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A Note on Primary and Weakly Primary Submodules

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Abstract. In this paper, we generalize primary submodules and weakly primary submodules which are called $P(N)$-locally primary submodules and $P(N)$-locally weakly primary submodules, respectively. Some properties of these generalizations of submodules are investigated.

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1. Introduction

The two generalization on prime submodules and weakly prime submodules, which are called $S(N)$-locally prime submodule and $S(N)$-weakly prime submodule, respectively where $S(N) = \{r \in R \mid rm \in N \text{ for some } m \notin N\}$ for a submodule N of an R-module M, have been broadly studied by A. K. Jabbar (see [2]). In this paper, our aim is to obtain the two generalization on primary submodules and weakly primary submodules of an R-module M.

Throughout this paper, we assume that all rings are commutative with identity $1 \neq 0$. An ideal I of R is called a proper ideal if $I \neq R$. Then the radical of a proper ideal I of R is denoted by $\sqrt{I}$ and $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some positive integer } n\}$. A proper ideal P of R is called prime (primary) if $ab \in P$ for some $a, b \in R$ implies that either $a \in P$ or $b \in P$ (either $a \in P$ or $b^n \in P$ for some positive integer n). A proper ideal P of R is said to be a weakly prime ideal if $0 \neq ab \in P$ for some $a, b \in R$ implies that either $a \in P$ or $b \in P$, and it is a weakly primary ideal if $0 \neq ab \in P$ for some $a, b \in R$ implies that either $a \in P$ or $b^n \in P$ for some positive integer n (see [3], [4]).

Let M be an R-module. A submodule N of M is called a proper submodule if $N \neq M$. A proper submodule N of M is called a prime submodule if $rm \in N$ for some $r \in R$ and $m \in M$ implies that either $m \in N$ or $rM \subseteq N$ and it is said to be a weakly prime submodule if $0 \neq rm \in N$ for some $r \in R$ and $m \in M$ implies that either $m \in N$ or $rM \subseteq N$. A subset S of R is said to be multiplicative closed set if $\emptyset \neq S$, $0 \notin S$ and

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whenever $a, b \in S$, then $ab \in S$. Let S be a multiplicative closed set in R. Then an R_S-module M_S is gotten under the operations $\frac{a}{s} + \frac{b}{u} = \frac{ua+sb}{su}$ and $\frac{r}{v}\frac{a}{s} = \frac{ra}{vs}$ for any $\frac{r}{v} \in R_S$ and $\frac{a}{s}, \frac{b}{u} \in M_S$ [5]. A proper submodule N of M is said to be $S(N)$-locally prime ($S(N)$-weakly prime) submodule if N_P is a prime (a weakly prime) submodule of M_P for each maximal ideal P with $S(N) \subseteq P$ [2].

In this paper, we study two generalizations of the primary submodules and the weakly primary submodules of M. A proper submodule N of M is said to be a primary submodule if $rm \in N$ for some $r \in R$, $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n and it is said to be a weakly primary submodule if $0 \neq rm \in N$ for some $r \in R$, $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n. The ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $(N : M)$ and $(0 : N) = \{r \in R \mid rN = 0\}$ where N is a submodule of M. Then the annihilator of M is $(0 : M)$ where $(0 : M) = \{r \in R \mid rM = 0\}$. An R-module M is called a faithful module if $(0 : M) = (0)$. It is known that if N is a primary submodule of M, then $(N : M)$ is a primary ideal of R and $\sqrt{(N : M)} = \{r \in R \mid r^n M \subseteq N \text{ for some positive integer } n\}$ is a prime ideal of R ([1], [7], [6]).

Now, we introduce the concepts that we will use. Let N be a proper submodule of M. An element $r \in R$ is said to be primary to N if $r^n m \in N$, where $m \in M$ and n is a positive integer, then $m \in N$. Then $r \in R$ is said to be not primary to N if $r^n m \in N$ for some positive integer n and for some $m \notin N$. Let us denote the set of all elements of R that are not primary to N by $P(N)$. Then we get $P(N) = \{r \in R \mid r^n m \in N$ for some positive integer n, for some element $m \notin N\}$. If $N = (0)$, then $P((0)) = \{r \in R \mid r^n m = 0$ for some positive integer n, for some $0 \neq m \in M\}$. A proper submodule N of M is said to be a P-primal if $P(N)$ forms an ideal of R.

