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Hypergroups Associated With Ternary Relations And The Associated Join Space Of The Spherical Geometry

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Abstract. The paper deals with hypergroupoids obtained from ternary relations.The correspondence between ternary relation and hypergroups, studied by the author, especially in the case when the relation ρ is symmetric and reflexive, is analyzed here in the most general context.A necessary and sufficient condition on a ternary relation ρ is obtained for the associated hypergroupoid to be a hypergroup or a join space. Extension of a hyperoperation associated with a ternary relation being employed to obtain a associated join space of the spherical geometry.

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1. Introduction

Hyper structure theory was born during the 8th Congress of Scandinavian Mathematicians in 1934, when F. Marty [17] defined hypergroups, a natural generalization of the concept of group, and began to analyze their properties and applied to them to non commutative groups, rational fractions and algebraic functions etc,. Since then various connection between hypergroups and other subjects of theoretical and applied mathematics have been established. The most important applications to geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy sets and rough sets, automata theory are found in [8]. The first association between binary relation and hyperstructures appeared in J. Nieminen [18], who studied hypergroups related to connected simple graphs. In the same direction P. Corsini [3] worked, considering different hyperoperations associated with graphs.

J. Chvalina [1] used ordered structures for the construction of semihypergroups and hypergroups. Later on, I. Rosenberg [20] introduced a hyperoperation obtained by a binary relation; the new hypergroupoid has been investigated by P. Corsini [5, 6], P. Corsini and V. Leoreanu [7]

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and recently by I. Cristea and M. Ştefanescu [10]. P.Corsini introduced a new hyperoperation obtained by a binary relation [4].

Another approach to the connection between hypergroups and ordered set is initiated by M. Ştefanescu [12] and recently hypergroupoids associated with n-ary ($n \geq 3$) relations are studied by I. Cristea and M. Ştefanescu [11] and I. Cristea [9] alone. B. Davvaz and T. Vougioklis[13] introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. I. Cristea [9] introduced a new hypergroupoid associated with n-ary relations and obtained necessary conditions for the new hypergropoid to be a hypergroup or a join space; the new hypergroupoid has been investigated by S. Govindarajan (the author) and G. Ramesh [14] to find sufficient condition if any so that the hypergroupoid introduced by I. Cristea to be a hypergroup or a join space. In the same direction, albeit with different hyperoperations went the paper by the author and G. Ramesh [15].

Next, the author and G. Ramesh established a new correspondence between n-ary relations and hypergroupoids and found conditions on a ternary relation ρ, such that H_ρ is a hypergroup or a join space [16]. In the same paper, the author and G. Ramesh studied these hypergroups under the union and intersection of relations. In the papers [14] and [16], we obtained a necessary and sufficient condition on the $n(n \geq 3)$-ary relation ρ for H_ρ to be a hypergroup, especially in the case when ρ is reflexive and symmetric relation. In this paper, the analysis of connection between hyperpgroups and the $n(n = 3)$-ary relations is continued. We confine with ternary relations, by remarking that any results obtained with regards to $n(n = 3)$-ary relation can be obtained analogously to the case of $n(n \geq 4)$-ary relation. Here, we consider the ternary relation in a most general context and, we find a necessary and sufficient conditions such that H_ρ is a hypergroup or a join space. Several extensions of hyperoperations being employed, among them, one for which coincides with W. Prenowitz [19] extension of hyperoperation associated with spherical geometry is remarkable, but in a little different form. That is, W. Prenowitz extended the hyperoperation by adjoining an ideal element e and $e \notin S$, but we set $e \in S$.

The n-ary relations were studied for their applications in theory of dependence space. Moreover, they are used in database Theory, providing a convenient tool for database modeling. The operations such as union, intersection, difference and Cartesian product on n-ary relations are useful to describe information manipulations in databases. These operations are useful in the construction of new database.

First we recall some necessary definitions:

For a non empty set H, we denote by $P^*(H)$ the set of all non empty subsets of H.

- A non empty set H, endowed with a mapping, called hyperoperation
- $\circ : H \times H \longrightarrow P^*(H)$ is called a hypergroupoid.
- A *hypergroupoid* which verifies the following conditions:

  (1) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$,

  (2) $x \circ H = H = H \circ x$, for all $x \in H$, (reproduction axiom)

  is called a *hypergroup*.

- If, for any $x, y \in H, x \circ y = H$, then $(H, \circ)$ is called the *total hypergroup*.
- If A and B are nonempty subsets of H, then we denote the set $A \circ B = \bigcup_{a \in A}^{b \in B} a \circ b$.
- If A and B are nonempty subsets of H, then we denote $A/B = \bigcup_{a \in A}^{b \in B} a/b$.
- A commutative hypergroupoid $(H; \circ)$ is called a *join space* if the following implication holds: for any $(a, b, c, d) \in H^4, a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset$ (transposition axiom).

For more details on hypergroup theory, see [2, 8, 21].

2. Properties of the n-ary Relations

In this section we present some basic notions about the n-ary relations defined on a nonempty set H, $n \in N$ a natural number such that $n \geq 3$, and $\rho \subseteq H^n$ is an n-ary relation on H.

