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Polynomial Integral Transform for Solving Differential Equations

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Abstract. In this paper, we propose Polynomial Integral Transform for solving differential equations. Unlike the Laplace Transform and others, the Polynomial Integral Transform solves differential equations with a little computational effort as well as time. In addition, the Polynomial Integral Transform entails a polynomial function as its kernel, which ensures the rapid convergence of the solution to a differential equation. Thus, this method transforms a linear differential equation into an algebraic equation, from which the solution is obtained. Moreover, we show the applicabilities of the Polynomial Integral Transform and its properties.

2010 Mathematics Subject Classifications: 44B24; 44B21

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1. Introduction

Most of the problems encountered in science, engineering and physics involve rates of change. Initial condition of the dependent variable is usually measured at a point. Finding solutions to these problems are often either difficult or not feasible at all. There are many approaches to search for solution to the differential equation. Special substitution techniques have been adopted in finding solution to differential equation with variable coefficients. The Cauchy-Euler method transforms a linear differential equation into an algebraic equation with the use of appropriate substitution technique. Other methods such as methods of undetermined coefficients, variation of parameters are limited in usage, see [8]. In addition, these classical methods for search of solutions to the differential equations are tedious and cumbersome as one has to look for the appropriate substitution expression. Thus, there is no single substitution expression for a single type of differential equation.

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Currently, integral transform method is the concern of mathematicians and scientists in general. Since the introduction of the Laplace integral transform, a number of integral transforms have been proposed for solving differential equations. An alternative integral transform, Laplace substitution method, for the construction of solutions of the partial differential equations was observed by [11].

The Sumudu integral transform

$$F(u) = s[f(t);u] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)dt, \;\; u \in (-\tau, \tau).$$

was proposed by [15] and applied to some controlled problems in engineering. This method faces the similar challenges as the Laplace integral transform. Thus, Sumudu integral transform solves a linear differential equation with constant coefficients. The author in [4], observed some properties of the Sumudu integral transform. Several studies have made use of the Sumudu integral transform to obtain the solutions of the differential equations. For example, see research papers by [5, 6, 9]. Recently, in [10], they compared both the Laplace integral transform and Sumudu integral transforms.

In [12], the authors introduced the Natural integral transform and applied it to obtain solutions of the differential equations. Unlike the Laplace and Sumudu integral transforms, the Natural integral transform entails

$$K(u, v, x) = e^{-\frac{vx}{u}},$$

where u and v are parameters, as its kernel, which transforms a linear differential equation into an algebraic equation. The duality of both the Natural and Laplace integral transforms was studied by [7]. Using the Natural integral transform, [2] sought the solution of differential equation on the spaces of generalized functions. Also, in [1], the author extended the applications of the Hartley transform of differential equation on the space of the generalized functions. In order to ensure the rapid convergence of solution of the differential equation, the Fresnel integral transform with variables in the Boehmains space has been obtained. For example, see a research paper by [3].

On the contrary, the integral transform method for the fractional difference equation has been obtained. The authors in [14], implemented the S-transforms for solving such problems in engineering.

We outline of this paper is as follows. In section 1, we give the introduction to integral transform methods. In this section, we discuss the integral transform methods for solving differential equations. In section 2, we present the definition and also, give the proof for the Polynomial Integral Transform. Using the Polynomial Integral Transform, we show that the solution of the differential equation converges for $x \in [1, \infty)$. The properties of the Polynomial Integral Transform is contained in section 3. In section 4, we apply the Polynomial Integral Transform to derivatives, some ordinary differential equations and partial differential equation. Section 5 contains the conclusion of this paper.

2. The Polynomial Integral Transform

In the previous section, we observed that the discussed integral transforms are either prototypes or have almost the same applicabilities as the Laplace integral transform. In addition, almost all of these integral transforms use exponential function of parameter(s) as their kernels. Using the exponential function kernel does not only requires complex mathematical structures but also takes a long time before the solution is obtained. One of the interesting papers which has drawn much attention in the 21^{st} century is the one given by [13]. Using the Mellin-Barnes integrals poses the similar challenges as the Laplace integral transform and its prototypes.

