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Proximity between selfadjoint operators and between their associated spectral measures

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Abstract. We study how the proximity between two selfadjoint bounded operators can be expressed as a proximity between the associated spectral measures. Between two operators, we use a classical distance. For projector-valued spectral measures, we introduce the notion of $\alpha-$equivalence, which is based on a partial order relation on the set of projectors. Assuming an hypothesis of commutativity, we show that the proximity between operators is equivalent with the proximity between the associated spectral measures. We develop the particular case where the operators are compact, and give some illustrations.

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1. Introduction

The question of the association of a spectral measure (s.m.) with an operator is a usefull technique, and often considered for the analysis of their properties ([4], [3]). Therefore, if P is an orthogonal projector, and λ a real, then $A = \lambda P$ is a selfadjoint bounded operator, which associated s.m. is $\mathcal{E} = \delta_0 P_\perp + \delta_\lambda P$. Besides, the null operator O is associated with the s.m. $\mathcal{E}_\mathbb{R} = \delta_0 I$.

As $\|A - O\| = |\lambda|$, A and O are two selfadjoint operators as close as we want, as far as we can get $|\lambda|$ as small as we want. Nevertheless, let us consider the proximity between their associated s.m.'s. For any B of a $\sigma-$field defined on $\mathbb{R}$, we have

$$\mathcal{E}(B) - \mathcal{E}_\mathbb{R}(B) = \delta_0(B)P_\perp + \delta_\lambda(B)P - \delta_0(B)P_\perp - \delta_0(B)P = \delta_\lambda(B)P - \delta_0(B)P.$$

So $\|\mathcal{E}(B) - \mathcal{E}_\mathbb{R}(B)\| = |\delta_\lambda(B) - \delta_0(B)|$, which maximum is obviously equal to 1.

This shows that the proximity between two s.m.'s, evaluated by $\sup\{\|\mathcal{E}(B) - \mathcal{E}_\mathbb{R}(B)\|; B \in \mathcal{B}_\mathbb{R}\}$, is not appropriate to be linked with the proximity between their associated operators.

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In [2], we have seen how the proximity between unitary operators could be equivalent with proximity between their associated s.m.'s.

In this paper, we develop tools for the study of the association between selfadjoint bounded operators and s.m.'s. Our study of proximity is placed in this context.

In order to fix notation, Section 2 is devoted to the recall of some notions, as random measure, spectral measure, space of the measurable applications of integrable square with respect to a random measure, and projector-valued spectral measure. We introduce in a third section the notion of $\alpha-$equivalence, for α positive real, between two s.m.'s. This concept of proximity can be translated by proximity between two selfadjoint bounded operators, what we will develop in Section 4. The fift section will be dedicated to the case of the compact operators, which is frequently encountered in numerical applications. A numerical illustration is given in the last section.

2. Recalls

2.1. Orthogonal families of projectors

In this text, H, $\mathcal{L}(H)$ and $\mathcal{P}(H)$ are respectively a $\mathbb{C}-$Hilbert space, the set of bounded endomorphisms of H (which is a Banach space for the norm $\|A\|_{\mathcal{L}} = \sup\{\|Ax\|; \|x\| = 1\}$), and the set of the orthogonal projectors on H.

Let P_1 and P_2 be two elements of $\mathcal{P}(H)$, P_1 *is said to be "less or equal" to* P_2, what we denote $P_1 \ll P_2$, if $P_1P_2 = P_1$.

This defines, on $\mathcal{P}(H)$, a partial order relation.

A *family* $\{P_n; n \in \mathbb{N}\}$ of elements of $\mathcal{P}(H)$ is said to be *orthogonal* when $P_n \circ P_m = 0$, for any pair (n, m) of distinct elements of $\mathbb{N}$. Then we can show that
a) for any X of H, the family $\{P_nX; n \in \mathbb{N}\}$ is summable;
b) the application $P : X \in H \mapsto \sum_{n\in\mathbb{N}} P_nX \in H$ is an element of $\mathcal{P}(H)$ which we name the *sum of the orthogonal family* of orthogonal projectors $\{P_n; n \in \mathbb{N}\}$ and which we denote $\sum_{n\in\mathbb{N}} P_n$;
c) for any n of $\mathbb{N}$, we have $P_n \circ P = P_n$.

We emphasize the fact that an orthogonal family of orthogonal projectors can be not summable (as a family of elements of the Banach space $\mathcal{L}(H)$).

The following first result, which can be easily proved, will be used in Section 3.

Lemma 2.1.1. *If $\{D_n; n \in \mathbb{N}\}$ and $\{D'_n; n \in \mathbb{N}\}$ are two orthogonal families of orthogonal projectors of respective sum D and D' such that $D_n \ll D'_n$, for any n of $\mathbb{N}$, then $D \ll D'$.*

2.2. Random measure

Let ξ be a $\sigma-$field of subsets of a set E. For any e of E, δ_e stands for the Dirac measure defined on ξ and concentrated on e.

A *random measure* (r.m.) defined on ξ, taking values in H, is an application Z from ξ into H such that:

a) $Z(A \cup B) = Z(A) + Z(B)$ and $< Z(A), Z(B) >= 0$, for any pair (A, B) of disjoint elements of ξ;
b) $\lim_{n\to\infty} Z(A_n) = 0$ for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of ξ which decreasingly converges to $\emptyset$.

We then have the following properties.

a) The application $\mu_Z : A \in \xi \mapsto \|Z(A)\|^2 \in \mathbb{R}_+$ is a bounded measure;
b) there exists one, and only one, isometry from $L^2(E, \xi, \mu_Z)$ onto $H_Z = \overline{\text{vect}}\{Z(A); A \in \xi\}$ such that for any A of ξ, ZA is the image of 1_A.

The *stochastic integral* of an element φ of $L^2(E, \xi, \mu_Z)$ with respect to the r.m. Z, is the image of φ by this isometry, and we note it $\int \varphi \mathrm{d}Z$.

For example, if $\{Z_n, n \in \mathbb{N}\}$ is a summable orthogonal family of elements of H, if $(e_n)_{n\in\mathbb{N}}$ is a sequence of elements of E such that $\{e_n\} \in \xi$, then, for any B of ξ, the family $\{\delta_{e_n}(B)Z_n, n \in \mathbb{N}\}$, of elements of H, is summable. The application $Z : B \in \xi \mapsto \sum_{n\in\mathbb{N}} \delta_{e_n}(B)Z_n \in H$ is a r.m.. If f is an element of $L^2(\mu_Z)$, then $\{f(e_n)Z_n, n \in \mathbb{N}\}$ is a summable family of elements of H, of sum $\int f \mathrm{d}Z$.

In the all text, (E', ξ') denotes a second measurable space. Let f be a measurable application from E into E'. We can affirm the following.

a) The application $f(Z) : A' \in \xi' \mapsto Z(f^{-1}A') \in H$ is a r.m., named *r.m. image* of Z by f;
b) $f(\mu_Z) = \mu_{f(Z)}$;
c) if φ' is a element of $L^2(E', \xi', \mu_{f(Z)})$, then $\varphi' \circ f$ belongs to $L^2(E, \xi, \mu_Z)$ and $\int \varphi' \mathrm{d}f(Z) = \int \varphi' \circ f \mathrm{d}Z$.

A stationary *continuous random function* (c.r.f.) $(X_t)_{t\in\mathbb{R}}$ is a family of elements of H such that the application $t \in \mathbb{R} \mapsto X_t \in H$ is continuous and such that $< X_t, X_{t'} >=< X_{t-t'}, X_0 >$ for any pair (t, t') of elements of $\mathbb{R}$.

There exists one, and only one r.m. Z, named *r.m. associated with the stationary c.r.f.* $(X_t)_{t\in\mathbb{R}}$, defined on $\mathcal{B}_\mathbb{R}$, such that $X_t = \int \mathrm{e}^{i.t} \mathrm{d}Z$, for any t of $\mathbb{R}$.

Then we will say that two c.r.f.'s $(X_t)_{t\in\mathbb{R}}$ and $(X'_t)_{t\in\mathbb{R}}$ are *stationarily correlated* when $< X_t, X'_{t'} >=< X_{t-t'}, X'_0 >$, for any pair (t, t') of elements of $\mathbb{R}$.

This property can be expressed in terms of associated r.m.'s: if Z and Z' are two r.m.'s associated with two stationary c.r.f.'s, stationarily correlated, then, for any pair (A, A') of disjoint elements of $\mathcal{B}_\mathbb{R}$, we have $< Z(A), Z'(A') >= 0$.

2.3. Spectral measure

A *spectral measure* (s.m.) $\mathcal{E}$ on ξ for H is an application from ξ into $\mathcal{P}(H)$ such that
a) $\mathcal{E}(E) = I_H$;
b) $\mathcal{E}(A \cup B) = \mathcal{E}(A) + \mathcal{E}(B)$, for any pair (A, B) of disjoint elements of ξ;
c) $\lim_{n\to\infty} \mathcal{E}(A_n)X = 0$, for any X of H and for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of ξ which decreasingly converges to $\emptyset$.

Then we easily check the following properties.

a) For any X of H, the application $Z_{\mathcal{E}}^X : A \in \xi \mapsto \mathcal{E}(A)X \in H$ is a r.m..

b) If f is a measurable application from E into E', the application $f(\mathcal{E}) : A' \in \xi' \mapsto \mathcal{E}(f^{-1}A') \in \mathcal{P}(H)$ is a r.m. on ξ' for H named *s.m. image* of $\mathcal{E}$ by f. For any X of H, we have $f(Z_{\mathcal{E}}^X) = Z_{f(\mathcal{E})}^X$.