2. $P(N)$-Locally Primary and $P(N)$-Locally Weakly Primary Submodules

Definition 1. *Let N be a proper submodule of an R-module M. Then N is called a $P(N)$-locally primary submodule of M if N_P is a primary submodule of M_P for all maximal ideal P where $P(N) \subseteq P$.*

Definition 2. *A proper submodule N of an R-module M is called a $P(N)$-locally weakly primary submodule of M if N_P is a weakly primary submodule of M_P for every maximal ideal P where $P(N) \subseteq P$.*

Lemma 1. *Let N be a proper submodule of an R-module M. Then $\sqrt{(N : M)} \subseteq P(N)$.*

Proof. Let $r \in \sqrt{(N : M)}$. Then $r^n M \subseteq N$ for some positive integer n. There exists $m \notin N$ such that $r^n m \in N$. Then $r \in P(N)$. Thus $\sqrt{(N : M)} \subseteq P(N)$.

The following propositions state that every primary submodule N is $P(N)$-locally primary submodule and every weakly primary submodule N is $P(N)$-locally weakly primary submodule.

Proposition 1. *A primary submodule N of an R-module M is a $P(N)$-locally primary submodule.*

Proof. Let P be a maximal ideal of R where $P(N) \subseteq P$. By the previous lemma, we say that $\sqrt{(N:M)} \subseteq P(N) \subseteq P$. Since $\sqrt{(N:M)} \cap (R\backslash P) = \emptyset$, then N_P is a primary submodule of M_P. Consequently, N is a $P(N)$-locally primary submodule.

Proposition 2. *A weakly primary submodule N of an R-module M is a $P(N)$-locally weakly primary submodule.*

Proof. Suppose that P is a maximal ideal of R where $P(N) \subseteq P$. From [2, Corollary 2.2], if $N \neq M$, then $N_P \neq M_P$, that is, N_P is a proper submodule of M_P. Let $0_P \neq \frac{r}{s}\frac{m}{p} \in N_P$ for some $\frac{r}{s} \in R_P$ and $\frac{m}{p} \in M_P$ (for some $r \in R,\ m \in M$ and $s, p \notin P$). Then there is a $q \notin P$ such that $qrm \in N$. Assume that $qrm = 0$. Then $\frac{r}{s}\frac{m}{p} = \frac{q}{q}\frac{r}{s}\frac{m}{p} = \frac{qrm}{qsp} = 0_P$, this is a contradiction. So $0 \neq qrm \in N$. As $\sqrt{(N:M)} \subseteq P(N) \subseteq P$, then $q \notin \sqrt{(N:M)}$. Thus $rm \in N$ since N is a weakly primary submodule.

It is clear that $rm \neq 0$. Hence $0 \neq rm \in N$ implies that $m \in N$ or $r^n M \subseteq N$ for some positive integer n. Thus we get $\frac{m}{p} \in N_P$ or $\frac{r^n}{s^n} M_P \subseteq N_P$ for some positive integer n by [2, Corollary 2.9]. Then we get that N_P is a weakly primary submodule of M_P. Consequently, N is a $P(N)$-locally weakly primary submodule.

Corollary 1. *Let N be a proper submodule of an R-module M. If N is a primary submodule, then N is a $P(N)$-locally weakly primary submodule.*

Proof. Assume that N is a primary submodule. Then N is a weakly primary submodule. Thus, N is a $P(N)$-locally weakly primary submodule by Proposition 2.

Note that if N is a $P(N)$-locally primary submodule of M, then N is a $P(N)$-locally weakly primary submodule of M.

Now, we give an example to show the converse is not true.