Definition 1 ([9, 11]). *The relation ρ is said to be :*

(1) reflexive if for any $x \in H$, the n-tuple $(x, x, \ldots, x) \in \rho$;

(2) n-transitive if it has the following property: if $(x_1, \ldots, x_n) \in \rho$, $(y_1, \ldots, y_n) \in \rho$ hold and if there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \leq n$, $1 \leq j_0 < n$, $x_{i_0} = y_{j_0}$, then the n-tuple $(x_{i_1}, \ldots, x_{i_k}, y_{j_{k+1}}, \ldots, y_{j_n}) \in \rho$ for any natural number $1 \leq k < n$ and $i_1, \ldots, i_k, j_{k+1}, \ldots, j_n$ such that $1 \leq i_1 < \ldots < i_k < i_0$, $j_0 < j_{k+1} < \ldots < j_n \leq n$;

(3) symmetric if $(x_1, x_2, \ldots, x_n) \in \rho$ implies $(x_n, x_{n-1}, \ldots, x_1) \in n\rho$

(4) strongly symmetric if $(x_1, \ldots, x_n) \in \rho$ implies $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \rho$ for any permutation σ of the set {1,...,n};

(5) n-ary preordering on H if it is reflexive and n-transitive;

(6) an n -equivalence on H if it is reflexive,strongly symmetric and n-transitive;

(7) diagonal n-ary relation on H if $\rho = \{(x, x, \ldots, x) \mid x \in H\}$.

2.1. Projections and Join Relations

Definition 2 ([9]). *Let ρ be an n-ary relation on a nonempty set H and $k < n$. The $(i_1, \ldots, i_k)$-projection of ρ, denoted by ρ_{i_1, i_k}, is a k- ary relation on H defined by: if $(a_1, a_2, a_3, a_n) \in \rho$, then $(a_{i_1}, \ldots, a_{i_k}) \in \rho_{i_1, \ldots, i_k}$*

Definition 3 ([9]). *Let ρ be an n-ary relation on a nonempty set H λ an m-ary relation on the same set H. The join relation of ρ and λ, denoted by $J_p(\rho, \lambda)$ where $1 < p < n$, $1 < p < m$ is an $m+n-p$ relation on H that consists of $m+n-p$-tuples $(a_1, \ldots, a_{m-p}, c_1, \ldots, c_p, b_1, \ldots, b_{n-p}) \in \rho$ and $(c_1, c_2, \ldots, c_p, b_1, \ldots, b_{n-p}) \in \lambda$.*

Let ρ be a ternary relation on H. The join relation $J_2(\rho,\rho)$ denoted by α is a 4-ary relation such that $(x,y,z)\in J_2(\rho,\rho) \Longleftrightarrow (x,y,z),(y,z,t)\in\rho$.

If we denote the join relation $J_2(\rho,\rho)$ by the symbol α then its projection relations are denoted by $\alpha_{_{1,2,4}}$ and $\alpha_{_{1,3,4}}$.

2.2. Some Results on Hypergroupoids Associated with n - ary Relations

To each ternary relation ρ on H, the hyperproduct is defined in [9] as follows:

(β) for any $x,y\in H, x\otimes_\rho y=\{z\in H \mid (x,z,y)\in\rho\}$.

The hyperproduct defined in (β) associated to ternary relation ρ is generalized to the case of an n-ary relation $\rho(n\geq 3)$, using projection as in the following:

(δ_1) For any $i\in\{2,\ldots,n-1\}$, $x\otimes_i y=\{z\in H \mid (x,z,y)\in\rho_{_{1,i,n}}\}$

(δ_2) $x\otimes_\rho y=\{z\in H \mid (x,z,y)\in\bigcup_{i=2}^{n-1}\rho_{_{1,i,n}}\}=\bigcup_{i=2}^{n-1}x\otimes_i y$.

This $(H,\otimes_\rho)$ is a hypergroupoid if and only if the projection $\rho_{1,n}$ is the total relation, that is $\rho_{1,n}=H\times H$. The necessary and sufficient condition on ρ for the hypergroupoid H_ρ to be a quasi hypergroup and necessary condition on ρ for H_ρ to be a semihypergroup are obtained by I. Cristea [9]. They are useful in the succeeding sections and so stated below.

Proposition 1 ([9, Proposition 11]). *Let ρ be an n-ary relation on H. Then $(H,\otimes_\rho)$ is a quasi-hypergroup if and only if $\rho_{1,n}=H\times H$, and there exists i,j with $2\leq i,\ j\leq n-1$, such that $\rho_{1,i}=\rho_{j,n}=H\times H$.*

Using the projection of $J_2(\rho,\rho)$, necessary condition for a hypergroupoid $(H,\otimes_\rho)$ to be a semihypergroup is obtained in [9] as follows:

Proposition 2 ([9, Proposition 16]). *Let ρ be a reflexive and symmetric ternary relation H. If $\rho\not\subset\alpha_{1,2,4}$ or $\rho\not\subset\alpha_{1,3,4}$, then the hyperoperation $\otimes_\rho$ is not associative.*

Corollary 1 ([9, Corollary 17]). *Let ρ be a reflexive and symmetric ternary relation H. If $(H,\otimes_\rho)$ is a semihypergroup, then $\rho\subset\alpha_{1,2,4}\cap\alpha_{1,3,4}$.*

Proposition 3. *Let ρ be a reflexive and symmetric n-ary relation H which satisfies the condition*

(S) $(x,a_1,\ldots a_{n-2},y)\in\rho \Longleftrightarrow (x,a_{\sigma(1)},\ldots a_{\sigma(n-2)},y)\in\rho$ for any permutation σ of the set {1,2,...,n-1}.

If, there exists $j\in\{2,\ldots,n-1\}$ such that $\rho_{1,j,n}\not\subset\alpha_{1,n,2n-2}$ or $\rho_{1,j,n}\not\subset\alpha_{1,n-1,2n-2}$, then the hyperoperation $\otimes_\rho$ is not associative.

Proposition 4 ([9, Proposition 14]). *Let ρ be a ternary relation on H such that $\rho_{1,3}=\rho_{1,2}=H\times H$ or $\rho_{1,3}=\rho_{2,3}$ If ρ is 3-transitive, then $(H;\otimes_\rho)$ is the total hypergroup.*

Example 1. *A ternary relation ρ is 3-transitive if and only if it satisfies the following conditions:*

(i) If $(x,y,z)\in\rho$, $(y,u,v)\in\rho$, then $(x,u,v)\in\rho$

(ii) If $(x,y,z)\in\rho$, $(z,u,v)\in\rho$, then $(x,y,u)\in\rho$, $(x,y,v)\in\rho$, $(x,u,v)\in\rho$, $(y,u,v)\in\rho$

(iii) If $(x,y,z)\in\rho$, $(u,z,v)\in\rho$, then $(x,y,v)\in\rho$.

