EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 10, No. 4, 2017, 638-644
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Coefficient estimates for the generalized subclass of analytic and bi-univalent functions

Haigen Xiao[1], Qinghua Xu[2,*]

[1] *The Binjiang Campus of the High School Attached to Jiangxi Normal University, China*
[2] *School of Science, Zhejiang University of Science and Technology, China*

Abstract. In this paper, we introduce and investigate an interesting subclass $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$, obtain estimates on the first two coefficients $|a_2|$ and $|a_3|$. The results presented in this paper generalize and improve some recent works of Frasin et al. [B.A.Frasin, M.K.Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. 24:1569-1573, 2011] and Srivastava et al. [Qing-Hua Xu, Ying-Chun Gui, H.M.Srivastava, coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. 25: 990-994, 2012].

2010 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Univalent functions, Bi-univalent functions, Coefficient bounds

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of the analytic function class $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

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*Corresponding author.

Email addresses: haigen2008@sina.com (H.G Xiao), xuqh@mail.ustc.edu.cn (Q.H.Xu)

and

$$f^{-1}(f(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given (1).

The coefficient bounds for the class Σ have been studied by Lewin [1], Brannan and Clunie [2], Netanyahu [3]. The coefficient estimate problem for $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} := \{1, 2, 3, \cdots\})$ is presumably still an open problem. In [4](see [5, 6, 7]), certain subclasses of the bi-univalent function class Σ were introduced, and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ were found.

Recently, Frasin et al.[8] introduced the following subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$.

Definition 1(see [8]). A function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \textit{and} \quad \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| \leq \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha \leq 1;\ \lambda \geq 1)$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right| \leq \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U};\ 0 < \alpha \leq 1;\ \lambda \geq 1),$$

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

Theorem 1(see [8]). Let $f(z)$ given by (1) be in the function class $\mathcal{B}_\Sigma(\alpha, \lambda)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}.$$

Definition 2(see [8]). A function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \textit{and} \quad \Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta \qquad (z \in \mathbb{U};\ 0 \leq \beta < 1; \lambda \geq 1)$$

and

$$\Re\left((1-\lambda)\frac{g(w)}{z} + \lambda g'(w)\right) > \beta \qquad (w \in \mathbb{U};\ 0 \leq \beta < 1;\ \lambda \geq 1),$$

where the function g is defined by (2).

Theorem 2(see [8]). Let $f(z)$ given by (1) be in the function class $\mathcal{B}_\Sigma(\beta,\lambda)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2\lambda+1}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{2\lambda+1}.$$

Here, in our present sequel to some of the aforecited works (especially [7, 8]), we introduce the following subclass of analytic functions.

Definition 3. Let $h,\ p : \mathbb{U} \to \mathbb{C}$ be functions such that

$$\min\{\Re(h(z)),\ \Re(p(z))\} > 0,\ (z \in \mathbb{U}) \quad \textit{and} \quad h(0) = p(0) = 1,$$

Also let f be an analytic function in $\mathbb{U}$ defined by (1). We say that $f \in \mathcal{B}_\Sigma^{h,p}(\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \textit{and} \quad (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \in h(\mathbb{U}) \quad (z \in \mathbb{U};\ \ \lambda \geq 1) \tag{3}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \in p(\mathbb{U}) \quad (w \in \mathbb{U};\ \ \lambda \geq 1), \tag{4}$$

where the function g is given by (2).

We note that for $\lambda = 1$, the class $\mathcal{B}_\Sigma^{h,p}(\lambda)$ reduces to the class $\mathcal{H}_\Sigma^{h,p}$ introduced and studied by Xu et al.[7].

Remark 1. There are many choices of the functions h and p which would provide interesting subclasses of analytic functions. For example, if we let

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad (z \in \mathbb{U};\quad 0 < \alpha \leq 1)$$

or

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathbb{U};\ 0 \leq \beta < 1),$$

it is easy to verify that $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{B}_\Sigma^{h,p}(\lambda)$, then

$$f \in \Sigma \quad \textit{and} \quad \left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right| \leq \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leq 1;\ \lambda \geq 1)$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\right)\right| \leq \frac{\alpha\pi}{2} \quad (w \in \mathbb{U};\ 0 < \alpha \leq 1;\ \lambda \geq 1),$$

or

$$f \in \Sigma \quad and \quad \Re\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right) > \beta \quad (z \in \mathbb{U};\ 0 \leq \beta < 1)$$

and

$$\Re\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\right) > \beta \quad (w \in \mathbb{U};\ 0 \leq \beta < 1;\ 0 \leq \beta < 1),$$

where the function g is given by (2).
This means that

$$f \in \mathcal{B}_\Sigma(\alpha, \lambda) \qquad \text{or} \qquad f \in \mathcal{B}_\Sigma(\beta, \lambda).$$

In this paper, stimulated by [7, 8], we introduce the following subclass of the bi-univalent function class Σ and obtain estimates on the first two coefficients $|a_2|$ and $|a_3|$. Our results would generalize and improve the related works of Frasin et al.[8] and Xu et al.[7] .