Example 1. *Consider the $\mathbb{Z}$-module $\mathbb{Z}_8$ and $N = (\overline{0})$. It is clear that $(\overline{0})$ is always weakly primary submodule but not a primary submodule. By Proposition 2, $(\overline{0})$ is a $P(\overline{0})$-locally weakly primary submodule. It is easily seen that $P(\overline{0}) = \{r \in \mathbb{Z} \mid r^n\overline{m} = \overline{0}$, for some positive integer n, for some $\overline{0} \neq \overline{m} \in \mathbb{Z}_8\} \subseteq (2) = P$. Now, we show that $(\overline{0})_P$ is not a primary submodule of $(\mathbb{Z}_8)_P$. It is clear that $\frac{2}{p}\frac{\overline{4}}{q} \in (\overline{0})_P$ for some $p, q \notin P$. Then there is an $u \notin P$ with $u2\overline{4} \in (0)$. Thus $2u \notin ((\overline{0}) : \mathbb{Z}_8)$ and $\overline{4} \notin (\overline{0})$. Then $\frac{2}{p} = \frac{2u}{pu} \notin ((\overline{0}) : \mathbb{Z}_8)_P \subseteq ((\overline{0})_P : (\mathbb{Z}_8)_P)$ and $\frac{\overline{4}}{q} \notin (\overline{0})_P$. Therefore, $(\overline{0})$ is not a $P(\overline{0})$-locally primary submodule.*

In the following example, we get that a submodule N is both $P(N)$-locally primary submodule of M and $P(N)$-locally weakly primary submodule of M but neither primary submodule of M nor weakly primary submodule of M.

Example 2. *Consider $R = \mathbb{Z}$-module $M = \mathbb{Z}_{12}$. Let N be the submodule of $\mathbb{Z}_{12}$ generated by $\overline{6}$. It is easly seen that $\overline{0} \neq 2\overline{3}(= 3\overline{2}) \in N$ but $2 \notin (N : M)$ and $\overline{3} \notin N$ ($3 \notin (N : M)$ and $\overline{2} \notin N$), that is, N is not a weakly primary submodule of M, hence N is not a primary submodule of M. Assume that N is not a $P(N)$-locally primary submodule of M. Then there exists a maximal ideal P of R with $P(N) \subseteq P$ where N_p is not a primary submodule of M_P. Note that $2, 3 \in P(N)$. Thus $1 \in P$, a contradiction. Therefore, N is a $P(N)$-locally primary submodule of M. Hence N is a $P(N)$-locally weakly primary submodule of M.*

Theorem 1. *Let N be a proper submodule of an R-module M. Then the following statements hold:*

1) N is a primary submodule if and only if $P(N) = \sqrt{(N : M)}$.

2) Let $P(0) \subseteq \sqrt{(N : M)}$. Then N is a primary submodule if and only if N is a weakly primary submodule.

Proof. *1)* $(\Longrightarrow)$: Assume that N is a primary submodule. Let $r \in P(N)$. Then $r^n m \in N$ for some positive integer n and for some $m \notin N$. Since N is a primary submodule, then $(r^n)^k M = r^{nk} M \subseteq N$ for some positive integer k, that is, $r \in \sqrt{(N : M)}$. Hence $P(N) \subseteq \sqrt{(N : M)}$. By Lemma 1, we get $P(N) = \sqrt{(N : M)}$.

$(\Longleftarrow)$: Suppose that $P(N) = \sqrt{(N : M)}$. Let $rm \in N$ and $m \notin N$ where $r \in R$, $m \in M$. Then $r \in P(N)$. Thus $r \in \sqrt{(N : M)}$, that is, $r^k M \subseteq N$ for some positive integer k. Consequently, N is a primary submodule.

2) $(\Longrightarrow)$: Clear.

$(\Longleftarrow)$: Assume that N is a weakly primary submodule. Let $r \in P(N)$. Then $r^n m \in N$ for some positive integer n and for some $m \notin N$. Suppose that $r^n m = 0$. Since $m \notin N$, then we get $m \neq 0$. So $r \in P(0)$. Thus $r \in \sqrt{(N : M)}$, by assumption. Hence $P(N) = \sqrt{(N : M)}$ by Lemma 1. Suppose that $0 \neq r^n m \in N$. Since $m \notin N$ and N is a weakly primary submodule, then $(r^n)^k M \subseteq N$ for some positive integer k, that is, $r \in \sqrt{(N : M)}$ and so $P(N) = \sqrt{(N : M)}$. By (1), N is a primary submodule.

Corollary 2. *Let N be a proper submodule of an R-module M with $P(N) = \sqrt{(N : M)}$. Then N is a $P(N)$-locally primary submodule and $P(N)$-locally weakly primary submodule.*

Proof. We get that N is a primary submodule by Theorem 1 (1). Then N is a $P(N)$-locally primary submodule by Proposition 1. Since N is primary submodule, then N is weakly primary submodule. Therefore, N is $P(N)$-locally weakly primary submodule by Proposition 2.