3. Hypergroups Associated with Ternary Relations

We begin by remarking that the hypergroupoid $(H, \otimes_\rho)$ is a semihypergroup if and only if: the following bi-implication holds:

$$(\beta_1)\ \begin{cases} \forall a, b, c, z \in H, \text{ there exists } x \in H, \text{ such that } (a, x, b), (x, z, c) \in \rho \\ \Longleftrightarrow \text{there exits } y \in H, \text{ such that } (a, z, y), (b, y, c) \in \rho \end{cases}$$

Now, let ρ be a ternary relation on H such that $\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H$. The expression (β_1) together with $\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H$ and Example 1 shows that ρ is 3-transitive. That is, if $(H, \otimes_\rho)$ is a quasi hypergroup and $\otimes_\rho$ satisfies the associative axiom, then ρ is 3-transitive.

Therefore, from Proposition 4 and from the above paragraph it derives that:

Theorem 1. *Let ρ be a ternary relation on H such that $\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H$. If $(H, \otimes_\rho)$ is a hypergroup, then it is the total hypergroup.*

However, we can prove the existence of a hypergroup $(H, \otimes_\rho)$ which is different from the total hypergroup by restricting elements of ρ.

Proposition 5. *Let ρ be a ternary relation on H such that $\rho_{_{1,3}} = H \times H$. Consider the following condition:*

$$(\tau_1)\ \begin{cases} (x, x, y), (x, y, y) \in \rho & \forall (x, y) \in H^2 \\ (x, y, z) \notin \rho & \text{if } x, y, z \text{ are distinct} \\ (x, y, x) \notin \rho & \text{if } x \neq y \end{cases}$$

$$(\tau_2)\ \begin{cases} \text{if } a, b, c, d \text{ are distinct, then } (a, b, c) \in \rho, (b, c, d) \in \rho \\ \Longleftrightarrow (a, b, d) \in \rho, (a, c, d) \in \rho. \end{cases}$$

Then $(H, \otimes_\rho)$ is a hypergroup if and only if: either (τ_1) alone holds or both (τ_1) and (τ_2) are simultaneously holds.

Proof. **Case(1):** Suppose (τ_1) alone holds. Then, $\forall (x, y) \in H^2$ $(x, x, y), (x, y, y) \in \rho$. Hence, $\forall (x, y) \in H^2$, $\{x, y\} \subset x \otimes_\rho y$. Therefore, $(H, \otimes_\rho)$ is a quasi-hypergroup. We clearly have

$$\forall (x, y, z) \in H^3, x \otimes_\rho (y \otimes_\rho z) = \bigcup_{a \in y \otimes_\rho z} x \otimes_\rho a = \{x, y, z\} = (x \otimes_\rho y) \otimes_\rho z.$$

So, $\otimes_\rho$ is associative, whence. $(H, \otimes_\rho)$ is a hypergroup.

Case(2): Suppose both (τ_1) and (τ_2) simultaneously hold.

It follows from **Case (1)** that $(H, \otimes_\rho)$ is a quasi-hypergroup. All that remains to be proved is that $\otimes_\rho$ associative, and we dispose of this by showing that

$$\forall (x, y, z) \in H^3, (x \otimes_\rho y) \otimes_\rho z \subset x \otimes_\rho (y \otimes_\rho z),$$

and conversely. Let $a \in (x \otimes_\rho y) \otimes_\rho z$. Thus $\Longrightarrow$ there exists $u \in x \otimes_\rho y$ such that $a \in u \otimes_\rho z$. Hence, for any $a \in (x \otimes_\rho y) \otimes_\rho z$, we have $(x, u, y) \in \rho$ and $(u, a, z) \in \rho$ (η).

We show that $a \in x \otimes_\rho (y \otimes_\rho z)$ to get that $(x \otimes_\rho y) \otimes_\rho z \subset x \otimes_\rho (y \otimes_\rho z)$.

We consider the following situations:

(i) Let $(x,u,y) \in \rho$ with $(u,a,z) \in \rho$. Using the $(u,a,z) \in \rho$ in (τ_2), we obtain that $(u,y,a),(y,a,z) \in \rho$. By the (τ_1) and from $(y,a,z) \in \rho$, it follows that $(x,a,a) \in \rho$ with $(y,a,z) \in \rho$. Thus, there exist $a = v \in y \otimes_\rho z$ with $a \in x \otimes_\rho a$. So, in this case $a \in x \otimes_\rho (y \otimes_\rho z)$.

(ii) Now, we consider the (η) with $x = y$. From (τ_1), we obtain, for any $x \in H$, $(x,x,x) \in \rho$. From (η), we obtain that $u = x$ and $(x,a,z) \in \rho$ it follows $a \in (x \otimes_\rho x) \otimes_\rho z$. Thus $a \in x \otimes_\rho (x \otimes_\rho z)$.

We have, $(x,a,z) \in \rho$ and (by the (τ_1)) $(x,a,a) \in \rho$. Thus, there exist $v = a \in H$ such that $(x,a,z) \in \rho$ with $(x,a,a) \in \rho$, so, in this case, $a \in x \otimes_\rho (x \otimes_\rho z)$.