Let us now examine more particularly the s.m.'s on $\mathcal{B}_{\mathbb{R}}$, the Borel $\sigma-$field of $\mathbb{R}$, for H.

Two s.m.'s $\mathcal{E}_1$ and $\mathcal{E}_2$, on $\mathcal{B}_{\mathbb{R}}$ for H, *commute* when, for any pair (A_1, A_2) of elements of $\mathcal{B}_{\mathbb{R}}$, $\mathcal{E}_1(A_1)$ and $\mathcal{E}_2(A_2)$ commute.

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two s.m.'s on $\mathcal{B}_{\mathbb{R}}$ for H, which commute, then there exists one s.m., and only one, on $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ for H, denoted $\mathcal{E}_1 \otimes \mathcal{E}_2$, such that $\mathcal{E}_1 \otimes \mathcal{E}_2(A_1 \times A_2) = \mathcal{E}_1(A_1) \circ \mathcal{E}_2(A_2)$, for any (A_1, A_2) of $\mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}}$.

We then name *convolution product* (or shortly *convolution*) of $\mathcal{E}_1$ and $\mathcal{E}_2$, and we denote it $\mathcal{E}_1 * \mathcal{E}_2$, the s.m. image of $\mathcal{E}_1 \otimes \mathcal{E}_2$ by the measurable application $\mathcal{S} : (\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R} \mapsto \lambda_1 + \lambda_2 \in \mathbb{R}$ (cf. [1]).

This convolution has got an identity element. More precisely, the application $\mathcal{E}_{\mathbb{R}} : A \in \mathcal{B}_{\mathbb{R}} \mapsto \delta_0(A) I_H \in \mathcal{P}(H)$ is a s.m. which commutes with any s.m. $\mathcal{E}$, on $\mathcal{B}_{\mathbb{R}}$ for H. Moreover, $\mathcal{E} * \mathcal{E}_{\mathbb{R}} = \mathcal{E}$.

When one of the elements of the product $\mathcal{E}_1 * \mathcal{E}_2$ is concentrated on a countable family of reals $\Lambda = \{\lambda_j; j \in \mathbb{N}\}$, we get a result which seems natural for a convolution.

If $\mathcal{E}_1$ is a s.m., on $\mathcal{B}_{\mathbb{R}}$ for H, such that $\mathcal{E}_1(\Lambda) = I_H$, and which commutes with a second s.m. $\mathcal{E}_2$, on $\mathcal{B}_{\mathbb{R}}$ for H, then, for any A of $\mathcal{B}_{\mathbb{R}}$, the set $\{\mathcal{E}_1(\{\lambda_j\}) \circ \mathcal{E}_2(A - \lambda_j); j \in \mathbb{N}\}$, is an orthogonal family of projectors which sum equals $(\mathcal{E}_1 * \mathcal{E}_2)(A)$.

Let us end these recalls with algebraic properties of the convolution.

If $\mathcal{E}$ is a s.m. on $\mathcal{B}_{\mathbb{R}}$ for H, if f_1 and f_2 are two measurable applications from $\mathbb{R}$ into itself, then the s.m.'s $f_1(\mathcal{E})$ and $f_2(\mathcal{E})$ commute and $(f_1 + f_2)(\mathcal{E}) = (f_1(\mathcal{E})) * (f_2(\mathcal{E}))$.

If we denote by w the measurable application $x \in \mathbb{R} \mapsto -x \in \mathbb{R}$, then, taking into account the previous results, for any s.m. $\mathcal{E}$ on $\mathcal{B}_{\mathbb{R}}$ for H, we can write $\mathcal{E} * (w(\mathcal{E})) = (I_{\mathbb{R}} + w)(\mathcal{E}) = O(\mathcal{E}) = \mathcal{E}_{\mathbb{R}}$. This means that any s.m. $\mathcal{E}$ has got its symmetric, the s.m. $w(\mathcal{E})$, for the convolution.

When it exists, the convolution is associative.

Finally, if $\mathcal{E}_1$ and $\mathcal{E}_2$ commute, then the s.m.'s $\mathcal{E}_1$ and $\mathcal{E}_1 * \mathcal{E}_2$ also commute.

2.4. The space $\mathcal{M}(E, \mathcal{E})$

Let $\mathcal{E}$ be a s.m. on ξ, $\sigma-$field of subsets of a set E, for H.

$\mathcal{M}(E, \mathcal{E})$ is the set of the measurable applications φ, from E into $\mathbb{C}$, such that

a) $\int |\varphi|^2 \mathrm{d}\mu_{Z_{\mathcal{E}}^X} < +\infty$, for any X of H;

b) the set of the reals $\{\int |\varphi|^2 \mathrm{d}\mu_{Z_{\mathcal{E}}^X}; \|X\| = 1\}$ is bounded.

Then, when φ is an element of $\mathcal{M}(E, \mathcal{E})$, we can consider the application $\mathcal{E}_\varphi : X \in H \mapsto \int \varphi \mathrm{d}Z_{\mathcal{E}}^X \in H$, and we have the following property.

Proposition 2.4.1. *For any φ of $\mathcal{M}(E, \mathcal{E})$, the application $\mathcal{E}_\varphi$ is linear and bounded.*

Proof. If φ is an element of $\mathcal{M}(E,\mathcal{E})$, then, for any (λ,λ',X,X') of $\mathbb{C}\times\mathbb{C}\times H\times H$, it belongs to $L^2(\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}}+\mu_{Z_{\mathcal{E}}^{X}}+\mu_{Z_{\mathcal{E}}^{X'}})$ (because $\int|\varphi|^2\mathrm{d}(\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}}+\mu_{Z_{\mathcal{E}}^{X}}+\mu_{Z_{\mathcal{E}}^{X'}}) = \int|\varphi|^2\mathrm{d}\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}}+\int|\varphi|^2\mathrm{d}\mu_{Z_{\mathcal{E}}^{X}}+\int|\varphi|^2\mathrm{d}\mu_{Z_{\mathcal{E}}^{X'}}$).

Taking into account the properties of density of the indicator functions, φ can be writen as follows.

$$\varphi=\lim_{n\to\infty}\sum_{j\in J_n}\alpha_{n,j}1_{B_{n,j}}, |J_n|<+\infty, B_{n,j}\in\mathcal{B}_{\mathbb{R}}, \text{ in } L^2(\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}}+\mu_{Z_{\mathcal{E}}^{X}}+\mu_{Z_{\mathcal{E}}^{X'}}). \quad (2.4.1)$$

As $\|.\|^2_{L^2(\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}}+\mu_{Z_{\mathcal{E}}^{X}}+\mu_{Z_{\mathcal{E}}^{X'}})}=\|.\|^2_{L^2(\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}})}+\|.\|^2_{L^2(\mu_{Z_{\mathcal{E}}^{X}})}+\|.\|^2_{L^2(\mu_{Z_{\mathcal{E}}^{X'}})}$, the equality (2.4.1) is exact in $L^2(\mu_{Z_{\mathcal{E}}^{\lambda X+\lambda' X'}})$, $L^2(\mu_{Z_{\mathcal{E}}^{X}})$, and $L^2(\mu_{Z_{\mathcal{E}}^{X'}})$. Integrating it successively with respect to the r.m.'s $Z_{\mathcal{E}}^{\lambda X+\lambda' X'}$, $Z_{\mathcal{E}}^{X}$ and $Z_{\mathcal{E}}^{X'}$, we have:

$$\mathcal{E}_\varphi(\lambda X+\lambda' X')=\lim_{n\to\infty}\sum_{j\in J_n}\alpha_{n,j}Z_{\mathcal{E}}^{\lambda X+\lambda' X'}(B_{n,j}),$$
$$\mathcal{E}_\varphi X=\lim_{n\to\infty}\sum_{j\in J_n}\alpha_{n,j}Z_{\mathcal{E}}^{X}(B_{n,j}),$$
$$\mathcal{E}_\varphi X'=\lim_{n\to\infty}\sum_{j\in J_n}\alpha_{n,j}Z_{\mathcal{E}}^{X'}(B_{n,j}).$$

Then the linearity comes from the fact that:

$$Z_{\mathcal{E}}^{\lambda X+\lambda' X'}(B_{n,j})=\mathcal{E}(B_{n,j})(\lambda X+\lambda' X')=\lambda\mathcal{E}(B_{n,j})X+\lambda'\mathcal{E}(B_{n,j})X'=\lambda Z_{\mathcal{E}}^{X}(B_{n,j})+\lambda' Z_{\mathcal{E}}^{X'}(B_{n,j}).$$

Finally, the continuity comes from the fact that, for any normed element X of H:

$$\|\mathcal{E}_\varphi X\|^2=\|\int\varphi\mathrm{d}Z_{\mathcal{E}}^{X}\|^2=\int|\varphi|^2\mathrm{d}\mu_{Z_{\mathcal{E}}^{X}}\leqslant\sup\{\int|\varphi|^2\mathrm{d}\mu_{Z_{\mathcal{E}}^{X}};\|X\|=1\}.\ \Box$$

Now it is clear that $\mathcal{M}(E,\mathcal{E})$ has got a vector space structure, as a subspace of the vector space of the measurable applications from E into $\mathbb{C}$. From the linearity of the stochastic integral, we deduce the following proposition.

Proposition 2.4.2. *The application $\varphi\in\mathcal{M}(E,\mathcal{E})\mapsto\mathcal{E}_\varphi\in\mathcal{L}(H)$ is linear.*

Let us now approach a result close to the transfert theorem.