By [2, Lemma 2.19], if P is a maximal ideal of R, then $(N : M)_P \subseteq (N_P : M_P)$. Now, we explain that $\sqrt{(N : M)_P} = \sqrt{(N_P : M_P)}$ when P is a maximal ideal of R with $P(N) \subseteq P$.

Proposition 3. *Let N be a proper submodule of an R-module M. Then $\sqrt{(N : M)_P} = \sqrt{(N_P : M_P)}$ for a maximal ideal P of R with $P(N) \subseteq P$.*

Proof. It is clear from [2, Lemma 2.19 and Lemma 2.20] since $S(N) \subseteq P(N)$ for some proper submodule N of M.

Lemma 2. *Let N be a proper submodule of an R-module M. Then $\sqrt{(N:M)_P} = (\sqrt{(N:M)})_P$ for any maximal ideal P of R with $P(N) \subseteq P$.*

Proof. Let $\frac{r}{p} \in \sqrt{(N:M)_P}$ for some $r \in R$ and $p \notin P$. Then $(\frac{r}{p})^n = \frac{r^n}{p^n} \in (N:M)_P$ for some positive integer n. There is a $q \notin P$ such that $qr^n \in (N:M)$, that is, $qr^n m \in N$ for every $m \in M$. Then $r^n m \in N$ for every $m \in M$ since $q \notin P(N)$. Thus $r \in \sqrt{(N:M)}$. Then $\frac{r}{p} \in (\sqrt{(N:M)})_P$. Conversely, assume that $\frac{r}{p} \in (\sqrt{(N:M)})_P(N:M)_P$ and so $\frac{r}{p} \in \sqrt{(N:M)_P}$.

Corollary 3. *Let N be a proper submodule of an R-module. If P is any maximal ideal of R with $P(N) \subseteq P$, then $(\sqrt{(N:M)})_P = \sqrt{(N_P:M_P)}$.*

Proof. It is clear from Proposition 3 and Lemma 2.

Proposition 4. *Let N be a proper submodule of an R-module M and $m \in M$. Then $\sqrt{(N:Rm)_P} = \sqrt{(N_P:(Rm)_P)}$ for a maximal ideal P of R with $P(N) \subseteq P$.*

Proof. It is clear from [5, Lemma 9.12].

If we put $N = 0$ in Proposition 4, we have the following corollary.

Corollary 4. *Let M be an R-module and $m \in M$. Then $\sqrt{(0:Rm)_P} = \sqrt{(0_P:(Rm)_P)}$ for a maximal ideal P of R with $P(0) \subseteq P$.*

Proposition 5. *Let N be a proper submodule of an R-module M and P be a maximal ideal of R with $P(N) \subseteq P$. Then the following statements hold:*

1) Let $P(0) \subseteq P(N)$. Then $\sqrt{(N:M)}$ is a weakly prime ideal of R if and only if $\sqrt{(N:M)_P}$ is a weakly prime ideal of R_P.

2) $\sqrt{(N:M)}$ is a prime ideal of R if and only if $\sqrt{(N:M)_P}$ is a prime ideal of R_P.

Proof. 1) $(\Longrightarrow)$: Suppose that $\sqrt{(N:M)}$ is a weakly prime ideal of R. If $\sqrt{(N:M)_P} = R_P$, then $\frac{1}{1} \in \sqrt{(N:M)_P} = (\sqrt{(N:M)})_P$ and so $q1 = q \in \sqrt{(N:M)}$ for some $q \notin P$. But by Lemma 1, $\sqrt{(N:M)} \subseteq P(N) \subseteq P$, which is a contradiction. So $\sqrt{(N:M)_P} \neq R_P$, that is, $\sqrt{(N:M)_P}$ is a proper ideal of R_P. Let $0 \neq \frac{r}{p}\frac{s}{q} \in \sqrt{(N:M)_P}$, where $r, s \in R$ and $p, q \notin P$. Then we have $\frac{r}{p}\frac{s}{q} = \frac{rs}{pq} \in (\sqrt{(N:M)})_P$, then there exists an $u \notin P$ such that $urs \in \sqrt{(N:M)}$. If $urs = 0$, then $\frac{r}{p}\frac{s}{q} = \frac{u}{u}\frac{r}{p}\frac{s}{q} = \frac{urs}{upq} = 0$, this is a contradiction. So $urs \neq 0$. Since $0 \neq urs \in \sqrt{(N:M)}$ and $\sqrt{(N:M)}$ is a weakly prime ideal of R, then $ur \in \sqrt{(N:M)}$ or $s \in \sqrt{(N:M)}$. Hence $\frac{r}{p} = \frac{u}{u}\frac{r}{p} \in (\sqrt{(N:M)})_P$ or $\frac{s}{q} \in (\sqrt{(N:M)})_P$, that is, $\frac{r}{p} \in \sqrt{(N:M)_P}$ or $\frac{s}{q} \in \sqrt{(N:M)_P}$.