An integral transform which uses polynomial function as its kernel requires a few time for computation as well as the convergence of the solution of the differential equation. In this paper, we introduce a Polynomial Integral Transform to solve differential equations in Hilbert space. This method entails a polynomial function as its kernel to transform the differential equation into the algebraic equation. The algebraic equation is then solved to obtain the solution of the differential equation. We state the Polynomial Integral Transform theorem for the ordinary differential equation.

Theorem 1 (A Polynomial Integral Transform). *Let $f(x)$ be a function defined for $x \geq 1$. Then the integral*

$$B(f(x)) = F(s) = \int_1^\infty f(\ln x).x^{-s-1}dx,$$

is the Polynomial Integral Transform of $f(x)$ for $x \in [1, \infty)$, provided the integral converges.

Proof. We consider the homogeneous Cauchy-Euler equation of the form:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = 0.$$

with the corresponding distinct roots

$$y(x) = c_1 e^{s_1 \ln x} + c_2 e^{s_2 \ln x} + \ldots + c_n e^{s_n \ln x},$$

where $c_1, c_2, \ldots, c_n$ are constants. Also, consider a constant linear differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0,$$

with a solution

$$y(x) = c_1 e^{s_1 t} + c_2 e^{s_1 t} + \ldots + c_n e^{s_n t}.$$

We can see that equation (3) has an integral transform

$$B(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Again, we see from equations (2) and (4) that

$$t = \ln x$$

Substituting equation (6) into equation (5), we obtain

$$\begin{aligned} B(f(x)) = &F(s) = \int_1^\infty f(\ln x).\frac{1}{x}e^{-s\ln x}dx \\ B(f(x)) = &\int_1^\infty f(\ln x).x^{-(s+1)}dx \end{aligned}$$

is the polynomial integral transform of $f(x)$ for $x \in [1, \infty)$, provided the integral converges.
□

2.1. The Convergence of the Polynomial Integral Transform

In this subsection, we show that the Polynomial Integral Transform converges for variable defined in $[1, \infty)$. By Taylor series expansion, we obtain

$$\begin{aligned} e^{\ln x^{-s-1}} =& 1 + \ln x^{-s-1} + \frac{(\ln x^{-(s+1)})^2}{2!} + \frac{(\ln x^{-(s+1)})^3}{3!} \\ &+ \frac{(\ln x^{-(s+1)})^4}{4!} + \ldots + \sum_{n=0}^{\infty}\frac{(\ln x^{-(s+1)})^n}{n!} + \ldots \\ e^{\ln x^{-s-1}} =& \sum_{n=0}^{\infty}\frac{(\ln x^{-(s+1)})^n}{n!} \end{aligned}$$

By the D'Lambert Ratio test, we obtain

$$\begin{aligned} &\lim_{n\to\infty}|(\sum_{n=0}^{\infty}\frac{(\ln x^{-(s+1)})^{n+1}}{(n+1)!} \div \sum_{n=0}^{\infty}\frac{(\ln x^{-(s+1)})^n}{n!})| \\ &\Rightarrow 0.\ln x^{-(s+1)} \\ &\Rightarrow 0. \end{aligned}$$

Then

$$\begin{aligned} B(f(x)) =& \sup_{1\le x<\infty}\int_1^\infty |f(\ln x).x^{-s-1}|dx \\ B(f(x)) \le& \sup_{1\le x<\infty}\int_1^\infty |f(\ln x)||x^{-s-1}|dx \\ B(f(x)) \le& M\int_1^\infty |f(\ln x)|dx, \text{ where } M > 0. \end{aligned}$$

It implies that the polynomial integral transform converges uniformly for a given s. The function $f(x)$ must be piecewise continuous. Thus, $f(x)$ has at most a finite number of discontinuities on any interval $1 \le x \le A$, and the limit of $f(x)$ exist at every point of discontinuity.

2.2. Existence of the Polynomial Integral Transform

In this subsection, we show that the Polynomial Integral Transform exists for $x \in [1, \infty)$. To see this, we state the existence theorem for the Polynomial Integral Transform.