(iii) Finally, we consider the (η) with $y = z$. Put $y = z$ in (η). Then, it follows that $(x,u,z) \in \rho$ and $(u,a,z) \in \rho$. By the (τ_2), we obtain that $(x,u,a) \in \rho$. Now, we have $(x,u,a) \in \rho$ and $(u,a,z) \in \rho$. Using $(x,u,a) \in \rho$ and $(u,a,z) \in \rho$ in (τ_2), we obtain that $(x,a,z) \in \rho$. Now, (τ_1) yields that $(z,z,z) \in \rho$. Hence, $(x,a,z) \in \rho$ with $(z,z,z) \in \rho$ and so $a \in x \otimes_\rho (z \otimes_\rho z)$.

We have proved that, for any $x,y,z,a \in H$, $a \in x \otimes_\rho (y \otimes_\rho z) \Longrightarrow a \in x \otimes_\rho (y \otimes_\rho z)$; i.e.

$$(x \otimes_\rho y) \otimes_\rho z \subset x \otimes_\rho (y \otimes_\rho z).$$

Similar argument establishes the other inclusion

$$x \otimes_\rho (y \otimes_\rho z) \subset (x \otimes_\rho y) \otimes_\rho z.$$

Conversely, we suppose that H_ρ is a semihypergroup and $(x,y,z) \notin \rho$ with x,y,z are all distinct. Assume to the contrary that $(x,x,z),(x,z,z) \notin \rho$, for some $x,z \in H$. Then, it follows that

$$\begin{aligned} z \notin & x \otimes_\rho (y \otimes_\rho z) = x \otimes_\rho \{y,z\} = \{x,y\} \cup x \otimes_\rho z \\ z \in & (x \otimes_\rho y) \otimes_\rho z = \{x,y\} \otimes_\rho z = \{y,z\} \cup x \otimes_\rho z \end{aligned}$$

which is a contradiction to that H_ρ is a semihypergroup. Thus (τ_1) holds.

Next, we prove that (τ_2) holds. Suppose to the contrary that (τ_2) does not holds. Let $(x,u,y),(u,y,z) \in \rho$ with x,u,y,z are all distinct. Suppose that $(x,y,z) \notin \rho$. We have from (τ_1) that $(z,z,z) \in \rho$ as $(z,z) \in H^2$. It follows that

$$y \notin x \otimes_\rho z \subset x \otimes_\rho (z \otimes_\rho z)$$

but

$$y \in u \otimes_\rho z \subset (x \otimes_\rho y) \otimes_\rho z$$

which is a contradiction to that H_ρ is a semihypergroup. Thus (τ_2) holds. □

Remark 1. *If H_ρ is a hypergroup and (τ_1) alone holds, then ρ is symmetric ternary relation on H.*

Proof. It follows by (τ_1) of Proposition 5. □

Remark 2. *If H_ρ is a hypergroup, then ρ is reflexive relation on H.*

Proof. According to Proposition 5, we have $\forall(x,y)\in H^2, (x,x,y)\in\rho$, so, for $(x,x)\in H^2$, $(x,x,x)\in\rho$, whence ρ is reflexive. □

Remark 3. *$(H,\otimes_\rho)$ is a quasihypergroup does not imply the condition (τ_1) of Proposition 5, as we can see in the following example.*

Example 2. *Let $H=\{x,y,z\}$ and*

$$\begin{aligned}\rho=&\{(x,x,x),(y,y,y),(z,z,z),(x,y,z),(z,y,x),\\&(x,x,y),(y,x,x),(x,x,z),(x,z,z),(y,z,z),(z,x,x),(z,z,x),(z,z,y)\}.\end{aligned}$$

We have clearly,

$$\begin{aligned}&\forall x\in H,\ x\otimes_\rho x=\{x\},\ x\otimes_\rho y=\{x\}=y\otimes_\rho x,\\&x\otimes_\rho z=\{x,y,z\}=z\otimes_\rho x,\ y\otimes_\rho z=\{z\}=z\otimes_\rho y.\end{aligned}$$

So, we obtain the hypergroupoid:

Table 1: Example 2 Hypergroupoid

$\otimes_\rho$	x	y	z
x	x	x	x,y,z
y	x	y	z
z	x,y, z	z	z

We have $\forall x\in H,\ x\otimes_\rho H=H=H\otimes_\rho x$. So, $(H;\otimes_\rho)$ is a quasi hypergroup, but we find also $(x,y,y),(y,y,x)\notin\rho$. Notice that $(x,y,z)\in\rho$ and $x\neq y\neq z$.

Moreover, $(H,\otimes_\rho)$ is a hypergroup.

Remark 4. *If $(x,y,z)\notin\rho$ with x,y,z are all distinct, then the condition (τ_2) is neither necessary nor sufficient for $(H,\otimes_\rho)$ to be a hypergroup as we see in the following example.*

Example 3. *Let us consider the ternary relation ρ on a non empty set H defined as follows:*

$$\rho=\{(x,x,y),\ (x,y,y)\mid(x,y)\in H^2\}.$$

We find

$$\rho_{1,3}=\rho_{1,2}=\rho_{2,3}=H\times H$$

and

$$\forall(x,y)\in H^2,\ x\otimes_\rho y=\{x,y\}=x\otimes_\rho y.$$

We clearly have

$$\forall(x,y,z)\in H^3,\ x\otimes_\rho(y\otimes_\rho z)=\{x,y,z\}=(x\otimes_\rho y)\otimes_{rho} z.$$

Moreover $(H,\otimes_\rho)$ is a quasi-hypergroup, since $\rho_{1,2}=\rho_{2,3}=H\times H$.

Now, we give an example to show that (τ_2) is a sufficient condition for $(H, \otimes_\rho)$ to be a semihypergroup though $(H, \otimes_\rho)$ is a quasi-hypergroup.