$(\Longleftarrow)$: Assume that $\sqrt{(N:M)_P}$ is a weakly prime ideal of R_P. If $\sqrt{(N:M)} = R$, then $\sqrt{(N:M)_P} = R_P$, a contradiction. So $\sqrt{(N:M)}$ is a proper ideal of R. Let $0 \neq ab \in \sqrt{(N:M)}$ for some $a, b \in R$. Then $\frac{ab}{1} = \frac{a}{1}\frac{b}{1} \in \sqrt{(N:M)_P}$. If $\frac{a}{1}\frac{b}{1} = 0$, then $qab = 0$ for some $q \notin P$. As $0 \neq ab$, then $q \in P(0)$. Thus $q \in P$, which is a contradiction. So $0 \neq \frac{a}{1}\frac{b}{1} \in \sqrt{(N:M)_P}$. Since $\sqrt{(N:M)_P}$ is a weakly prime ideal of R_P, then $\frac{a}{1} \in \sqrt{(N:M)_P}$ or $\frac{b}{1} \in \sqrt{(N:M)_P}$. Therefore $pa \in \sqrt{(N:M)}$ for some $p \notin P$ or $sb \in \sqrt{(N:M)}$ for some $s \notin P$. As $p \notin P$ and $s \notin P$, then $p, s \notin P(N)$. Consequently, $a \in \sqrt{(N:M)}$ or $b \in \sqrt{(N:M)}$.

2) $(\Longrightarrow)$: Assume that $\sqrt{(N:M)}$ is a prime ideal of R. In a similar way, we get $\sqrt{(N:M)_P}$ is a proper ideal of R_P. Now, let $\frac{r}{p}\frac{s}{q} \in \sqrt{(N:M)_P}$, where $r, s \in R$ and $p, q \notin P$. Then we have $\frac{rs}{pq} \in (\sqrt{(N:M)})_P$, then $urs \in \sqrt{(N:M)}$ for some $u \notin P$. Since $\sqrt{(N:M)}$ is a prime ideal of R, then $ur \in \sqrt{(N:M)}$ or $s \in \sqrt{(N:M)}$. Consequently, $\frac{r}{p} = \frac{u}{u}\frac{r}{p} \in (\sqrt{(N:M)})_P$ or $\frac{s}{q} \in (\sqrt{(N:M)})_P$, that is, $\frac{r}{p} \in \sqrt{(N:M)_P}$ or $\frac{s}{q} \in \sqrt{(N:M)_P}$.

$(\Longleftarrow)$: Suppose that $\sqrt{(N:M)_P}$ is a prime ideal of R_P. From (1), it is clear that $\sqrt{(N:M)}$ is a proper ideal of R. Then $\frac{ab}{1} = \frac{a}{1}\frac{b}{1} \in \sqrt{(N:M)_P}$ for some $a, b \in R$ and since $\sqrt{(N:M)_P}$ is a prime ideal of R_P, then $\frac{a}{1} \in \sqrt{(N:M)_P}$ or $\frac{b}{1} \in \sqrt{(N:M)_P}$. Thus $pa \in \sqrt{(N:M)}$ for some $p \notin P$ or $sb \in \sqrt{(N:M)}$ for some $s \notin P$. As $p \notin P$ and $s \notin P$, then $p, s \notin P(N)$. Therefore, $a \in \sqrt{(N:M)}$ or $b \in \sqrt{(N:M)}$.