2. Main results and their proofs

In this section, we state and prove our results involving the bi-univalent function class $\mathcal{B}_\Sigma^{h,p}(\lambda)$ given by Definition 3.

Theorem 3. Let $f(z)$ given by (1) be in the function class $f \in \mathcal{B}_\Sigma^{h,p}(\lambda)$. Then

$$|a_2| \leq \sqrt{\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda)}} \quad and \quad |a_3| \leq \frac{|h''(0)|}{2(1+2\lambda)}. \tag{5}$$

Proof. It follows from (3) and (4) that

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z) = h(z) \quad (z \in \mathbb{U}) \tag{6}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w) = p(w) \quad (w \in \mathbb{U}), \tag{7}$$

where h and p satisfy the conditions of Definition 3, Furthermore, the functions $h(z)$ and $p(w)$ have the following series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots ,$$

respectively. Now, equating the coefficients in (6) and (7), we get

$$(1+\lambda)a_2 = h_1, \tag{8}$$

$$(1+2\lambda)a_3 = h_2, \tag{9}$$

$$-(1+\lambda)a_2 = p_1 \tag{10}$$

and

$$(1+2\lambda)(2a_2^2 - a_3) = p_2. \tag{11}$$

From (8) and (10), we get

$$h_1 = -p_1 \qquad 2(1+\lambda)^2 a_2^2 = h_1^2 + p_1^2. \tag{12}$$

Also, from (9) and (11), we find that

$$2(1+2\lambda)a_2^2 = h_2 + p_2, \tag{13}$$

which gives us the desired estimate on $|a_2|$ as asserted in (5).

Next, in order to find the bound on $|a_3|$, by subtracting (11) from (9), we get

$$2(1+2\lambda)a_3 - 2(1+2\lambda)a_2^2 = h_2 - p_2. \tag{14}$$

Upon substituting the value of a_2^2 from (13)into (14), it follows that

$$a_3 = \frac{h_2}{1+2\lambda}, \tag{15}$$

as claimed. This completes the proof of Theorem 1.

3. Corollaries and consequences

In view of Remark 1, if we set

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}; \quad 0 < \alpha \leq 1)$$

and

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathbb{U}; \quad 0 \leq \beta < 1)$$

in Theorem 3, respectively, we can readily deduce the following two corollaries, which we merely state here without proof.

Corollary 1. Let $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{B}_\Sigma(\alpha,\lambda)$. Then

$$|a_2| \le \sqrt{\frac{2}{2\lambda+1}}\alpha \qquad \textit{and} \qquad |a_3| \le \frac{2\alpha^2}{2\lambda+1}. \tag{16}$$

Remark 2. It is easy to prove that

$$\sqrt{\frac{2}{1+2\lambda}}\alpha \le \frac{2\alpha}{\sqrt{(\lambda+1)^2+\alpha(1+2\lambda-\lambda^2)}} \qquad (0<\alpha\le 1;\ \lambda\ge 1)$$

and

$$\frac{2\alpha^2}{1+2\lambda} \le \frac{4\alpha^2}{(\lambda+1)^2}+\frac{2\alpha}{2\lambda+1} \qquad (0<\alpha\le 1;\ \lambda\ge 1),$$

which, in conjunction with Corollary 1, would obviously yield an improvement of Theorem 1.

Corollary 2. Let $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{B}_\Sigma(\beta,\lambda)$. Then

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{2\lambda+1}} \quad \textit{and} \quad |a_3| \le \frac{2(1-\beta)}{2\lambda+1}. \tag{17}$$

Remark 3. It is obvious that

$$\frac{2(1-\beta)}{2\lambda+1} \le \frac{4(1-\beta)^2}{(\lambda+1)^2}+\frac{2(1-\beta)}{2\lambda+1} \qquad (0\le\beta<1;\ \lambda\ge 1),$$

which, in conjunction with Corollary 2, would lead us to an improvement of Theorem 2.

Setting $\lambda=1$ in Theorem 3, we get the following estimate, which was obtained by Xu et al. [7].

Corollary 3 (see [7]). Let $f(z)$ given by (1) be in the bi-univalent function class $\mathcal{H}_\Sigma^{h,p}$. Then

$$|a_2| \le \sqrt{\frac{|h''(0)|+|p''(0)|}{12}} \quad \textit{and} \quad |a_3| \le \frac{|h''(0)|}{6}. \tag{18}$$

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