Example 4. *On the set* $H = \{1,2,3,4\}$*, we consider the ternary relation* ρ *defined as follows:*

$$\rho = \{(x,x,y),\ (x,y,y) \mid (x,y) \in H^2\} \cup \{(1,2,1),(1,2,3),(1,3,2),(1,2,4),(2,3,2),$$
$$(2,3,1),(2,3,4),(3,2,1),(4,2,1),(4,3,2)\}.$$

We obtain the hypergroupoid:

Table 2: Example 4 Hypergroupoid

$\otimes_\rho$	1	2	3	4
1	1,2	1,2,3	1,2,3	1,2,4
2	1,2,3	2,3	2,3	2,3,4
3	1,2,3	2, 3	3	3,4
4	1,2,4	2,3,4	3,4	4

Clearly, we have

$$\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H.$$

So, $(H, \otimes_\rho)$ *is a quasi-hypergroup, but* $\otimes_\rho$ *is not associative. Indeed, we have* $(1,2,3) \in \rho$ *and* $(2,3,4) \in \rho$*, but* $(1,3,4) \notin \rho$ *while* $(1,2,4) \in \rho$*.*

Therefore,

$$1 \otimes_\rho (4 \otimes_\rho 4) = \{1,2,4\} \neq \{1,2,3,4\} = (1 \otimes_\rho 4) \otimes_\rho 4.$$

If we suppose $(1,3,4) \in \rho$ *then* (τ_2) *is satisfied, and then* $\otimes_\rho$ *is associative.*

Proposition 6. *Let* ρ *be a reflexive and symmetric ternary relation* H *such that* $\rho_{1,3} = H \times H$*. Then* $(H, \otimes_\rho)$ *is a hypergroup, which is different from the total hypergroup if and only if :*

(1) $\rho_{1,2} = \rho_{2,3} = H \times H$

(2) $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$

(3) either (τ_1) *alone holds or both* (τ_1) *and* (τ_2) *are simultaneously holds.*

Proof. ($\Longrightarrow$) It results by Proposition 5 and Corollary 1.
($\Longleftarrow$) The conditions of Proposition 5 are verified. Further, (3) $\Longrightarrow$ (2). □

3.1. Hypergroups Associated with Reflexive Symmetric Ternary Relations

Recall now what a spherical geometry is (see [19], [8, defn. 25, p.37 in]). This definition is useful in order to characterize the hypergroup H_ρ

Definition 4. *An (abstract) spherical geometry is a system* (S, R)*, where* S *is a set of elements called points and* R *is a ternary relation on* S *called betweenness, which satisfies the following postulates:*

(i) If $(x,y,z)\in R$, then x,y,z are distinct;

(ii) if $(x,y,z)\in R$, then $(z,y,x)\in R$;

(iii) for any x, there exists a unique x' such that $x\neq x'$ and the following implication holds:

$$(x,u,v)\in R\Longrightarrow(u,v,x')\in R;$$

(iv) if $y\neq x$ and $y\neq x'$, then there exists u such that $(x,u,y)\in R$.

W. Prenowitz [19] defined the following hyperoperation on S:

$$\forall(x,y)\in S^2, y\neq x, y\neq x', x\circ y=\{t\mid(x,t,y)\in R\}, x\circ x=\{x\}.$$

Now, let us set: $R=\rho$. Again, recall that the join relation $J_2(\rho,\rho)$ denoted by α is a 4-ary relation such that

$$(x,y,z)\in J_2(\rho,\rho)\Longleftrightarrow(x,y,z),(y,z,t)\in\rho.$$

Suppose that ρ is betweenness relation. Then the postulates *(iii)* and *(iv)* of Definition 4 together with the definition of join relation and projection relations shows that, for any $(x,y,z)\in\rho$, there exists a unique x' such that $x\neq x'$ and $(x,y,x')\in\alpha_{1,2,4},(x,z,x')\in\alpha_{1,3,4}$. We observe that the product $\underset{\rho}{\otimes}$ and $\circ$ are identical and the relation $\rho=R$ if and only if ρ is reflexive and symmetric ternary relation on H. Hence, we obtain the following proposition:

Proposition 7. *Let ρ be a reflexive and symmetric ternary relation on H with $|H|\geq 3$ such that $\rho_{1,3}=H\times H$. If ρ satisfies the following the postulates, then $\underset{\rho}{\otimes}$ is not associative*

(i) If $(x,y,z)\in\rho$, then x,y,z are distinct;

(ii) if $x\neq y$, then $(x,y,x)\notin\rho$;

(iii) for any $(x,y,z)\in\rho$, there exists a unique x' such that $x\neq x'$ and $(x,y,x')\in\alpha_{1,2,4}$, $(x,z,x')\in\alpha_{1,3,4}$.

Proof. The ρ verifies the postulates of the betweenness relation on H. Now, the proof follows from [8, Theorem 27, p.39]. □

Let ρ be a reflexive ternary relation H such that $\rho_{1,3}=H\times H$. Let $(x,a,y)\in\rho$ be arbitrary and x,a,y are distinct. Let $(H;\circ_\rho)$ be the hypergroupoid defined as follows.

$$\begin{aligned}\forall(x,y)\in H^2, x\neq y, x\circ_\rho y&=\{a\mid(x,a,y)\in\rho\}\cup\{x,y\}\\ \forall x\in H,\ x\circ_\rho a&=\{x\}.\end{aligned}$$

Indeed, $\circ_\rho$ is an extension of $\otimes_\rho$.

If we suppose,

$$\forall\ (x,y)\in H^2,\ (x,y,x)\notin\rho \text{ when } x\neq y; \tag{1}$$

then, we have clearly:

$$\forall x\in H, x\circ_\rho x=\{x\}=x\underset{\rho}{\otimes}x\Longleftrightarrow(x,x,x)\in\rho$$

$$\forall(x,y)\in H^2, x\neq y, x\circ_\rho y=\{x,a,y\}=x\underset{\rho}{\otimes}y\cup\{x,y\}\Longleftrightarrow\forall x\neq y,(x,a,y)\in\rho$$
$$\forall x\in H,\ x\circ_\rho a=\{x\}=x\underset{\rho}{\otimes}a\Longleftrightarrow(x,x,a)\in\rho$$

If $(x',a',y')\in\rho$ and x',a',z',a are distinct, then we extend the product as follows:

$$a\circ_\rho a'=\{a,a'\}\text{ whenever }\nexists\,u\in H\text{ such that }(a,u,a')\in\rho.$$

Again, if $(a,u,a')\in\rho$, then we define the product recursively as above.