Proposition 2.4.3. *When $\mathcal{E}$ is a s.m. on ξ, $\sigma-$field of subsets of a set E for H, and when f is a measurable application from E into E', then, for any φ of $\mathcal{M}(E',f(\mathcal{E}))$, we can affirm that $\varphi\circ f$ belongs to $\mathcal{M}(E,\mathcal{E})$. Moreover, we have $(f(\mathcal{E}))_\varphi=\mathcal{E}_{\varphi\circ f}$.*

Proof. The properties of a r.m. image allow us, for any X of H, to obtain:

$$\int|\varphi\circ f|^2\mathrm{d}\mu_{Z_{\mathcal{E}}^{X}}=\int|\varphi|^2\circ f\mathrm{d}\mu_{Z_{\mathcal{E}}^{X}}=\int|\varphi|^2\mathrm{d}f(\mu_{Z_{\mathcal{E}}^{X}})=\int|\varphi|^2\mathrm{d}\mu_{Z_{f(\mathcal{E})}^{X}},$$

so $\varphi\circ f$ belongs to $\mathcal{M}(E,\mathcal{E})$, because φ is an element of $\mathcal{M}(E',f(\mathcal{E}))$.

Moreover,

$$(f(\mathcal{E}))_\varphi X=\int\varphi\mathrm{d}Z_{f(\mathcal{E})}^{X}=\int\varphi\mathrm{d}f(Z_{\mathcal{E}}^{X})=\int\varphi\circ f\mathrm{d}Z_{\mathcal{E}}^{X}=(\mathcal{E}_{\varphi\circ f})(X),$$

for any X of H, what ends the proof. $\Box$

Let us now introduce a new notion.

Definition 2.4.1. *We say that a s.m. $\mathcal{E}$, on $\mathcal{B}_{\mathbb{R}}$ for H, is bounded when there exists a real $a>0$ such that $\mathcal{E}([-a,a[)=I_H$.*

This characteristic is stable by convolution as follows.

Lemma 2.4.1. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two bounded r.m.'s, on $\mathcal{B}_{\mathbb{R}}$ for H, which commute, then $\mathcal{E}_1*\mathcal{E}_2$ is also bounded.*

Proof. Because $\mathcal{E}_1$ and $\mathcal{E}_2$ are bounded, there exists two strictly positive reals a_1 and a_2 such that $I_H = \mathcal{E}_1([-a_1, a_1[) = \mathcal{E}_2([-a_2, a_2[)$. If we denote $a = a_1 + a_2$, as $[-a_1, a_1[\times[-a_2, a_2[\subset S^{-1}[-a, a[$, we can write

$$I_H = \mathcal{E}_1 \otimes \mathcal{E}_2([-a_1, a_1[\times[-a_2, a_2[) \ll \mathcal{E}_1 \otimes \mathcal{E}_2(S^{-1}[-a, a[) = \mathcal{E}_1 * \mathcal{E}_2([-a, a[) \leqslant I_H,$$

then $I_H = \mathcal{E}_1 * \mathcal{E}_2([-a, a[)$, what ends the proof. □

Let us end this section by results which we will use later.

For any n of $\mathbb{N}$, it is clear that the application $j^n : \lambda \in \mathbb{R} \mapsto \lambda^n \in \mathbb{C}$ is measurable and, when μ is a bounded measure defined on $\mathcal{B}_{\mathbb{R}}$ having a compact support, then $\text{vect}\{j^n, n \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$. So, when $\mathcal{E}$ is a bounded s.m. on $\mathcal{B}_{\mathbb{R}}$ for H, then $\overline{\text{vect}}\{j^n, n \in \mathbb{N}\} = L^2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_{Z_{\mathcal{E}}^X})$, this for any X of H. Moreover, j^n belongs to $\mathcal{M}(\mathbb{R}, \mathcal{E})$, so we can consider the applications $\mathcal{E}_{j^n}$.

Lemma 2.4.2. *If $\{D_p; p \in \mathbb{N}\}$ is an orthogonal family of projectors of sum I, if $(\mu_p)_{p\in\mathbb{N}^*}$ is a real sequence which decreasingly strictly converges to 0 and if we set $\mu_0 = 0$, then we can affirm that*

a) for any B of $\mathcal{B}_{\mathbb{R}}$, $\{\delta_{\mu_p}(B)D_p; p \in \mathbb{N}\}$ is an orthogonal family of projectors;

b) the application $\mathcal{E} : B \in \mathcal{B}_{\mathbb{R}} \mapsto \sum_{p\in\mathbb{N}} \delta_{\mu_p}(B)D_p \in \mathcal{P}(H)$ is a bounded s.m.;

c) $\{\mu_p D_p; p \in \mathbb{N}\}$ is a family of elements of $\mathcal{L}(H)$, summable of sum $\mathcal{E}_j$.

Proof. For any B of $\mathcal{B}_{\mathbb{R}}$, it is clear that $\{\delta_{\mu_p}(B)D_p; p \in \mathbb{N}\}$ is an orthogonal family of projectors. If we denote by $\mathcal{E}(B)$ its sum, for any X of H, $\{\delta_{\mu_p}(B)D_pX; p \in \mathbb{N}\}$ is a summable family of sum $(\mathcal{E}(B))X$. As the set of the indexes is $\mathbb{N}$, we can write

$$(\mathcal{E}(B))X = \lim_n \sum_{p=0}^{n} \delta_{\mu_p}(B)D_pX,$$

and then

$$\|(\mathcal{E}(B))X\|^2 = \lim_n \|\sum_{p=0}^{n} \delta_{\mu_p}(B)D_pX\|^2 = \lim_n \sum_{p=0}^{n} \delta_{\mu_p}(B)\|D_pX\|^2$$
$$= \sum_{p\in\mathbb{N}} \delta_{\mu_p}(B)\|D_pX\|^2. \qquad (2.4.2)$$

Of course, $\mathcal{E}(\mathbb{R}) = I$ (because $\sum_{p\in\mathbb{N}} D_p = I$). Moreover, if (B_1, B_2) is a pair of disjoint elements of $\mathcal{B}_{\mathbb{R}}$, we have

$$\mathcal{E}(B_1 \cup B_2) = \mathcal{E}(B_1) + \mathcal{E}(B_2),$$

because for any p of $\mathbb{N}$ we have:

$$\delta_{\mu_p}(B_1 \cup B_2)D_pX = \delta_{\mu_p}(B_1)D_pX + \delta_{\mu_p}(B_2)D_pX.$$

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_{\mathbb{R}}$ which decreasingly converges to $\emptyset$ and X an element of H. In order to prove that $\mathcal{E}$ is a s.m., it remains to be proved that $\lim_n(\mathcal{E}(B_n))X = 0$.

For this, let us first recall that, if $(a_p)_{p\in\mathbb{N}}$ is a sequence of elements of $\mathbb{R}_+$ such that $\sum_{p\in\mathbb{N}} a_p < +\infty$, if $\{f_{n,p}; (n, p) \in \mathbb{N} \times \mathbb{N}\}$ is a family of positive reals such that, for any p of $\mathbb{N}$, $\lim_n f_{n,p} = 0$, and if $f_{n,p} < a_p$ for any (n, p) of $\mathbb{N} \times \mathbb{N}$, then on one side, for any n of $\mathbb{N}$, the family $\{f_{n,p}; p \in \mathbb{N}\}$ is summable, and on another side, $\lim_n \sum_{p\in\mathbb{N}} f_{n,p} = 0$.

Now let us apply this result to $a_p = \|D_pX\|^2$ and $f_{n,p} = \delta_{\mu_p}(B_n)\|D_pX\|^2$, for any (n, p) of $\mathbb{N} \times \mathbb{N}$. It comes

$$\lim_n \sum_{p\in\mathbb{N}} \delta_{\mu_p}(B_n)\|D_pX\|^2 = 0,$$

or, taking into account the result of (2.4.2),

$$\lim_n \|(\mathcal{E}(B_n))X\|^2 = 0,$$

so

$$\lim_n(\mathcal{E}(B_n))X = 0.$$

So $\mathcal{E}$ is a s.m., and is clearly bounded (because, if we choose a such that $\{\mu_p; p \in \mathbb{N}\} \subset [-a, a[$, then $\mathcal{E}(\complement[-a, a[) = 0)$. When X is an element of H, from recalls of Section 2.2, as $\{D_p X; p \in \mathbb{N}\}$ is a summable orthogonal family of elements of H, the family $\{j(\mu_p)D_p X; p \in \mathbb{N}\}$, and then the family $\{\mu_p D_p X; p \in \mathbb{N}\}$, is summable of sum $\int j \mathrm{d}Z^X_{\mathcal{E}}$, and hence of sum $\mathcal{E}_j(X)$. As the set of indices is $\mathbb{N}$, we can write

$$\mathcal{E}_j(X) = \lim_{p\to+\infty} \sum_{k=0}^{p} \mu_k D_k X.$$

Let us consider ε an element of $\mathbb{R}^*_+$. There exists an integer N_ε such that $\mu_{N_\varepsilon} \leqslant \varepsilon$ (because $\lim_n \mu_n = 0$). For any finite part J of $\mathbb{N}$, disjoint of $\{0, 1, \ldots, N_\varepsilon - 1, N_\varepsilon\}$, we can write

$$\| \sum_{p\in J} \mu_p D_p\| \leqslant \max\{\mu_p; p \in J\} < \mu_{N_\varepsilon} \leqslant \varepsilon.$$

This allows us to affirm that $\{\mu_p D_p; p \in \mathbb{N}\}$ is a summable family of elements of $\mathcal{L}(H)$. As the set of indices of this summable family is $\mathbb{N}$, it comes

$$(\sum_{p\in\mathbb{N}} \mu_p D_p)X = (\lim_p \sum_{k=0}^{p} \mu_p D_p)X = \lim_p \sum_{k=0}^{p} \mu_p D_p X = \mathcal{E}_j(X),$$

taking into account what precedes. This ends the proof of point c). □

2.5. Spectral measure associated with an operator

Many monographs (see, for instance, Dunford and Schwartz, 1963, Riesz and Nagy, 1991) evoque an association between a s.m. $\mathcal{E}$ and a selfadjoint operator A which allows to express this last one as an integral: $A = \int \lambda \mathrm{d}\mathcal{E}(\lambda)$. More precisely, we check the following.