Theorem 2. *Let $f(x)$ be a piecewise continuous function on $[1, \infty)$ and of exponential order, then the polynomial integral transform exists.*

Proof. By the definition of polynomial integral transform, we obtain

$$\begin{aligned}
I =& \int_1^\infty f(x).x^{-(s+1)}dx \\
I =& \int_1^A f(x).x^{-(s+1)}dx + \int_A^\infty f(x).x^{-(s+1)}dx \\
I =& I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \int_1^A f(x).x^{-(s+1)}dx$$

and

$$I_2 = \int_A^\infty f(x).x^{-(s+1)}dx.$$

The integral I_1 exists since $f(x)$ is piecewise continuous. Taking

$$\begin{aligned}
I_2 =& \int_A^\infty f(x).x^{-(s+1)}dx \\
I_2 =& \int_A^\infty f(x).x^{-(s+1)}dx \leq M \int_A^\infty e^{\alpha x}.x^{-(s+1)}dx.
\end{aligned}$$

By the Taylor series expansion, we obtain

$$e^{\alpha x} \approx \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{n!}.$$

Substituting the expression for $e^{\alpha x}$ in equation (7), we obtain

$$\begin{aligned}
I_2 \approx& M \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_A^\infty x^{-(s+1-n)}dx \\
I_2 \approx& M \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_1^\infty x^{-(s+1-n)}dx \\
I_2 =& M \sum_{n=0}^{\infty} \frac{M\alpha^n}{n!(s-n)}, \quad s > n
\end{aligned}$$

$$\int_1^\infty f(x).x^{-(s+1)}dx = M\sum_{n=0}^{\infty}\frac{M\alpha^n}{n!(s-n)}, \quad s>n$$

This completes the proof. □

3. Properties of the Polynomial Integral Transform

In this section, we give the properties of the Polynomial Integral Transform.

Theorem 3. *The Polynomial Integral Transform is a linear operator.*

Proof. Suppose that $f(x)$ and $g(x)$ are functions and α_1 and α_2 are real constants.

$$\begin{aligned}
B(\alpha_1 f(x)+\alpha_2 g(x)) =& \int_1^\infty (\alpha_1 f(\ln x)+\alpha_2 g(\ln x)).x^{-s-1}dx \\
B(\alpha_1 f(x)+\alpha_2 g(x)) =& \alpha_1\int_1^\infty f(\ln x).x^{-s-1}dx+\alpha_2\int_1^\infty g(\ln x).x^{-s-1}dx \\
B(\alpha_1 f(x)+\alpha_2 g(x)) =& \alpha_1 B(f(x))+\alpha_2 B(g(x))
\end{aligned}$$

□

Theorem 4. *The Inverse Polynomial Integral Transform is a also linear operator.*

Proof. Taking the inverse integral transform of the both sides of the above equation, we obtain

$$\begin{aligned}
\alpha_1 f(x)+\alpha_2 g(x) =& B^{-1}(\alpha_1(f(x))+\alpha_2 B(g(x))) \\
\alpha_1 f(x)+\alpha_2 g(x) =& \alpha_1 B^{-1}((f(x)))+\alpha_2 B^{-1}(L(g(x))) \\
\alpha_1 f(x)+\alpha_2 g(x) =& \alpha_1 B^{-1}(F(s))+\alpha_2 B^{-1}(G(s)) \\
\alpha_1 f(x)+\alpha_2 g(x) =& B^{-1}(\alpha_1 F(s)+\alpha_2 G(s)),
\end{aligned}$$

where $B(f(x))=F(s)$ and $B(g(x))=G(s)$, respectively. □

Theorem 5 (First shifting theorem)**.** *If $B(f(x))=B(s)$, then $B(e^{ax}f(x))=B(s-a)$, for $s>1$.*

Proof. Let

$$\begin{aligned}
B(e^{ax}f(x)) =& \int_1^\infty e^{a\ln x}f(\ln x).x^{-s-1}dx \\
B(e^{ax}f(x)) =& \int_1^\infty x^a f(\ln x).x^{-s-1}dx \\
B(e^{ax}f(x)) =& \int_1^\infty f(\ln x).x^{-(s-a+1)}dx
\end{aligned}$$

$$B(e^{ax}f(x)) = B(s-a).$$

□

Theorem 6 (Second shifting theorem). *Let*

$$H_c(x) = \begin{cases} 0 & 0 \le x < c \\ 1 & x \ge c \end{cases}$$

be a unit step function. Then

$$B(H_c f(x-c)) = F(s-c)$$

Proof. By applying the Polynomial Integral Transform, we obtain

$$B(H_c(x)f(x-c)) = \int_1^\infty H_c(\ln x) f(\ln(x-c)).x^{-s-1}dx$$
$$B(H_c(x)f(x-c)) = \lim_{t\to\infty}\int_1^t 1.f(\ln(x-c)).x^{-s-1}dx$$
$$B(H_c(x)f(x-c)) = \lim_{t\to\infty}\int_1^t f(\ln(x-c)).x^{-s-1}dx$$