Proposition 8. *Let ρ be a reflexive and symmetric ternary relation on H with $|H|\geq 3$ such that $\rho_{1,3}=H\times H$. If ρ satisfies the following conditions:*

(i) any $(a,b,c)\in\rho$ with a,b,c distinct $\Longrightarrow(x,b,y)\in\rho$, $\forall(x,y)\in H^2$ and $x\neq y$

(ii) $\forall(x,y)\in H^2$, $x\neq y$, $(x,y,x)\notin\rho$

(iii) $\forall x\in H$, $(x,x,b)\in\rho$

then the extension $(H;\circ_\rho)$ of $(H;\underset{\rho}{\otimes})$ defined by setting

$$\forall(x,y)\in H^2, x\neq y, x\circ_\rho y=\{z\mid(x,z,y)\in\rho\}\cup\{x,y\}$$

is a join space.

Proof. Suppose $(x,z,y)\in\rho$. Then, we clearly have

$$\forall(x,y)\in H^2, x\neq y, x\circ_\rho y=\{z\mid(x,z,y)\in\rho\}\cup\{x,y\}$$
$$\forall x\in H,\ x\circ_\rho x=\{x\}$$
$$\forall x\in H,\ x\circ_\rho z=\{x\}$$

Since, for any $x,y\in H$, $\{x,y\}\subset x\circ_\rho y$; it follows that $(H;\circ_\rho)$ is a quasihypergroup.

Moreover, since ρ is symmetric, it follows that $x\circ_\rho y=y\circ_\rho x$, for any $x,y\in H$, and therefore $(H;\circ_\rho)$ is commutative.

Now, we prove that the hyper operation $<\circ_\rho>$ is associative. We have clearly,

$$\begin{aligned}x\circ_\rho(y\circ_\rho z)=&x\circ_\rho\{y,z',z\mid(y,z',z)\in\rho\}\\=&x\circ_\rho y\cup x\circ_\rho z'\cup x\circ_\rho\{z'\mid(y,z',z)\in\rho\}\\=&x\circ_\rho y\cup x\circ_\rho z,\text{ since }x\circ_\rho z'=\{x\}\subset x\circ_\rho y\\=&x\circ_\rho y\cup x\circ_\rho\{x,z'',z\mid(x,z'',z)\in\rho\}\\&,\text{ since }x\circ_\rho z=\{z''\mid\{x,z'',z)\in\rho\}\cup\{x,z\}\\=&(x\circ_\rho y)\circ_\rho z,\text{ since }\{x,z''\}\subset(x\circ_\rho y)\end{aligned}$$

It remains to check the condition of the join space. Set $a,b,c,d\in H$ such that $a/b\cap c/d\neq\emptyset$; then there exists $x\in a/b\cap c/d$, that is

$$a\in x\circ_\rho b=\{z\mid(x,z,b)\in\rho\}\cup\{x,b\}\text{ and }c\in x\circ_\rho d=\{z\mid(x,z,d)\in\rho\}\cup\{x,d\}.$$

We consider the following situations:

(i) Let a,b,c,d,z are distinct and $z \in a \circ_\rho d$. Recall that, for any $x \neq y \neq z$, $(x,z,y) \in \rho \Longrightarrow (x',z,y') \in \rho$, for all $x' \neq y'$. Then, it follows that $z \in b \circ_\rho c$ and therefore $z \in a \circ_\rho d \cap b \circ_\rho c$. Similarly, if $z \in b \circ_\rho c$ then $z \in a \circ_\rho d \cap b \circ_\rho c$.

(ii) If a,b,c,d,z are distinct and $a = x$ then $c \in a \circ_\rho d$, and since $c \in b \circ_\rho c$, it follows that $c \in a \circ_\rho d \cap b \circ_\rho c$.

(iii) If a,b,c,d,z are distinct and $c = x$ then $a \in c \circ_\rho b$. By the symmetry of ρ, it follows that $a \in b \circ_\rho c$, and since $a \in a \circ_\rho d$, it follows that $a \in a \circ_\rho d \cap b \circ_\rho c$.

By the same way, we establish that $a \in a \circ_\rho d \cap a \circ_\rho c$ and $c \in a \circ_\rho c \cap a \circ_\rho c$ in the case $a = b$ and $c = d$ respectively.

We can conclude that $a \circ_\rho d \cap b \circ_\rho c \neq \emptyset$, so $(H;\circ_\rho)$ is a join space. □

Let $< \circ_\rho >$ be the extension of $< \otimes_\rho >$ as it is defined in Proposition 8 and ρ be a ternary relation on H. Let α be the join relation $J_2(\rho,\rho)$. Using the projections of α, we give the following proposition.

Proposition 9. *Let ρ be a reflexive and symmetric ternary relation on H with $|H| \geq 3$ such that $\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H$. If ρ satisfies the following condition:*

(1) If x,y,z are distinct and $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$, then $(x,y,z) \in \rho$ and conversely;

then $(H;\circ_\rho)$ is a hypergroup.

Proof. Since $\rho_{1,2} = \rho_{2,3} = H \times H$, it follows, by Proposition 1, that $(H;\circ_\rho)$ is a quasihypergroup.

It remains to check the associative axiom. First of all, we shall check the following equality:

$$\forall (x,y,z) \in H^3, x \circ_\rho (y \circ_\rho z) = x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z. \tag{2}$$

We suppose that $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$ and x,y,z are distinct, then $(x,y,z) \in \rho$ and x,y,z are distinct. Then (1) $\Longrightarrow (x,y,z) \in \rho$ and x,y,z are distinct.