Let A be a bounded selfadjoint operator. There exists one, and only one, bounded s.m. $\mathcal{E}$, named *s.m. associated with* A, such that, for any X of H, $AX = \int j \mathrm{d}Z^X_{\mathcal{E}}$. This s.m. is such that $\|A\| = \inf\{a \in \mathbb{R}_+; \mathcal{E}([-a, a]) = I_H\}$.

Without pretention of giving exhaustive explanations, we will give some indications on the way the s.m. $\mathcal{E}$ is defined. If A is a bounded selfadjoint operator, it is easy to verify that $(\mathrm{e}^{itA}(X))_{t\in\mathbb{R}}$ is a stationary c.r.f., and we denote by Z^X its associated r.m.. So we obtain a family of stationary c.r.f.'s, which are pairwise stationarily correlated. From this fact, we deduce, on one hand, that for any B of $\mathcal{B}_\mathbb{R}$, the application $\mathcal{E}(B) : X \in H \mapsto Z^X(B) \in H$ is a projector, and on another hand, that the application $\mathcal{E} : B \in \mathcal{B}_\mathbb{R} \mapsto \mathcal{E}(B) \in \mathcal{P}(H)$ is a bounded s.m. such that $A = \mathcal{E}_j$, or, in other words, such that $AX = \int j \mathrm{d}Z^X_{\mathcal{E}}$, for any X of H. For any $a > \|A\|$, we can write

$$A = \lim_{m\to\infty} \sum_{k=0}^{k=m-1} (-a + k\tfrac{2a}{m})\mathcal{E}([-a + k\tfrac{2a}{m}, -a + (k+1)\tfrac{2a}{m}[),$$

in $\mathcal{L}(H)$, expression which evoques a Riemann sum associated with an integral of the type $\int \lambda \mathrm{d}\mathcal{E}(\lambda)$.

Let us now examine the following preliminary result.

Lemma 2.5.1. *If $\mathcal{E}$ is the s.m. associated with the bounded selfadjoint operator A, then, for any n of $\mathbb{N}$, we have $A^n = \mathcal{E}_{j^n}$.*

Proof. The proof is obtained by induction. In fact, if n is an integer such that $A^n = \mathcal{E}_{j^n}$, then, for any X of H, we have:

$$< A^{n+1}X, X > = < A^n X, AX > = < \mathcal{E}_{j^n}(X), \mathcal{E}_j(X) > = < \int j^n \mathrm{d}Z^X_{\mathcal{E}}, \int j \mathrm{d}Z^X_{\mathcal{E}} >$$
$$= < j^n, j >_{L^2(\mu_{Z^X_{\mathcal{E}}})} = \int j^{n+1} \mathrm{d}\mu_{Z^X_{\mathcal{E}}} = < \int j^{n+1} \mathrm{d}Z^X_{\mathcal{E}}, \int_\mathbb{R} \mathrm{d}Z^X_{\mathcal{E}} > = < \mathcal{E}_{j^{n+1}}X, X >.$$

Then $A^{n+1} = \mathcal{E}_{j^{n+1}}$, what means that the property is true for $n+1$. $\square$

So the proposition follows.

Proposition 2.5.1. *If A is a bounded selfadjoint operator of associated s.m. $\mathcal{E}$, if T is an element of $\mathcal{L}(H)$ such that $T \circ A = A \circ T$, then, for any B of $\mathcal{B}_{\mathbb{R}}$, T and $\mathcal{E}(B)$ commute.*

Proof. As $\mu_{Z_{\mathcal{E}}^{TX}} + \mu_{Z_{\mathcal{E}}^{X}}$ has a compact support, for any B of $\mathcal{B}_{\mathbb{R}}$, 1_B can be writen as

$$1_B = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} j^{n_{j,m}},\ |J_m| < +\infty, \text{ in } L^2(\mu_{Z_{\mathcal{E}}^{TX}} + \mu_{Z_{\mathcal{E}}^{X}}). \qquad (2.5.1)$$

Equality (2.5.1) is exact in $L^2(\mu_{Z_{\mathcal{E}}^{TX}})$ and in $L^2(\mu_{Z_{\mathcal{E}}^{X}})$, its integrations successively with respect to the r.m.'s $Z_{\mathcal{E}}^{TX}$ and $Z_{\mathcal{E}}^{X}$ give:

$$(\mathcal{E}(B))TX = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} A^{n_{j,m}} TX, \qquad (2.5.2)$$

and

$$(\mathcal{E}(B))X = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} A^{n_{j,m}} X,$$

hence

$$T(\mathcal{E}(B))X = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} T A^{n_{j,m}} X,$$

what, taking into account result (2.5.2), and as $A^{n_{j,m}} \circ T = T \circ A^{n_{j,m}}$, allows us to write

$$\mathcal{E}(B) \circ T = T \circ \mathcal{E}(B).\ \square$$

This property has got its converse.

Proposition 2.5.2. *If A is a bounded selfadjoint operator of associated s.m. $\mathcal{E}$, if T is an element of $\mathcal{L}(H)$ such that $T \circ \mathcal{E}(B) = \mathcal{E}(B) \circ T$, then, for any B of $\mathcal{B}_{\mathbb{R}}$, T and A commute.*

Proof. Taking into account the property of density of the indicator functions, the element j of $L^2(\mu_{Z_{\mathcal{E}}^{TX}} + \mu_{Z_{\mathcal{E}}^{X}})$ can be writen as:

$$j = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} 1_{B_{j,m}},\ B_{j,m} \in \mathcal{B}_{\mathbb{R}},\ |J_m| < +\infty, \text{ in } L^2(\mu_{Z_{\mathcal{E}}^{TX}} + \mu_{Z_{\mathcal{E}}^{X}}). \qquad (2.5.3)$$

The equality (2.5.3) is true in $L^2(\mu_{Z_{\mathcal{E}}^{TX}})$ and in $L^2(\mu_{Z_{\mathcal{E}}^{X}})$, by integration with respect to the r.m.'s $Z_{\mathcal{E}}^{TX}$ and $Z_{\mathcal{E}}^{X}$, we have:

$$ATX = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} (\mathcal{E}(B_{j,m}))TX, \qquad (2.5.4)$$

and

$$AX = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} (\mathcal{E}(B_{j,m}))X,$$

hence

$$TAX = \lim_{m\to\infty} \sum_{j\in J_m} \alpha_{j,m} T(\mathcal{E}(B_{j,m}))X,$$

what allows, thanks to (2.5.4) and to the fact that $T \circ \mathcal{E}(B_{j,m}) = \mathcal{E}(B_{j,m}) \circ T$, to write:

$$A \circ T = T \circ A.\ \square$$

The following property is obtained combining these two last results.

Proposition 2.5.3. *Two bounded selfadjoint operators A and A' commute if, and only if, their associated bounded s.m.'s commute.*

We are now able to examine the main result of this section.

Proposition 2.5.4. *If A and A' are two bounded selfadjoint operators which commute, of respective associated s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, then $\mathcal{E} * \mathcal{E}'$ is the s.m. associated with the operator $A + A'$.*

Proof. Let us first notice that, because A and A' commute, the bounded s.m.'s $\mathcal{E}$ and $\mathcal{E}'$ also commute. So we can consider the s.m.'s $\mathcal{E} \otimes \mathcal{E}'$ and $\mathcal{E} * \mathcal{E}'$, this last one being bounded.

Let us denote by P (resp. P') the measurable application $(\lambda, \lambda') \in \mathbb{R}^2 \mapsto \lambda \in \mathbb{R}$ (resp. $(\lambda, \lambda') \in \mathbb{R}^2 \mapsto \lambda' \in \mathbb{R}$). We easily verify that $P(\mathcal{E} \otimes \mathcal{E}') = \mathcal{E}$ (resp. $P'(\mathcal{E} \otimes \mathcal{E}') = \mathcal{E}'$).

As $\mathcal{E}$ is a bounded s.m., j belongs to $\mathcal{M}(\mathbb{R}, \mathcal{E})$, so to $\mathcal{M}(\mathbb{R}, P(\mathcal{E} \otimes \mathcal{E}'))$. Proposition 2.4.3 allows us to affirm that $j \circ P$ belongs to $\mathcal{M}(\mathbb{R} \times \mathbb{R}, \mathcal{E} \otimes \mathcal{E}')$ and that $\mathcal{E}_j = (\mathcal{E} \otimes \mathcal{E}')_{j \circ P}$. In a same way, we get $\mathcal{E}'_j = (\mathcal{E} \otimes \mathcal{E}')_{j \circ P'}$. The linearity of the application $f \in \mathcal{M}(\mathbb{R} \times \mathbb{R}, \mathcal{E} \otimes \mathcal{E}') \mapsto (\mathcal{E} \otimes \mathcal{E}')_f \in \mathcal{L}(H)$ allows us to write:

$$\begin{aligned} \mathcal{E}_j + \mathcal{E}'_j = & \quad (\mathcal{E} \otimes \mathcal{E}')_{j \circ P} + (\mathcal{E} \otimes \mathcal{E}')_{j \circ P'} \\ = & \quad (\mathcal{E} \otimes \mathcal{E}')_{j \circ P + j \circ P'} \\ = & \quad (\mathcal{E} \otimes \mathcal{E}')_{j \circ S} \end{aligned} \tag{2.5.5}$$

Besides, as j is an element of $\mathcal{M}(\mathbb{R}, \mathcal{E} * \mathcal{E}')$, so of $\mathcal{M}(\mathbb{R}, S(\mathcal{E} \otimes \mathcal{E}'))$, from Proposition 2.4.3, $j \circ S$ belongs to $\mathcal{M}(\mathbb{R} \times \mathbb{R}, \mathcal{E} \otimes \mathcal{E}')$ and $(S(\mathcal{E} \otimes \mathcal{E}'))_j = (\mathcal{E} \otimes \mathcal{E}')_{j \circ S}$.

