cdots + c_{im}) - \sum_{i=0}^{n} c_i.$$

Considering that $detA = 1$, there are 2^{n-1} critical points [4].

4. Perfect Morse Functions

First, we will give the basic notions.

Definition 4. *The Poincaré polynomial of the n-dimensional manifold M is defined to be*

$$P_M(t) = \sum_{k=0}^{n} b_k(M) t^k \tag{12}$$

where $b_k(M)$ is the k-th Betti number of M.

Definition 5. *Let $f : M \to \mathbb{R}$ be a Morse function. Then, the Morse polynomial of f is defined to be*

$$P_f(t) = \sum_{k=0}^{n} \mu_k t^k \tag{13}$$

where μ_k is the number of critical points of f of index k.

Theorem 2 (The Morse Inequality). *Let $f : M \to \mathbb{R}$ be a Morse function on a smooth manifold M. Then, there exists a polynomial $R(t)$ with non-negative integer coefficients such that*

$$P_f(t) = P_M(t) + (1+t)R(t).$$

One may refer to [7] for proof.

A Morse function $f : M \to \mathbb{R}$ is called a perfect Morse function if $P_f(t) = P_M(t)$ [7].

Now, we show that the function f_C on $SO(n)$ defined in the previous section is also a perfect Morse function.

Theorem 3. *The function*

$$f_C : SO(n) \to \mathbb{R}, \quad f_C(A) := \langle C, A \rangle$$

is a perfect Morse function where $C \in S_n(\mathbb{R})$.

Proof. First we show that the Morse polynomial is,

$$P_{f_C}(t) = (1+t)(1+t^2)\cdots(1+t^{n-1}). \tag{14}$$

We use induction method. For making it easier, we label the function f_C with n as $f_{C\,n} : SO(n) \to \mathbb{R}$.

Trivially, for $n = 1$, $P_{f_{C1}}(t) = 1$ and for $n = 2$, $P_{f_{C2}}(t) = 1 + t$. Assume that $P_{f_{Cn}}(t) = (1+t)(1+t^2) + \ldots + (1+t^{n-1})$. Then, we need to show that $P_{f_{Cn+1}}(t)$ satisfies the form (14).

We may consider that $SO(n+1)$ gets all the critical points from $SO(n)$ with extra bottom entry ($(n+1)$-th diagonal entry), which is either +1 or -1. Say the set of all these points are C^+_{n+1} and C^-_{n+1} respectively.

Let $A \in C^-_{n+1}$. Then we have $\tilde{A} \in O(n)$ such that, A is the matrix $\tilde{A}$ with extra bottom entry -1. Then, by the definition of index, we obtain

$$ind(A) = ind(\tilde{A}).$$

Thus, for the elements of C^-_{n+1} the equation (14) holds. Let $A \in C^+_{n+1}$. Then we have $\tilde{A} \in SO(n)$ such that, A is the matrix with $\tilde{A}$ with the bottom entry +1. Thus, by the definition of index, we obtain

$$ind(A) = ind(\tilde{A}) + n.$$

So, by the definition of Morse polynomial, we gain

$$P_{f_{Cn+1}}(t) = P_{f_{Cn}}(t)(1+t^n) = (1+t)(1+t^2)\ldots(1+t^{n-1})(1+t^n). \tag{15}$$

Now, we find out the Poincaré polynomial of $SO(n)$. The graded abelian group $H_*(SO(n), \mathbb{Z}_2)$ is isomorphic to the graded group coming from the exterior algebra [2]

$$\wedge_{\mathbb{Z}_2}[e_1, e_2, \ldots, e_{n-1}].$$

Let say $A(n) = \wedge_{\mathbb{Z}_2}[e_1, e_2, \ldots, e_{n-1}]$ where the degree of e_i, $|e_i| = i$. Then, we obtain

$$|e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}| = \sum_{j=1}^{k} |e_{i_j}| = \sum_{j=1}^{k} i_j.$$

If we define $a(n)_k = dim\mathbb{Z}_2(A(n)_k)$, then by the result in [2], $a(n)_k$ is nothing but the k-th Betti number of $SO(n)$. Hence, the polynomial

$$P(A(n)) = \sum_{i=0}^{\infty} a(n)_i t^i$$

is the Poincaré polynomial of $SO(n)$.

Now, our claim is that The Poincaré polynomial of $SO(n)$ is

$$P(A(n)) = (1+t)(1+t^2)\ldots(1+t^{n-1}).$$

Let $B(A(n))$ be the basis of $A(n)$. For instance, $B(A(1)) =$ trivial, $B(A(2)) = \{1, e_1\}$, $B(A(3)) = \{1, e_1, e_2, e_1 \wedge e_2\}$ etc.

In this sense, we obtain

$$B(A(n+1)) = (B(A(n)) \wedge e_n) \sqcup B(A(n)).$$

We use induction method. Indeed, here we have very similar arguments with the previous claim. The variable e_n has the same role with " the extra bottom entry ±1". Then, we have the polynomial $P(A(n)) = \sum_{b \in B(A(n))} a(n)_b t^{|b|}$. Trivially, $P(A(1)) = 1$ and $P(A(2)) = 1 + t$. By the induction hypothesis, assume that

$$P(A(n)) = \sum_{b \in B(A(n))} a(n)_b t^{|b|} = (1+t)(1+t^2)\cdots(1+t^{n-1}).$$

For the polynomial $P(A(n+1))$, pick an element $b \in B(A(n+1))$. Then, b is in either $B(A(n))$ or $B(A(n)) \wedge e_n$. For $b \in B(A(n+1))$, trivially, $P(A(n+1))$ has the desired form. If $b \in B(A(n)) \wedge e_n$, then by the definition of degree, there is $\tilde{b} \in B(A(n))$ such that $|b| = |\tilde{b}| + n$. Thus, by the definition of $P(A(n))$, we obtain

$$P(A(n+1)) = (1+t)(1+t^2)\cdots(1+t^n) \tag{16}$$

which completes the proof. □

Thereby, we have shown that, for the given Morse function $f_C : SO(n) \to \mathbb{R}, P_M(t) = P_{f_C}(t)$, meaning that f_C is a perfect Morse function.

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