Now, we verify:

$$u \in x \circ_\rho (y \circ_\rho z) \Longleftrightarrow u \in x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z.$$

$\Longrightarrow$ There exists $v \in y \circ_\rho z$ such that $u \in x \circ_\rho v$. Hence $(x,u,v) \in \rho$ with $(y,v,z) \in \rho$. Recall that,

$$(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \Longleftrightarrow (x,y,t,z) \in J_2(\rho,\rho) \Longleftrightarrow (x,y,t),(y,t,z) \in \rho. \tag{3}$$

Set $t = v$. Then, we have $(x,y,v),(y,v,z) \in \rho$, whence $(x,u,v) \in \rho$ and $(x,y,v) \in \rho$.

By the condition specified and by the (3), if

$$(a,t,b) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \text{ and } (a,s,b) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \in \rho,\ t \neq s$$

then

$$(a,t,b) \in \rho \text{ and } (a,s,b) \in \rho;$$

it follows that $(a,t,s) \in \rho$ or $(a,s,t) \in \rho$. Therefore, from $(x,u,v) \in \rho$, $(x,y,v) \in \rho$ and $u \neq y$ it results $(x,u,y) \in \rho$ or $(x,y,u) \in \rho$.

If $(x,u,y) \in \rho$, then $u \in x \circ_\rho y$. If $(x,y,u) \in \rho$, then $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4} \Longrightarrow (y,u,z) \in \rho$, whence $u \in y \circ_\rho z$. If $u = y$, then $u = y \in y \circ_\rho y$ (by the reflexivity of ρ). Therefore, $u \in x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z$.

$\Longleftarrow$ Suppose $u \in x \circ_\rho y$. Then $(x,u,y) \in \rho$. Let $(x,y,z) \in \rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$.

Since $(x,u,y) \in \rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$, it follows that there exists a $v \in H$ such that $(x,u,v) \in \rho$, $(u,v,z) \in \rho$. Thus $u \in x \circ_\rho v$. Again,

$$(x,y,z) \in \rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4} \Longrightarrow (y,v,z) \in \rho.$$

Hence, from $(x,u,v) \in \rho$ and $(y,v,z) \in \rho$, it follows that $u \in x \circ_\rho v$ with $v \in y \circ_\rho z$. Therefore $u \in x \circ_\rho (y \circ_\rho z)$.

Now, suppose $u = y$. Then we obtain $(x,y,v),(y,v,z) \in \rho$ since $(x,y,z) \in \alpha_{1,2,4} \cap \alpha_{1,3,4}$. Hence, from $(x,y=u,v) \in \rho$ and $(y,v,z) \in \rho$, it follows $u \in x \circ_\rho v$ with $v \in y \circ_\rho z$. Therefore $u \in x \circ_\rho (y \circ_\rho z)$.

Finally, suppose $u \in y \circ_\rho z$. By the definition of the product $\circ_\rho$, $u \in x \circ_\rho u$, whence $(x,u,u) \in \rho$ and so $u \in x \circ_\rho (y \circ_\rho z)$.

Thus, we have established the following result:

$$u \in x \circ_\rho (y \circ_\rho z) \Longleftrightarrow u \in x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z.$$

By the same way, we prove the following result:

$$u \in (x \circ_\rho y) \circ_\rho z \Longleftrightarrow u \in x \circ_\rho y \cup y \circ_\rho y \cup y \circ_\rho z.$$

We find $x \circ_\rho (y \circ_\rho z) = (x \circ_\rho y) \circ_\rho z$.

This completes the proof. □

4. The Associated Join Space Of The Spherical Geometry

Proposition 10. *Let ρ be a reflexive and symmetric ternary relation on H with $|H| \geq 3$ such that $\rho_{1,2} = \rho_{2,3} = \rho_{1,3} = H \times H$. Let ρ satisfies the following postulates:*

(i) If $(x,y,z) \in \rho$, then x,y,z are distinct;

(ii) for any x, there exists a unique x' such that $(x,y,x') \in \alpha_{1,3,4}$;

(iii) if $x \neq y$, then $(x,y,x) \notin \rho$.

Then the extension $(H;\tilde{\circ}_\rho)$ of $(H;\otimes_\rho)$ defined by the following setting is a join space. $\forall x \neq y$, choose $t \in H$ such that $x \neq t \neq x'$.

Set $\forall x \neq y$, $x\tilde{\circ}_\rho y = \{x,t,y\} \forall x \in H$, $x\tilde{\circ}_\rho t = \{x\} = x\tilde{\circ}_\rho e = \{x\}$ and for any $t,t' \in x\tilde{\circ}_\rho y$, set $t\tilde{\circ}_\rho t' = \{t,t'\}$

Proof. Suppose ρ be a reflexive and symmetric ternary relation on H. Then, from the hypothesis, we obtain the following implications:

(i) If $(x, y, z) \in \rho$, then x, y, z are distinct;

(ii) if $(x, y, z) \in \rho$, then $(z, y, x) \in \rho$ are distinct.

(iii) For any x, there exists a unique x' such that $(x, y, x') \in \alpha_{1,3,4} \Longrightarrow$ for any x, there exists a unique x' and an element $u \in H$ such that $(x, u, y) \in \rho$ and $(u, y, x') \in \rho$,

(iv) if $x \neq y$, then $(x, y, x) \notin \rho \Longrightarrow \forall x \in H$, $x \,\widetilde{\circ}_\rho\, x = \{x\}$.

Hence, ρ is the *betweenness* relation on H.

Since $\rho_{1,2} = \rho_{2,3} = H \times H$, it follows, by Proposition 1, that $(H; \widetilde{\circ}_\rho)$ is a quasihypergroup.