So, taking into account (2.5.5), we have

$$\mathcal{E}_j + \mathcal{E}'_j = (S(\mathcal{E} \otimes \mathcal{E}'))_j = (\mathcal{E} * \mathcal{E}')_j,$$

or also

$$A + A' = (\mathcal{E} * \mathcal{E}')_j,$$

what ends the proof. □

3. The $\alpha-$equivalence

In this section, we will examine a proximity relation between s.m.'s on $\mathcal{B}_{\mathbb{R}}$ for H. We will frequently use the fact that, if B is a compact subset of $\mathbb{R}$ and α an element of $\mathbb{R}_+^*$, then $B + [-\alpha, \alpha]$ is compact.

Definition 3.1. *Let α be an element of $\mathbb{R}_+^*$, we say that two s.m.'s $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\alpha-$equivalent, what we denote $\mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{E}_2$, when, for any compact B, we have*
i) $\mathcal{E}_1(B) \ll \mathcal{E}_2(B + [-\alpha, \alpha])$;
ii) $\mathcal{E}_2(B) \ll \mathcal{E}_1(B + [-\alpha, \alpha])$.

Remark. When two s.m.'s are $\alpha-$equivalent, for any compact B, we have: $\mathcal{E}_1(B) \ll \mathcal{E}_2(B + [-\alpha, \alpha]) \ll \mathcal{E}_1(B + [-2\alpha, 2\alpha])$. As far as α is small, the compact sets B, $B + [-\alpha, \alpha]$ and $B + [-2\alpha, 2\alpha]$ are close together. It is also the case for the projectors $\mathcal{E}_1(B)$ and $\mathcal{E}_1(B + [-2\alpha, 2\alpha])$. So the projector $\mathcal{E}_2(B + [-\alpha, \alpha])$, which is between $\mathcal{E}_1(B)$ and $\mathcal{E}_1(B + [-2\alpha, 2\alpha])$, is close to $\mathcal{E}_1(B)$. This induces the proximity between $\mathcal{E}_1(B)$ and $\mathcal{E}_2(B)$.

Apparently, the relation of $\alpha-$equivalence is symmetric, so the following property is close to a property of transitivity.

Proposition 3.1. *If three s.m.'s $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are such that $\mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{E}_2$ and $\mathcal{E}_2 \overset{\alpha'}{\sim} \mathcal{E}_3$, where α and α' are elements of $\mathbb{R}_+^*$, then $\mathcal{E}_1 \overset{\alpha+\alpha'}{\sim} \mathcal{E}_3$.*

Proof. It is the result of the relations:

$$\mathcal{E}_1(B) \ll \mathcal{E}_2(B + [-\alpha, \alpha]) \ll \mathcal{E}_3((B + [-\alpha, \alpha]) + [-\alpha', \alpha']) = \mathcal{E}_3(B + [-(\alpha + \alpha'), \alpha + \alpha']),$$

and, in a symmetric way:
$\mathcal{E}_3(B) \ll \mathcal{E}_2(B+[-\alpha',\alpha']) \ll \mathcal{E}_1((B+[-\alpha',\alpha'])+[-\alpha,\alpha]) = \mathcal{E}_1(B+[-(\alpha+\alpha'),\alpha+\alpha'])$,
for any compact B. □

The $\alpha-$equivalence provides a kind of continuity as follows.

Proposition 3.2. *If $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{R}_+^*$ which decreasingly converges to α, element of $\mathbb{R}_+^*$, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two s.m.'s such that $\mathcal{E}_1 \overset{\alpha_n}{\sim} \mathcal{E}_2$, for any n of $\mathbb{N}$, then $\mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{E}_2$.*

Proof. Let B be a compact. It is known that $B+[-\alpha,\alpha] = \cap_{n\in\mathbb{N}}(B+[-\alpha_n,\alpha_n])$.
As $(B+[-\alpha_n,\alpha_n])_{n\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathcal{B}_{\mathbb{R}}$, for any X of H, the properties of continuity of the r.m.'s allow us to write:

$$\begin{aligned}\lim_{n\to\infty}\mathcal{E}_1(B+[-\alpha_n,\alpha_n])X &= \lim_{n\to\infty} Z^X_{\mathcal{E}_1}(B+[-\alpha_n,\alpha_n])\\ &= Z^X_{\mathcal{E}_1}(\cap_{n\in\mathbb{N}}(B+[-\alpha_n,\alpha_n]))\\ &= Z^X_{\mathcal{E}_1}(B+[-\alpha,\alpha])\\ &= \mathcal{E}_1(B+[-\alpha,\alpha])X,\end{aligned}$$

then

$$\lim_{n\to\infty}(\mathcal{E}_2(B))(\mathcal{E}_1(B+[-\alpha_n,\alpha_n]))X = (\mathcal{E}_2(B))(\mathcal{E}_1(B+[-\alpha,\alpha]))X.$$

As $(\mathcal{E}_2(B))(\mathcal{E}_1(B+[-\alpha_n,\alpha_n]))X = \mathcal{E}_2(B)$, because $\mathcal{E}_2(B) \ll \mathcal{E}_1(B+[-\alpha_n,\alpha_n])$, what precedes lets us write $(\mathcal{E}_2(B))X = (\mathcal{E}_2(B))(\mathcal{E}_1(B+[-\alpha,\alpha]))X$.
It is then clear that $\mathcal{E}_2(B) \ll \mathcal{E}_1(B+[-\alpha,\alpha])$.
The relation $\mathcal{E}_2(B) \ll \mathcal{E}_1(B+[-\alpha,\alpha])$ can be proved in a similar way, what ends the proof. □

When a s.m. is concentrated on the neigbourhood of 0, it is close to $\mathcal{E}_{\mathbb{R}}$, this is what is expressed in the following result.

Proposition 3.3. *A s.m. $\mathcal{E}$ is $\alpha-$equivalent to $\mathcal{E}_{\mathbb{R}}$ if and only if $\mathcal{E}([-\alpha,\alpha]) = I_H$.*

Proof. Let $\mathcal{E}$ be a s.m. such that $\mathcal{E}([-\alpha,\alpha]) = I_H$. Let us consider a compact B. If $0 \in B+[-\alpha,\alpha]$, then $\mathcal{E}(B) \ll I_H = \mathcal{E}_{\mathbb{R}}(B+[-\alpha,\alpha])$. If $0 \notin B+[-\alpha,\alpha]$, then $B\cap[-\alpha,\alpha] = \emptyset$ and then $0 = \mathcal{E}(B\cap[-\alpha,\alpha]) = (\mathcal{E}(B))(\mathcal{E}([-\alpha,\alpha])) = \mathcal{E}(B) \ll \mathcal{E}_{\mathbb{R}}(B+[-\alpha,\alpha])$.
In both cases we have $\mathcal{E}(B) \ll \mathcal{E}_{\mathbb{R}}(B+[-\alpha,\alpha])$.
In order to prove that $\mathcal{E}_{\mathbb{R}}(B) \ll \mathcal{E}(B+[-\alpha,\alpha])$, we also have two possibilities.
Either $\mathcal{E}_{\mathbb{R}}(B) = I_H$, either $\mathcal{E}_{\mathbb{R}}(B) = 0$.
In the first case, $0 \in B$ and then $[-\alpha,\alpha] \subset B+[-\alpha,\alpha]$, so $\mathcal{E}_{\mathbb{R}}(B) = I_H = \mathcal{E}([-\alpha,\alpha]) \ll \mathcal{E}(B+[-\alpha,\alpha])$.
In the second case, $\mathcal{E}_{\mathbb{R}}(B) = 0 \ll \mathcal{E}(B+[-\alpha,\alpha])$.
So we can conclude to the $\alpha-$equivalence between the s.m.'s $\mathcal{E}$ and $\mathcal{E}_{\mathbb{R}}$.
As for the converse, it comes from the relations
$I_H = \mathcal{E}_{\mathbb{R}}(\{0\}) \ll \mathcal{E}(\{0\}+[-\alpha,\alpha]) = \mathcal{E}([-\alpha,\alpha]) = I_H$. □

Let us now start the study of the transmission of the $\alpha-$equivalence through convolution. First of all, let us introduce a preliminary result.

Lemma 3.1. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two s.m.'s which commute. If $\mathcal{E}$ is $\alpha-$equivalent with $\mathcal{E}_{\mathbb{R}}$, then $\mathcal{E}' * \mathcal{E} \overset{\alpha}{\sim} \mathcal{E}'$.*

Proof. For any compact B of $\mathbb{R}$,

$$(\mathbb{R}\times[-\alpha,\alpha])\cap S^{-1}B\subset(B+[-\alpha,\alpha])\times[-\alpha,\alpha],$$

and we can write, from the fact that $\mathcal{E}([-\alpha,\alpha])=I_H$:

$$\begin{aligned}(\mathcal{E}'*\mathcal{E})(B) &= (\mathcal{E}'\otimes\mathcal{E})(S^{-1}B)\\ &= (\mathcal{E}'\otimes\mathcal{E})(\mathbb{R}\times[-\alpha,\alpha])(\mathcal{E}'\otimes\mathcal{E})(S^{-1}B)\\ &= (\mathcal{E}'\otimes\mathcal{E})((\mathbb{R}\times[-\alpha,\alpha])\cap S^{-1}B)\\ &\ll (\mathcal{E}'\otimes\mathcal{E})((B+[-\alpha,\alpha])\times[-\alpha,\alpha])\\ &= \mathcal{E}'(B+[-\alpha,\alpha]).\end{aligned}$$

Moreover, as $B\times[-\alpha,\alpha]\subset S^{-1}(B+[-\alpha,\alpha])$, we also have

$$\begin{aligned}\mathcal{E}'(B) &= \mathcal{E}'\otimes\mathcal{E}(B\times[-\alpha,\alpha])\\ &\ll \mathcal{E}'\otimes\mathcal{E}(S^{-1}(B+[-\alpha,\alpha]))\\ &= \mathcal{E}'*\mathcal{E}(B+[-\alpha,\alpha]),\end{aligned}$$

what allows to conclude. □

Let us now examine the case where a s.m. is concentrated on a countable family.