We set

$$u = x - c$$

and substituting u into right hand side of the above equation, we obtain

$$B(H_c(x)f(x-c)) = \lim_{t\to\infty}\int_{1-c}^{t-c} f(\ln u).(u+c)^{-s-1}du$$
$$B(H_c(x)f(x-c)) = \lim_{t\to\infty}\int_1^t f(\ln(v-c)).v^{-s-1}dv$$
$$B(H_c(x)f(x-c)) = F(s-c),$$

where $v = u + c$. □

Theorem 7. *If $f(x)$ is a piecewise continuous function on $[0,\infty)$, but not of exponential order, then a polynomial integral transform*

$$B(f(x)) \to 0 \text{ as } s \to \infty.$$

Proof. Let

$$|B(f(x))| = |\int_1^\infty f(\ln x)x^{-s-1}dx|$$

$$|B(f(x))| \leq \int_1^\infty |f(\ln x)x^{-s-1}|dx$$
$$|B(f(x))| = \int_1^\infty f(\ln x)|x^{-(s+1)}|dx.$$

But, we observe that:

$$|x^{-(s+1)}| \to 0 \text{ as } s \to \infty.$$

It follows that

$$B(f(x)) \to 0 \text{ as } s \to \infty.$$

□

4. The Polynomial Integral Transform of Derivatives

In this section, we give the Polynomial Integral Transform of derivatives of the function $f(x)$ with respect to x.

Theorem 8. *If $f, f', \ldots f^{n-1}$ are continuous on $[1, \infty)$ and if $f^n(x)$ is piecewise continuous on $[1, \infty)$, then*

$$B(f^{(n)}(x)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0),$$

where $F(s) = B(f(x))$.

Proof. Let

$$B(f'(x)) = \int_1^\infty f'(\ln x).x^{-s-1}dx$$

Using integration by parts, we obtain

$$L(f'(x)) = \lim_{t\to\infty} [f(\ln x)x^{-s}]_1^t + \lim_{t\to\infty} \int_1^t f(\ln x)\frac{1}{x}x^{-s}dx$$
$$L(f'(x)) = sF(s) - f(0).$$

Proceeding a similar as above, we obtain

$$B(f''(x)) = \int_1^\infty f''(\ln x).x^{-s-1}dx$$
$$B(f''(x)) = -f'(0) + sL(f'(x))$$

Substituting the expression of $f'(x)$ into the above equation, we obtain

$$B(f''(x)) = s^2 F(s) - sf(0) - f'(0).$$

By induction, we obtain

$$B(f^{(n)}(x)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0),$$

where $F(s) = B(f(x))$. □

Corollary 1. *Suppose f is a piecewise function and let $F(s)$ by the Polynomial Integral Transform by equation (4). Then, we obtain*

$$L(x^n f(x))(s) = (-1)^n F^{(n)}(s)$$

Proof. By applying the Polynomial Integral Transform, we obtain

$$\begin{aligned}
F'(s) =& \frac{d}{ds}\int_1^\infty f(\ln x).x^{-s-1}dx \\
F'(s) =& \int_1^\infty f(\ln x).x^{-1}\frac{\partial}{\partial s}e^{\ln x^{-s}}dx \\
F'(s) =& \int_1^\infty f(\ln x).x^{-1}\frac{\partial}{\partial s}e^{-s\ln x}dx \\
F'(s) =& -\int_1^\infty \ln x f(\ln x).x^{-s-1}dx \\
-F'(s) =& L(xf(x))
\end{aligned}$$

Proceeding in a similar manner, we obtain

$$\begin{aligned}
F''(s) =& \frac{d}{ds}\int_1^\infty \ln x f(\ln x).x^{-s-1}dx \\
F''(s) =& L(x^2 f(x)) \\
& \vdots \\
(-1)^n F^{(n)}(s) =& L(x^n f(x))(s),
\end{aligned}$$

where $n = 1, 2, \ldots$. □

4.1. Applications of Polynomial Integral Transform to Linear Ordinary Differential Equation with Constant Coefficients

We apply the Polynomial Integral Transform to obtain the solutions of the ordinary differential equations as follows:

Example 1.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y(x) = 0, \quad y(0) = 0, \quad y'(0) = -7$$

Using the polynomial integral transform, we obtain

$$\begin{aligned}
B(y''(x) - y'(x) - 6y(x)) =& B(0) \\
y(x) =& B^{-1}\left(\frac{-1}{(s-3)} + \frac{2}{(s+2)}\right) \\
y(x) =& -e^{3x} + 2e^{-2x}.
\end{aligned}$$

Example 2.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y(x) = x, \quad y(0) = 0, \quad y'(0) = 1$$

Applying polynomial integral transform to the above equation, we obtain

$$\begin{aligned} B(y''(x) - 3y'(x) + 2y(x)) &= B(x) \\ y(x) =& B^{-1}\left(\frac{3}{4}.\frac{1}{s} + \frac{1}{4}.\frac{1}{s^2} - 2.\frac{1}{(s-1)} + \frac{5}{4}.\frac{1}{(s-2)}\right) \\ y(x) =& \frac{3}{4} + \frac{1}{2}x - 2e^x + \frac{5}{4}e^{2x} \end{aligned}$$

4.2. A Polynomial Integral Transform in Two Variables

In order to obtain analytic solution of the partial differential equations PDEs, we extend the polynomial integral transform to solve the functions in two dimensions as below:

Theorem 9. *Let f be a function defined for $x, t \geq 1$. Then the integral*

$$B_x B_t(f(x,t);(p,s)) = F(p,s) = \int_1^\infty \int_1^\infty f(\ln x, \ln t).x^{-p-1}t^{-s-1}dxdt,$$

is the integral transform of $f(x,t)$ for $x, t \in [1, \infty)$, provided the integral converges.

Proof. It follows from Theorem 1. □

We then apply Polynomial Integral Transform to transform partial derivatives. By the definition of the Polynomial Integral Transform in two variables, we obtain following results:

$$\begin{aligned} B_x B_t\left(\frac{\partial f(x,t)}{\partial x};(p,s)\right) =& pF(p,s) - F(0,s) \\ B_x B_t\left(\frac{\partial^2 f(x,t)}{\partial x \partial t};(p,s)\right) =& psF(p,s) - pF(p,0) - sF(0,s) + f(0,0) \\ B_x B_t\left(\frac{\partial^2 f(x,t)}{\partial x^2};(p,s)\right) =& p^2F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x} \\ B_x B_t\left(\frac{\partial^2 f(x,t)}{\partial t^2};(p,s)\right) =& s^2F(p,s) - sF(p,0) - \frac{\partial f(p,0)}{\partial t} \end{aligned}$$

We consider a wave equation below:

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} =& \frac{\partial^2 w}{\partial x^2} \quad x, t \geq 0 \\ w(0,t) =& g(t), \quad \lim_{x \to \infty} w(x,t) = 0, \quad x, t \geq 0 \\ w(x,0) =& \frac{\partial w}{\partial t}(x,0) = 0, \end{aligned}$$

where w is the deflection of a string released from rest on the x-axis. Applying the polynomial integral transform in two variables, we obtain

$$\begin{aligned} B(w_{tt}) =& B(w_{xx}) \\ \Rightarrow W_{xx} - s^2 W =& 0 \\ W(x,s) =& A(s)e^{sx} + B(s)e^{-sx} \\ W(x,s) =& G(s)e^{-sx} \\ w(x,t) =& B^{-1}(W(x,s)) \\ w(x,t) =& xu(t-1)g(t-1), \end{aligned}$$

where $g(t) = B^{-1}(G(s))$.

5. Conclusion

We observed that the Polynomial Integral Transform solves differential equation with a few computations as well as time. Unlike the Laplace Integral Transform and others, the Polynomial Integral Transform involves a polynomial function as its kernel, which is easier and transforms complicated functions into algebraic equations. The solution of the differential equation is then obtained from the algebraic equation. Also, using the Polynomial Integral Transform, the convergence of the solution of the differential equation is faster as compared with the Laplace integral transform and others. We observed that the Polynomial Integral Transform is defined on the interval $[1, \infty)$.

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