Proposition 6. *Let M be a faithful cyclic R-module and N be a proper submodule of M with $P(0) \subseteq P(N)$. If N is a $P(N)$-locally weakly primary submodule of M, then $\sqrt{(N:M)}$ is a weakly prime ideal of R.*

Proof. Let P be a maximal ideal of R with $P(N) \subseteq P$. By [2, Proposition 2.18], M_P is a faithful cyclic R_P-module. Then N_P is a weakly primary submodule of M_P. Thus by [1, Proposition 2.3], $\sqrt{(N_P : M_P)}$ is a weakly prime submodule of M_P. By Proposition 3, $\sqrt{(N:M)_P}$ is a weakly prime submodule of M_P. By Proposition 5 (1), $\sqrt{(N:M)}$ is a weakly prime ideal of R.

Proposition 7. *Let M be an R-module. Let N be a P-primal and a $P(N)$-locally weakly primary submodule of M not primary submodule of M. If $P(0) \subseteq P(N)$ and I is an ideal of R such that $I \subseteq \sqrt{(N:M)}$, then $IN = 0$. Particularly, $\sqrt{(N:M)}N = 0$.*

Proof. Suppose that $P(0) \subseteq P(N)$ and I is an ideal of R such that $I \subseteq \sqrt{(N:M)}$. Since N is a P-primal, then $P(N)$ is an ideal of R. As $1 \notin P(N)$, then $P(N)$ is a proper ideal. Hence there is a maximal ideal P of R such that $P(N) \subseteq P$. Then, N_P is a weakly primary submodule of M_P because N is a $P(N)$-locally weakly primary submodule of M. Our aim is to show that N_P is not a primary submodule of M_P. Assume that N_P is a primary submodule of M_P. Let $rm \in N$ for some $r \in R$, $m \in M$. Then $\frac{rm}{1} = \frac{r}{1}\frac{m}{1} \in N_P$. By assumption, $\frac{m}{1} \in N_P$ or $(\frac{r}{1})^n M_P \subseteq N_P$ for some positive integer n. By using a similar technique in the previous proofs, $m \in N$ or $r^n M \subseteq N$ for some positive integer n since

$P(N) \subseteq P$, but this contradicts with N which is not a primary submodule of M. By [2, Lemma 2.19], $I_P \subseteq \sqrt{(N:M)_P} \subseteq \sqrt{(N_P:M_P)}$. By [1, Corollary 3.4], $I_P N_P = 0$. We get $\frac{r}{1}\frac{m}{1} = \frac{rm}{1} = 0$ for every $r \in I$ and every $m \in N$. Therefore $qrm = 0$ for some $q \notin P$. If $rm \neq 0$, then $q \in P(0)$ and so $q \in P$, which is a contradiction. Hence $rm = 0$, that is, $IN = 0$. Particularly, by putting $I = \sqrt{(N:M)}$, we have $\sqrt{(N:M)}N = 0$.

Proposition 8. *([2, Proposition 2.16]) Let M be an R-module and P be a maximal ideal of R. If $\overline{I}$ is an ideal of R_P and $\overline{N}$ is a submodule of M_P, then*
1) $I = \{a \in R \mid \frac{a}{1} \in \overline{I}\}$ is an ideal of R and $\overline{I} = I_P$.
2) $N = \{m \in M \mid \frac{m}{1} \in \overline{N}\}$ is a submodule of M and $\overline{N} = N_P$.

Theorem 2. *Let N be a P-primal submodule of an R-module M with $P(0) \subseteq P(N)$. Then N is a $P(N)$-locally weakly primary submodule of M if and only if $0 \neq ID \subseteq N$ for some ideal I of R and some submodule D of M implies $I \subseteq \sqrt{(N:M)}$ or $D \subseteq N$.*