Lemma 3.2. *Let $\Lambda=\{\lambda_n;n\in\mathbb{N}\}$ be a countable family of reals, and $\mathcal{E}$ be a s.m. such that $\mathcal{E}(\Lambda)=I_H$. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two $\alpha-$equivalent s.m.'s which commute with $\mathcal{E}$, then $\mathcal{E}*\mathcal{E}_1\overset{\alpha}{\sim}\mathcal{E}*\mathcal{E}_2$.*

Proof. Let B be a compact of $\mathbb{R}$. From Section 2.3, we can affirm that

i) $\{\mathcal{E}(\{\lambda_n\})\mathcal{E}_1(B-\lambda_n);n\in\mathbb{N}\}$ is an orthogonal family of projectors which sum is $\mathcal{E}*\mathcal{E}_1B$;

ii) $\{\mathcal{E}(\{\lambda_n\})\mathcal{E}_2(B+[-\alpha,\alpha]-\lambda_n);n\in\mathbb{N}\}$ is an orthogonal family of projectors which sum is $\mathcal{E}*\mathcal{E}_2(B+[-\alpha,\alpha])$.

As the s.m.'s $\mathcal{E}$ and $\mathcal{E}_1$ commute and from

$$\mathcal{E}_1(B-\lambda_n)\circ\mathcal{E}_2(B+[-\alpha,\alpha]-\lambda_n)=\mathcal{E}_1(B-\lambda_n),$$

(because $\mathcal{E}_1\overset{\alpha}{\sim}\mathcal{E}_2$) we can write

$$\begin{aligned}\mathcal{E}(\{\lambda_n\})\mathcal{E}_1(B-\lambda_n)\mathcal{E}(\{\lambda_n\})\mathcal{E}_2(B+[-\alpha,\alpha]-\lambda_n) &= \mathcal{E}(\{\lambda_n\})\mathcal{E}_1(B-\lambda_n)\mathcal{E}_2(B+[-\alpha,\alpha]-\lambda_n)\\ &= \mathcal{E}(\{\lambda_n\})\mathcal{E}_1(B-\lambda_n),\end{aligned}$$

and so

$$\mathcal{E}(\{\lambda_n\})\mathcal{E}_1(B-\lambda_n)\ll\mathcal{E}(\{\lambda_n\})\mathcal{E}_2(B+[-\alpha,\alpha]-\lambda_n),$$

this for any $n\in\mathbb{N}$. From the recalls of Section 2.1, we can affirm that

$$\mathcal{E}*\mathcal{E}_1(B)\ll\mathcal{E}*\mathcal{E}_2(B+[-\alpha,\alpha]).$$

Conversely, we could prove that

$$\mathcal{E}*\mathcal{E}_2(B)\ll\mathcal{E}*\mathcal{E}_1(B+[-\alpha,\alpha]),$$

what allows to conclude. □

A discretization of $\mathbb{R}$ allows to approximate any s.m. by a s.m. concentrated on a countable family. For this, let us denote by $\mathcal{L}_n$ the measurable application $x\in\mathbb{R}\mapsto\frac{[nx]}{n}\in\mathbb{R}$, where n is an element of $\mathbb{N}^*$ and $[nx]$ the integer part of nx. Then the proposition follows.

Proposition 3.4. *For any n of $\mathbb{N}^*$ and for any s.m. $\mathcal{E}$, we can affirm that*

a) $\mathcal{L}_n\mathcal{E}*w\mathcal{E}\overset{\frac{1}{n}}{\sim}\mathcal{E}_{\mathbb{R}}$;

b) $\mathcal{L}_n\mathcal{E}\overset{\frac{1}{n}}{\sim}\mathcal{E}$.

Proof. It is easy to verify that, for any x of $\mathbb{R}$, $-\frac{1}{n}<(\mathcal{L}_n+w)(x)\leqslant 0$,

we deduce from this that $(\mathcal{L}_n + w)^{-1}[-\frac{1}{n}, \frac{1}{n}] = \mathbb{R}$.
So it comes
$(\mathcal{L}_n + w)\mathcal{E}([-\frac{1}{n}, \frac{1}{n}]) = \mathcal{E}((\mathcal{L}_n + w)^{-1}[-\frac{1}{n}, \frac{1}{n}]) = \mathcal{E}(\mathbb{R}) = I_H$.
Then, from Proposition 3.3, $(\mathcal{L}_n + w)\mathcal{E} \overset{\frac{1}{n}}{\sim} \mathcal{E}_{\mathbb{R}}$, or in other words
$(\mathcal{L}_n\mathcal{E}) * (w\mathcal{E}) \overset{\frac{1}{n}}{\sim} \mathcal{E}_{\mathbb{R}}$, and Point a) is proved. As for Point b), it is a consequence of Point a) and of Lemma 3.1. □

This last property lets us generalize both Lemma 3.1 and Lemma 3.2.

Proposition 3.5. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two $\alpha-$equivalent s.m.'s, which commute with a third s.m. $\mathcal{E}$, then we have $\mathcal{E} * \mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{E} * \mathcal{E}_2$.*

Proof. For any n of $\mathbb{N}^*$, we can affirm that $\mathcal{L}_n\mathcal{E} * w\mathcal{E} \overset{\frac{1}{n}}{\sim} \mathcal{E}_{\mathbb{R}}$.
As the s.m.'s $\mathcal{L}_n\mathcal{E} * w\mathcal{E}$ and $\mathcal{E} * \mathcal{E}_1$ commute, Lemma 3.1 allows us to write

$$(\mathcal{L}_n\mathcal{E} * w\mathcal{E}) * (\mathcal{E} * \mathcal{E}_1) \overset{\frac{1}{n}}{\sim} \mathcal{E} * \mathcal{E}_1,$$

or, taking into account the associativity of the convolution,

$$\mathcal{L}_n\mathcal{E} * \mathcal{E}_1 \overset{\frac{1}{n}}{\sim} \mathcal{E} * \mathcal{E}_1. \tag{3.1}$$

In a similar way, we can prove that

$$\mathcal{L}_n\mathcal{E} * \mathcal{E}_2 \overset{\frac{1}{n}}{\sim} \mathcal{E} * \mathcal{E}_2. \tag{3.2}$$

As $\mathcal{L}_n\mathcal{E}$ is a s.m. concentrated on a countable family, and as $\mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{E}_2$, Lemma 3.2 lets us write:

$$\mathcal{L}_n\mathcal{E} * \mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{L}_n\mathcal{E} * \mathcal{E}_2. \tag{3.3}$$

From relations (3.1), (3.2) and (3.3), we deduce

$$\mathcal{E} * \mathcal{E}_1 \overset{\alpha+\frac{2}{n}}{\sim} \mathcal{E} * \mathcal{E}_2.$$

As the sequence $(\alpha + \frac{2}{n})_{n\in\mathbb{N}^*}$ decreasingly converges to α, Proposition 3.2 allows us to conclude. □

4. Proximity between operators and $\alpha-$equivalence

If $\mathcal{E}$ is the s.m. associated with a bounded selfadjoint operator A, then $\mathcal{E}([-\|A\|, \|A\|]) = I_H$, and Proposition 3.3 induces the following result.

Proposition 4.1. *If $\mathcal{E}$ is the s.m. associated with a bounded selfadjoint operator A, then $\mathcal{E} \overset{\|A\|}{\sim} \mathcal{E}_{\mathbb{R}}$.*

This means that if A is *close* to the null operator O, then the s.m. $\mathcal{E}$ is *close* to $\mathcal{E}_{\mathbb{R}}$, the s.m. associated with O. Indeed, as $Z^X_{\mathcal{E}_{\mathbb{R}}} = \delta_0(.)X$, it comes $(\mathcal{E}_{\mathbb{R}})_j X = \int j \mathrm{d}\delta_0(.)X = j(0)X = 0$, for any X of H.

So it seams that the proximity between operators can be transposed to s.m.'s. The following result lets us approach this aspect.

Lemma 4.1. *If A and A' are two bounded selfadjoint operators which commute, of respective associated s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, then $\mathcal{E} * (w\mathcal{E}')$ is the s.m. associated with the bounded selfadjoint operator $A - A'$.*

Proof. It is clear that if $\mathcal{E}'$ is the s.m. associated with A', then $w\mathcal{E}'$ is the s.m. associated with $-A'$. Indeed, from one side, $w\mathcal{E}'$ is bounded, and from another side, we have

$$(w\mathcal{E}')_j X = \int j \mathrm{d}Z^X_{w\mathcal{E}'} = \int j \mathrm{d}w(Z^X_{\mathcal{E}'}) = \int j \circ w \mathrm{d}Z^X_{\mathcal{E}'} = \int -j \mathrm{d}Z^X_{\mathcal{E}'} = -A'X,$$

for any X of H.