Moreover, since ρ is symmetric, it follows that $x \,\widetilde{\circ}_\rho y = y \,\widetilde{\circ}_\rho x$, for any $x, y \in H$, and therefore $(H; \widetilde{\circ}_\rho)$ is commutative.

Now, we prove that the hyper operation $\langle \widetilde{\circ}_\rho \rangle$ is associative. We shall check the following inclusion:

$$\forall (x, y, z) \in H^3, (x \,\widetilde{\circ}_\rho\, y) \,\widetilde{\circ}_\rho\, z \subset x \,\widetilde{\circ}_\rho\, (y \,\widetilde{\circ}_\rho\, z). \tag{4}$$

Let $u \in (x \,\widetilde{\circ}_\rho\, y) \,\widetilde{\circ}_\rho\, z$. Then, there exists $v \in x \,\widetilde{\circ}_\rho\, y$ such that $u \in x \,\widetilde{\circ}_\rho\, v$. Hence, $(x, v, y) \in \rho$ with $(v, u, z) \in \rho$. We distinguish the following cases:

(i) If $x \neq y$ and $y \neq x'$ then $(v, y, x') \in \rho$; it follows $(x, y, x') \in \alpha_{1,3,4} \subset \rho$. Set: $y = e$. Hence, from $x \,\widetilde{\circ}_\rho\, e = \{x\}$ and $v \in x \,\widetilde{\circ}_\rho\, y$, it follows $v = x$. Thus, $(x, u, z) \in \rho$. On the other hand, from $z \,\widetilde{\circ}_\rho\, y = z$ and from the symmetry, it follows $(y, z, z) \in \rho$. Therefore, $u \in x \,\widetilde{\circ}_\rho\, (y \,\widetilde{\circ}_\rho\, z)$.

By the same way, we check the following other inclusion:

$$\forall (x, y, z) \in H^3, (x \,\widetilde{\circ}_\rho\, y) \,\widetilde{\circ}_\rho\, z \supseteq x \,\widetilde{\circ}_\rho\, (y \,\widetilde{\circ}_\rho\, z) \tag{5}$$

(ii) If $y = x'$, then

$$\begin{aligned}
(x \,\widetilde{\circ}_\rho\, y) \,\widetilde{\circ}_\rho\, z =& \{x, t, x'\} \,\widetilde{\circ}_\rho\, z \\
=& \{x, e, x'\} \,\widetilde{\circ}_\rho\, z \\
=& x \,\widetilde{\circ}_\rho\, z, \cup \{z\} \,\cup\, x' \,\widetilde{\circ}_\rho\, z \\
=& \{x, e, z\} \,\cup\, \{z\} \,\cup\, x' \,\widetilde{\circ}_\rho\, z \\
=& \{x, e, z\} \,\cup\, \{x', e, z, \} = \{x, e, x', z\} \\
=& x \,\widetilde{\circ}_\rho\, (x' \,\widetilde{\circ}_\rho\, z)
\end{aligned}$$

so $(H; \widetilde{\circ}_\rho)$ is a commutative hypergroup.

Now, let us check the following implication:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \,\widetilde{\circ}_\rho\, d \,\cap\, b \,\widetilde{\circ}_\rho\, c \neq \emptyset.$$

Notice that $\forall a \in H$, a has a unique inverse a' and $\forall (a, b) \in H^2$, $a/b = a \,\widetilde{\circ}_\rho\, b'$.

So,

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \,\widetilde{\circ}_\rho\, b' \,\cap\, c \,\widetilde{\circ}_\rho\, d' \neq \emptyset$$

whence $\{a\} \,\cap\, b \,\widetilde{\circ}_\rho\, (c \,\widetilde{\circ}_\rho\, d') \neq \emptyset$, hence $a/d' \cap\, b \,\widetilde{\circ}_\rho\, c \neq \emptyset$, that is, $a \,\widetilde{\circ}_\rho\, d \,\cap\, b \,\widetilde{\circ}_\rho\, c \neq \emptyset$ (By [8, Theorem (64, 2), p.12]).

Therefore, $(H; \widetilde{\circ}_\rho)$ is a join space. Thus, we obtain a join space with identity e' in H. □

To illustrate the application of this proposition, let us consider a ternary relation defined on the set $H = \{1, 2, 3, 4\}$ with

$$\rho \supset \{(1,2,3),(2,3,4),(1,3,2),(3,2,4),(3,2,1),(4,3,2),(2,3,1),(4,2,3)\}$$

and

$$\rho_{1,3} = \rho_{1,2} = \rho_{2,3} = H \times H.$$

Let $x = 1, x' = 4$. Choose $t = 2$ and $t' = 3$ and set: $\forall x \in H$, $x \circ_\rho x = \{x\}$. Applying the hyperoperation defined in the proposition, we obtain the hypergroupoid:

Table 3: Join Space Associated with Spherical Geometry

$\otimes_\rho$	1	2	3	4
1	1	1,2,3	1,2,3	1,2, 3,4
2	1,2,3	2	2,3	2,3,4
3	1,2,3	2, 3	3	2,3,4
4	1,2,3,4	2,3,4	2,3,4	4

Clearly, we have $(H; \circ_\rho)$ a join space.

We have consider only one element $x = 1$ and its inverse $x' = 4$. For the remaining elements $2, 3$, we can easily find their inverses in infinite case which is necessary for the spherical geometry.

5. Conclusion

We have characterized all the ternary relation ρ such that the hypergroupoid H_ρ is a hypergroup or a join space. We have stated some connections between a general ternary relation ρ and hypergroups and then proved them. Moreover, a correspondence between this hypergroupoid and the join space obtained by W. Prenowitz from betweenness (ternary) relation has been established.

In future work we intend to study the properties of hypergroupoid associated with the Cartesian product and join of two ternary relations. Moreover, We try to associate a directed connected graph with the hypergroupoids which have been considered in this paper.

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