Proof. $(\Longrightarrow)$: Assume that N is a $P(N)$-locally weakly primary submodule of M. Let $0 \neq ID \subseteq N$ for some ideal I of R and some submodule D of M. Since N is P-primal, then $P(N)$ is an ideal of R. As $1 \notin P(N)$, then $P(N)$ is a proper ideal. So we have $P(N) \subseteq P$ for some maximal ideal P of R. Thus N_P is a weakly primary submodule of M_P. Now, I_P is an ideal of R_P and D_P is a submodule of M_P with $(ID)_P = I_P D_P \subseteq N_P$. Suppose that $I_P D_P = 0_P$. Then $\frac{r}{1}\frac{m}{1} = \frac{rm}{1} = 0$ for every $r \in I$ and every $m \in D$. So there exists a $q \notin P$ such that $qrm = 0$. If $rm \neq 0$, then $q \in P(0)$. Thus $q \in P$, which is a contradiction. So $rm = 0$, hence $ID = 0$, that is a contradiction. Then $0_P \neq I_P D_P \subseteq N_P$. Since N is a $P(N)$-locally weakly primary submodule of M, then N_P is a weakly primary submodule of M_P. By [1, Theorem 3.6], either $I_P \subseteq \sqrt{(N_P:M_P)}$ or $D_P \subseteq N_P$. Since $P(N) \subseteq P$, then $I \subseteq \sqrt{(N:M)}$ or $D \subseteq N$.

$(\Longleftarrow)$: Let P be a maximal ideal of R with $P(N) \subseteq P$. Since N is a proper ideal of R, then there is an $a \notin N$, but $\frac{a}{1} \in M_P$. If $\frac{a}{1} \in N_P$, then $qa \in N$ such that $q \notin P$. As $a \notin N$, then $q \in P(N)$, that is, $q \in P$, which is a contradiction. So $\frac{a}{1} \notin N_P$. Hence N_P is a proper ideal of R_P. Let $\overline{I}$ be an ideal of R_P and $\overline{D}$ be a submodule of M_P with $0_P \neq \overline{ID} \subseteq N_P$. By [2, Proposition 2.16], $\overline{I} = I_P$, for some ideal I of R and $\overline{D} = D_P$, for some submodule D of M. So $0_P \neq I_P D_P \subseteq N_P$, that is, $0_P \neq (ID)_P \subseteq N_P$. Since $P(N) \subseteq P$, then $ID \subseteq N$. Also $0 \neq ID$. On the contrary, $(ID)_P = 0_P$. By the hypothesis, we have either $I \subseteq \sqrt{(N:M)}$ or $D \subseteq N$. If $I \subseteq \sqrt{(N:M)}$, then $\overline{I} = I_P \subseteq \sqrt{(N:M)_P}$. If $D \subseteq N$, then $\overline{D} = D_P \subseteq N_P$. From [1, Theorem 3.6], N_P is a weakly primary submodule of M_P. Therefore, N is a $P(N)$-locally weakly primary submodule of M.

Corollary 5. *Let N be a P-primal submodule of an R-module M with $P(0) \subseteq P(N)$. Then N is a $P(N)$-locally weakly primary submodule of M if and only if N is a weakly primary submodule of M.*

Proof. It is clear from Theorem 2 and [1, Theorem 3.6].

Theorem 3. *Let M be an R-module and N be a P-primal submodule of M with $P(0) \subseteq P(N)$. Then the following statements are equivalent:*

1) N is a $P(N)$-locally weakly primary submodule of M.

2) For any $m \notin N$, $\sqrt{(N:Rm)} = \sqrt{(N:M)} \cup (0:Rm)$.

3) For any $m \notin N$, $\sqrt{(N:Rm)} = \sqrt{(N:M)}$ or $\sqrt{(N:Rm)} = (0:Rm)$.