So, from Proposition 2.5.4, the s.m. associated with $A - A'$ is $\mathcal{E} * (w\mathcal{E}')$. □

We are now able to enunciate a property which generalizes the previous result.

Proposition 4.2. *If A and A' are two bounded selfadjoint operators which commute, of respective associated s.m.'s $\mathcal{E}$ and $\mathcal{E}'$, then $\mathcal{E} \overset{\|A-A'\|}{\sim} \mathcal{E}'$.*

Proof. As $\mathcal{E} * w\mathcal{E}'$ is the s.m. associated with the bounded selfadjoint operator $A - A'$, Proposition 4.1 allows us to write

$$\mathcal{E} * w\mathcal{E}' \overset{\|A-A'\|}{\sim} \mathcal{E}_{\mathbb{R}}.$$

As $\mathcal{E}'$ commute with $\mathcal{E} * w\mathcal{E}'$, Lemma 3.1 allows us to affirm that

$$\mathcal{E}' * (\mathcal{E} * w\mathcal{E}') \overset{\|A-A'\|}{\sim} \mathcal{E}',$$

so that

$$\mathcal{E} \overset{\|A-A'\|}{\sim} \mathcal{E}',$$

and the property is proved. □

The following property is, in some way, the converse of the previous one.

Proposition 4.3. *Let $\mathcal{E}$ and $\mathcal{E}'$ be respectively the s.m.'s associated with the bounded selfadjoint operators A and A', which commute. If $\mathcal{E}$ and $\mathcal{E}'$ are $\alpha-$equivalent, then $\|A - A'\| \leqslant \alpha$.*

Proof. The hypothesis of the Proposition can be writen $\mathcal{E} \overset{\alpha}{\sim} \mathcal{E}'$. As $w\mathcal{E}'$ commutes with $\mathcal{E}$ (because $\mathcal{E}$ commutes with $\mathcal{E}'$), Proposition 3.5 allows us to write

$$w\mathcal{E}' * \mathcal{E} \overset{\alpha}{\sim} w\mathcal{E}' * \mathcal{E}',$$

that is

$$\mathcal{E} * w\mathcal{E}' \overset{\alpha}{\sim} \mathcal{E}_{\mathbb{R}}.$$

From Proposition 3.3, we have then

$$\mathcal{E} * w\mathcal{E}'([-\alpha, \alpha]) = I_H.$$

But, as $\|A - A'\| = \inf\{a \in \mathbb{R}^*_+, \mathcal{E} * w\mathcal{E}'([-a, a]) = I_H\}$, because $\mathcal{E} * w\mathcal{E}'$ is the s.m. associated with $A - A'$. We have then $\|A - A'\| \leqslant \alpha$, what allows to conclude. □

The notion of $\alpha-$equivalence is a good translation of the proximity of s.m.'s associated with two operators, because of the equivalence with the closeness of the associated operators.

Remark. From Proposition 4.2, when two self-adjoint bounded operators which commute are close together, the same happens for their respectively associated s.m.'s, according to the $\alpha-$equivalence. We examine here an example where, when the commutativity is not satisfied, the proximity of the operators does not imply the proximity of the associated s.m.'s. It illustrates the necessity of this hypothesis for this proposition.

Let us consider two elements y and h of H such that $< y, h >= 0$, $\|y\|^2 + \|h\|^2 = 1$, $\|y\|.\|h\| \neq 0$, and $\|h\| < \frac{1}{4}$. Let $x = y + h$ and $x' = y - h$.

It is clear that $P = x \otimes x$ and $P' = x' \otimes x'$ are orthogonal projectors such that $\|P - P'\| \leqslant 2\|h\| < \frac{1}{2}$.

We can easily verify that $< x, x' > \neq 0$ and that $\{x, x'\}$ is a free family.

From these last two points we can deduce that P and P' do not commute. So the selfadjoint bounded operators $A = \lambda P$ and $A' = \lambda P'$, λ being an element of $\mathbb{R}^*$, also do not commute. The s.m.'s respectively associated with A and A' are, with obvious notation:

$$\mathcal{E} = \delta_0(.)P_\perp + \delta_\lambda(.)P \text{ and } \mathcal{E}' = \delta_0(.)P'_\perp + \delta_\lambda(.)P'.$$

If we assume that $\mathcal{E}' \overset{\|A-A'\|}{\sim} \mathcal{E}$, then we have:

$P'_\perp = \mathcal{E}'\{0\} \ll \mathcal{E}(\{0\} + [-\|A - A'\|, \|A - A'\|]) = \mathcal{E}([-\|A - A'\|, \|A - A'\|]) = P_\perp + \delta_\lambda([-\|A - A'\|, \|A - A'\|])P = P_\perp$ (because $\lambda \notin [-\|A - A'\|, \|A - A'\|]$).

So $P \ll P'$ and then $PP' = P'P = P$, what is the opposite of the hypothesis made at the beginning. So the relation $\mathcal{E}' \overset{\|A-A'\|}{\sim} \mathcal{E}$ is false.

5. The case of compact operators

In this section we will examine the particular case of the compact selfadjoint positive operators. This particular case plays an important role in various fields of mathematics, and in particular, in statistics, the covariance operators belong to this family. It is well known that any compact operator is the limit, in $\mathcal{L}(H)$, of a sequence of finite rank operators $(A_n)_{n\in\mathbb{N}}$. Moreover, when A is positive and selfadjoint, the operators A_n are linear combinations of projectors. More precisely, we have the following.

If A is a positive selfadjoint compact operator, which image is of infinite dimension, we can say that

− there exists a real sequence $(\lambda_p)_{p\in\mathbb{N}}$ which strictly decreasingly converges to 0,

− there exists a family $\{P_p; p \in \mathbb{N}\}$ of orthogonal projectors such that, for any p of $\mathbb{N}$, $P_p \neq O$ and $\dim \mathrm{Im} P_p < +\infty$.

This real sequence and this family of projectors are such that

a) the family $\{\lambda_p P_p; p \in \mathbb{N}\}$ of elements of $\mathcal{L}(H)$, is summable of sum A;

b) $\{\lambda_p; p \in \mathbb{N}\}$ is the family of the eigenvalues of A, different from 0;

c) for any p of $\mathbb{N}$, $\mathrm{Im} P_p$ is the eigenspace of A associated with the eigenvalue λ_p;

d) if we denote by D the projector, sum of the orthogonal family of projectors $\{P_p; p \in \mathbb{N}\}$, then $\mathrm{Ker} A = \mathrm{Im}(I - D)$.

Let us set $\mu_0 = 0$, $D_0 = I - D$ and, for any p of $\mathbb{N}^*$, $\mu_p = \lambda_{p-1}$ and $D_p = P_{p-1}$. Then we can affirm that

a) for any p of $\mathbb{N}^*$, μ_p is the p^{th} largest eigenvalue of A;

b) $\mathrm{Im} D_p$ is the eigenspace associated with the eigenvalue μ_p;

c) the family $\{\mu_p D_p; p \in \mathbb{N}^*\}$, of elements of $\mathcal{L}(H)$, is summable of sum A;

d) $\mathrm{Im} D_0 = \mathrm{Ker} A$;

e) $\{D_p; p \in \mathbb{N}\}$ is an orthogonal family of projectors of sum I.

With these notation, we can write the following result.

Proposition 5.1. *If, for any p of $\mathbb{N}^*$, we denote by μ_p and D_p respectively the $p-th$ greatest eigenvalue and the associated eigenprojector of A, positive selfadjoint operator, if*

we set $\mu_0 = 0$ *and* $D_0 = I - \sum_{p\in\mathbb{N}^*} D_p$*, then,*
i) for any B *of* $\mathcal{B}_\mathbb{R}$, $\{\delta_{\mu_p}(B)D_p; p \in \mathbb{N}\}$ *is an orthogonal family of projectors;*
ii) the application $\mathcal{E} : B \in \mathcal{B}_\mathbb{R} \mapsto \sum_{p\in\mathbb{N}} \delta_{\mu_p}(B)D_p \in \mathcal{P}(H)$ *is the bounded s.m. associated with* A*.*

Proof. It is clear that $\{D_p; p \in \mathbb{N}\}$ is a family of orthogonal projectors of sum $D_0 + \sum_{p\geq 1} D_p = I - D + \sum_{n\in\mathbb{N}} P_n = I$.

As $(\mu_p)_{p\in\mathbb{N}^*}$ is a real sequence which strictly converges, Lemma 2.4.2 lets us affirm that

a) for any B of $\mathcal{B}_\mathbb{R}$, $\{\delta_{\mu_p}(B)D_p; p \in \mathbb{N}\}$ is an orthogonal family of projectors;

b) the application $\mathcal{E} : B \in \mathcal{B}_\mathbb{R} \mapsto \sum_{p\in\mathbb{N}} \delta_{\mu_p}(B)D_p \in \mathcal{P}(H)$ is a bounded s.m.;

c) $\{\mu_p D_p; p \in \mathbb{N}\}$ is a family of elements of $\mathcal{L}(H)$, of sum $\mathcal{E}_j$.

To complete the proof, we need to verify that $\mathcal{E}_j = A$, that is that the summable families $\{\lambda_p P_p; p \in \mathbb{N}\}$ and $\{\mu_p D_p; p \in \mathbb{N}\}$ are of same sum.

Therefore, as the set of indices of these families is $\mathbb{N}$, we can write

$$\begin{aligned}\mathcal{E}_j = \lim_n \textstyle\sum_{p=0}^n \mu_p D_p &= \mu_0 D_0 + \lim_n \textstyle\sum_{p=1}^n \mu_p D_p \\ &= \lim_n \textstyle\sum_{p=1}^n \lambda_{n-1} P_{n-1} \\ &= \lim_n \textstyle\sum_{k=0}^{n-1} \lambda_k P_k \\ &= \lim_n \textstyle\sum_{k=0}^{n} \lambda_k P_k \\ &= A,\end{aligned}$$

what ends the proof. □

Remark. From properties of a partition of the set of the indexes of a summable family, we can affirm that, for any B of $\mathcal{B}_\mathbb{R}$, the family $\{D_n; n \in \mathbb{N}$ such that $\mu_n \in B\}$ is summable of sum $\mathcal{E}(B)$.

Let us now examine how the results of the previous section will express in terms of eigenspaces of two compact operators.

Proposition 5.2. *Let* A_1 *and* A_2 *be two selfadjoint compact operators which commute. For any* (i, p) *of* $\{1, 2\} \times \mathbb{N}^*$*, let us denote respectively by* μ_p^i *and* D_p^i *the* $p-th$ *largest eigenvalue of* A_i *and the associated eigenprojector. Then for any* p *of* $\mathbb{N}^*$*, we have*

$$D_p^1 \ll \sum_{k:\mu_p^1-\alpha\leqslant\mu_k^2\leqslant\mu_p^1+\alpha} D_k^2 \ll \sum_{k:\mu_p^1-2\alpha\leqslant\mu_k^1\leqslant\mu_p^1+2\alpha} D_k^1, \text{ with } \alpha = \|A_1 - A_2\|.$$

Proof. This proposition is a direct consequence of Proposition 4.2. Indeed, let us denote by $\mathcal{E}_1$ and $\mathcal{E}_2$ the respective bounded s.m.'s associated with the operators A_1 and A_2. As $\|A_1 - A_2\| \leqslant \alpha$, and as A_1 and A_2 commute, we have $\mathcal{E}_1 \overset{\alpha}{\sim} \mathcal{E}_2$. Then, for any p of $\mathbb{N}^*$,

$$\mathcal{E}_1(\{\mu_p^1\}) \ll \mathcal{E}_2([\mu_p^1 - \alpha; \mu_p^1 + \alpha]) \ll \mathcal{E}_1([\mu_p^1 - 2\alpha; \mu_p^1 + 2\alpha]).$$

In other terms, taking into account the previous remark:

$$D_p^1 \ll \textstyle\sum_{k:\mu_p^1-\alpha\leqslant\mu_k^2\leqslant\mu_p^1+\alpha} D_k^2 \ll \sum_{k:\mu_p^1-2\alpha\leqslant\mu_k^1\leqslant\mu_p^1+2\alpha} D_k^1. \ \square$$

Of course, if A_1 and A_2 are close enough such that in the interval $[\mu_p^1 - 2\alpha; \mu_p^1 + 2\alpha]$ there is no other eigenvalue of A_1 than μ_p^1, then we have the following.

Corollary 5.1. *Let* A_1 *and* A_2 *be two selfadjoint compact operators which commute. Let* $\alpha = \|A_1 - A_2\|$ *and, for any* (i, p) *of* $\{1, 2\} \times \mathbb{N}^*$*, let us denote respectively by* μ_p^i *and* D_p^i *the* $p-th$ *largest eigenvalue of* A_i *and the associated eigenprojector. We can affirm* *that if* $\mu_1^1 - \mu_2^1 > 2\alpha$, *then* $D_1^1 = \sum_{k:\mu_p^1-\alpha\leqslant\mu_k^2\leqslant\mu_p^1+\alpha} D_k^2$. *Moreover, for any integer* $p \geq 2$, *if* $\mu_{p-1}^1 - \mu_p^1 > 2\alpha$ *and if* $\mu_p^1 - \mu_{p+1}^1 > 2\alpha$, *we have* $D_p^1 = \sum_{k:\mu_p^1-\alpha\leqslant\mu_k^2<\mu_p^1\leqslant\alpha} D_k^2$.

Proof. It is enough to notice that if $\mu_1^1 - \mu_2^1 > 2\alpha$, then $\{k \in \mathbb{N}; \mu_1^1 - 2\alpha \leqslant \mu_k^1 \leqslant \mu_1^1 + 2\alpha\} = \{1\}$ and then $\sum_{k:\mu_p^1-\alpha\leqslant\mu_k^1<\mu_p^1\leqslant\alpha} D_k^1 = D_1^1$, so the first point is proved, thanks to Proposition 5.2. The proof of the second point is analog if we remark that, from hypotheses, we have $\{k \in \mathbb{N}; \mu_p^1 - 2\alpha \leqslant \mu_k^1 \leqslant \mu_p^1 + 2\alpha\} = \{p\}$. □

Remark. There exists at least one eigenvalue, of A_2, which belongs to $[\mu_p^1 - \alpha, \mu_p^1 + \alpha]$.

6. Numerical illustration

Let $\{P_j, j = 1, \ldots, k\}$ be a set of orthogonal projectors from $\mathbb{R}^p$ into $\mathbb{R}^p$, $\{\lambda_j, j = 1, \ldots, k\}$ be a set of real values, and, for any j of $\{1, \ldots, k\}$, $(\lambda_n^j)_{n\in\mathbb{N}}$ be a sequence of real values converging to λ_j. Then, the sequence of the selfadjoint operators $(A_n)_{n\in\mathbb{N}}$, where $A_n = \sum_{j=1}^k \lambda_n^j P_j$, converges to the selfadjoint operator $A = \sum_{j=1}^k \lambda_j P_j$. Each A_n obviously commutes with A.

So $\left((e^{itA_n})_{t\in\mathbb{R}}\right)_{n\in\mathbb{N}}$ is a sequence of sets of unitary operators which converges to $(e^{itA})_{t\in\mathbb{R}}$.

If X is a random vector which takes values in $\mathbb{R}^p$, $Y_n = \mathrm{Re}(\left((e^{itA_n})X\right)_{t\in\mathbb{R}})$, where Re stands for the real part, is a continuous random function, and the sequence $(Y_n)_{n\in\mathbb{N}}$ converges to the continuous random function $Y = \mathrm{Re}(\left((e^{itA})X\right)_{t\in\mathbb{R}})$.

The associated s.m. of this last c.r.f. is $\mathcal{E} = \sum_{j=1}^k \delta_{\lambda_j}(.)P_jX$, and the r.m. is $\mu_{Z_X} = \sum_{j=1}^k \delta_{\lambda_j}(.)\|P_jX\|^2$.

In order to give a simple graph illustration, we consider here the case where $p = 2$, and compute simulated sequences of Y_n, with X randomized from the normalized Gaussian distribution, $(\lambda_1, \lambda_2) = (0.4, 2)$, and $(\lambda_n^1, \lambda_n^2) = (0.4 + 1/2n, 2 - 1/n)$.

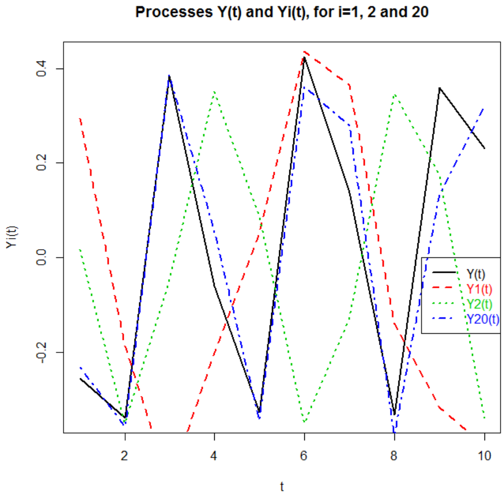

Figure 1: Variations of $Y_n(t)$ for three values of n, t varying, in regard with the limit process $Y(t)$

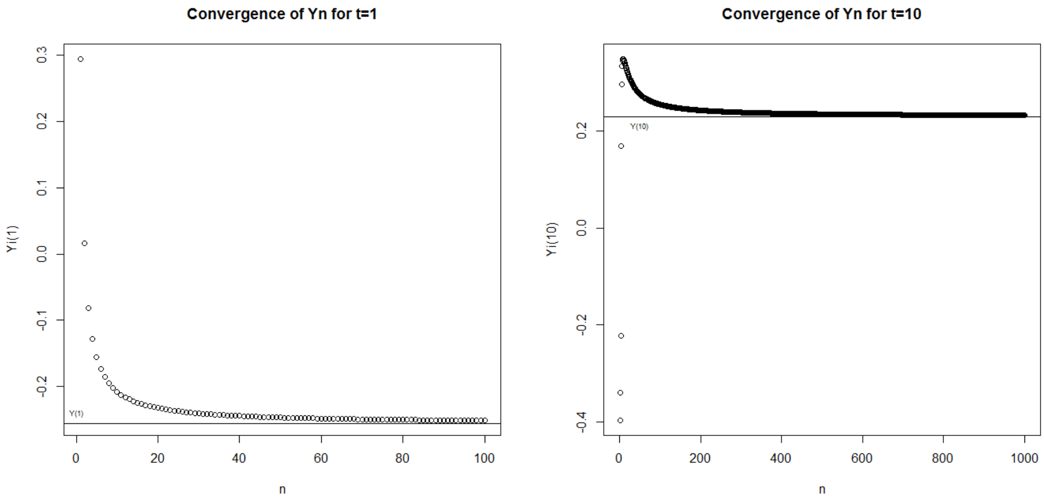

Figure 2: Variations of $Y_n(t)$ for two values of t, n varying, in regard with the limit value $Y(t)$

In Figure 1, we see that the most n is high, the nearest the curve Y_n is close to the curve Y. As for Figure 2, it shows how, for t small (equal to 1) and then t higher (equal to 10), convergence is reached. We can notice that the more t is high, the fastest convergence is reached.

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