Proof. (*1*$\Longrightarrow$*2*): Let N be a $P(N)$-locally weakly primary submodule of M and let $m \notin N$. Since N is P-primal, then $P(N)$ is an ideal of R. As $1 \notin P(N)$, then $P(N)$ is a proper ideal. So we have $P(N) \subseteq P$ for some maximal ideal P of R. Hence N_P is a weakly primary submodule of M_P. As $m \in M$, then $\frac{m}{1} \in M_P$, but $\frac{m}{1} \notin N_P$. If $\frac{m}{1} \in N_P$, then $pm \in N$ for some $p \notin P$. Since $p \notin P(N)$, then $m \in N$, this is a contradiction. By [3, Theorem 2.15], $\sqrt{(N_P : R_P\frac{m}{1})} = \sqrt{(N_P : M_P)} \cup (0_P : R_P\frac{m}{1})$ and from [2, Corollary 2.9], $\sqrt{(N_P : (Rm)_P)} = \sqrt{(N_P : M_P)} \cup (0_P : (Rm)_P)$. Then by Proposition 3, Proposition 4 and Corollary 4, $\sqrt{(N:Rm)_P} = \sqrt{(N:M)_P} \cup (0:Rm)_P$. Let $r \in \sqrt{(N:Rm)}$. Then $\frac{r}{1} \in \sqrt{(N:Rm)_P}$ and so $\frac{r}{1} \in \sqrt{(N:M)_P}$ or $\frac{r}{1} \in (0:Rm)_P$. If $\frac{r}{1} \in \sqrt{(N:M)_P}$, then $\frac{r^n}{1} \in (N:M)_P$ for some positive integer n and thus $qr^n \in (N:M)$ for some $q \notin P$, that is, $qr^nM \subseteq N$. Assume that $r^nM \nsubseteq N$. Then $r^nm \notin N$ for some $m \in M$, however $qr^nm \in N$. Hence $q \in P(N)$. Then $q \in P$, which is a contradiction. So $r^nM \subseteq N$ for some positive integer n, that is, $r \in \sqrt{(N:M)}$. If $\frac{r}{1} \in (0:Rm)_P$, then $pr \in (0:Rm)$ for some $p \notin P$. Thus $prRm = 0$. Assume that $rRm \neq 0$. Then $rsm \neq 0$ for some $s \in R$, but $prsm = 0$. Therefore $p \in P(0)$. As $P(0) \subseteq P$, then $p \in P$, which is a contradiction. So $rRm = 0$. Then $r \in (0:Rm)$. Hence $r \in \sqrt{(N:M)} \cup (0:Rm)$. Conversely, let $r \in \sqrt{(N:M)} \cup (0:Rm)$. If $r \in \sqrt{(N:M)}$, then $r^nM \subseteq N$ for some positive integer n and so we get $r^nRm \subseteq r^nM \subseteq N$. Thus $r \in \sqrt{(N:Rm)}$. If $r \in (0:Rm)$, then $rRm = 0 \subseteq N$. Thus $r \in (N:Rm) \subseteq \sqrt{(N:Rm)}$.

(*2*$\Rightarrow$*3*): Clear.

(*3*$\Rightarrow$*1*): Let P be a maximal ideal of R with $P(N) \subseteq P$. Let $\frac{m}{p} \notin N_P$ where $m \in M$, $p \notin P$. Then $m \notin N$. By the condition of the theorem, $\sqrt{(N:Rm)} = \sqrt{(N:M)}$ or $\sqrt{(N:Rm)} = (0:Rm)$ for some $m \notin N$. If $\sqrt{(N:Rm)} = \sqrt{(N:M)}$, then $\sqrt{(N:Rm)_P} = \sqrt{(N:M)_P}$ and from Proposition 3 and Proposition 4 $\sqrt{(N_P:(Rm)_P)} = \sqrt{(N_P:M_P)}$. By [2, Proposition 2.8] , $\sqrt{(N_P : R_P\frac{m}{p})} = \sqrt{(N_P:M_P)}$. If $\sqrt{(N:Rm)} = (0:Rm)$, then $\sqrt{(N:Rm)_P} = (0:Rm)_P$ and by Proposition 4 and Corollary 4, $\sqrt{(N_P:(Rm)_P)} = (0:(Rm)_P)$. By [2, Proposition 2.8], $\sqrt{(N_P : R_P\frac{m}{p})} = (0 : R_P\frac{m}{p})$. By [3, Theorem 2.15], N_P is a weakly primary submodule of M_P. Thus N is a $P(N)$-locally weakly primary submodule of M.

Theorem 4. *Let M be an R-module and N be a P-primal submodule of M with $P(0) \subseteq P(N)$. Then the following statements are equivalent:*

1) N is a $P(N)$-locally weakly primary submodule of M.

2) $0 \neq ID \subseteq N$ for any ideal I of R and any submodule D of M implies either $I \subseteq \sqrt{(N:M)}$ or $D \subseteq N$.

3) $\sqrt{(N:Rm)} = \sqrt{(N:M)} \cup (0:Rm)$ for any $m \notin N$.

4) $\sqrt{(N:Rm)} = \sqrt{(N:M)}$ *or* $\sqrt{(N:Rm)} = (0:Rm)$ *for any* $m \notin N$.

Proof. It is clear from Theorem 2 and Theorem 3.

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