EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 3, 2016, 250-265
ISSN 1307-5543 – www.ejpam.com

On "Essential" Subsemimodules and Weakly Co-Hopfian Semimodules

El Hadji Demba Wade Diop*, Djiby Sow

Department de Mathematiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, BP 5005 Dakar Fann, Sénégal

Abstract. Two different notions of "essential" subsemimodules were introduced in the theory of semimodules over a semiring with identity, in order to generalize the same notion of "essential" submodules in the theory of modules over a ring with identity.
In this paper, we introduce a new class of essential subsemimodules called weakly essential subsemimodules. We prove that this new class contains the others two kind of classes of "essential" subsemimodules. Futhermore, we studie the properties of weakly essential subsemimodules. For applications we introduce and investigate the co-hopfian semimodules with this new definition of semi-essential.

2010 Mathematics Subject Classifications: 16Y60
Key Words and Phrases: Semiring, subsemimodule, R-congruence relation, essential, cancellative, subtractive, direct sum, cohopfian semimodules

1. Introduction

The theory of semimodules over semirings with identity (see Golan [6], Abuhlail [1], Takahashi [9–11]) can be regarded as a generalization of the theory of modules over rings with identity (see Anderson-Fuller[2], Lam [8], Wisbauer [15]). Many results for semirings and semimodules also hold for rings and modules, but not conversely [5, 12–14].

The concept of "essential submodule" in an R-module M [2], introduced by Johnson, Eckman and Schpof [4, 7], plays an important role in the context of commutative and noncommutative algebras.

In module theory, for a ring R, a submodule N of a module M is said to be essential (denoted by $N \lhd M$) if $K \cap N = 0 \Longrightarrow K = 0$ for all submodule K of M. A monomorphism $f : M \longrightarrow M'$ is said to be essential if $f(M) \lhd M'$. It is known that an R-monomorphism $f : M \longrightarrow M'$ of left R-modules is essential iff for any R-homomorphism $g : M' \longrightarrow M$", $g \circ f$ is a monomorphism implies that g is a monomorphism. A submodule N of a module M is essential iff the injective map $i : N \longrightarrow M : x \mapsto x$ is essential.

*Corresponding author.

Email addresses: `elhjoop@gmail.com` (E. Diop), `sowdjibab@yahoo.fr` (D. Sow)

The previous characterisations of "essential" are not equivalents in semimodules theory. Hence this notion of "essential" was generalized in semimodules theory in two different ways.

In Golan book's [6], it was proposed the following definitions. An R-monomorphism $f : M \longrightarrow M'$ of left R-semimodules is essential if for any R-homomorphism $g : M' \longrightarrow M''$, $g \circ f$ is a monomorphism implies that g is a monomorphism. A subsemimodule N of a left R-semimodule M is essential (or large) in M if the inclusion map $i_N : N \longrightarrow M$ is an essential R-monomorphism. Note that $f : M \longrightarrow M'$ is an essential R-homomorphism if and only if $f(M)$ is a large subsemimodule of M'.

Another way for defining the notion of "essential" is proposed in [5] as follows. A left R-subsemimodule N of M is said to be semi-essential in M, written as $N \lhd_s M$, if for every R-subsemimodule K of M: $N \cap K = 0 \Rightarrow K = 0$. A monomorphism (respectively: semimonomorphism) $f : M \longrightarrow M'$ of left R-semimodules is said to be semi-essential if: $f(M) \lhd_s M'$.

These two differents notions of "essential subsemimodules" in the theory of semimodules (see [5]) are the same in the theory of modules.

Also, it was proved [5] that the class of essential subsemimodules is not contained and doesn't contain the class of semi-essential subsemimodules. Furthermore, the intersection of these two classes is not empty.

In this paper, we investigate a new class of "essential" subsemimodules (called here: semi-weakly-essential subsemimodules).

In modules theory, it is well known that the congruence relations are defined by the submodules but not in theory of semimodules. So in the new class we consider the congruence relations defined only by the subsemimodules.

We show that this new class contains the two known classes of "essential" subsemimodules. Futhermore, we study some interesting properties of this new class.

As applications we introduce three notions of semi-weakly-co-hopfian semimodules.

All semirings are associative with identity 1 (if R is a semiring, we assume that $1 \neq 0$), all semimodules are unital and all semiring extensions contain the common identity.

Throughout this paper, for semimodule theoretic notions and notations we will follow [1] and [6]. In the following, we recall some definitions and notations that will be used in this paper.

This work is organized as follows:

- In section 1: Preliminaries: We give some results which we will use in the sequel.
- In section 2: New notions of essential: Some properties of semi-weakly-essential subsemimodules are investigated.
- In section 3: Applications: Three types of semi-weakly-co-hopfian semimodules are introduced.

2. Preliminaries

We recall briefly some basic notions about semimodules.

We denote by $N \leq M$, if N is a subsemimodule of a semimodule M and by homomorphism, we mean a homomorphism of left R-semimodules.

Throughout this paper, we consider the left R-semimodules, but the results are also true for right R-semimodules and the proofs are similar.

Definition 1. *Let M be a left R-semimodule.*

- *An R-congruence relation on a semimodule M is an equivalence relation ρ on M such that $m\rho m'$ and $n\rho n' \Longrightarrow (m+n)\rho(m'+n')$ and $(rm)\rho(rm')$, $\forall m, m', n, n' \in M$ and $r \in R$.*
- *The congruence relation ρ defined on M by $m\rho m' \Longleftrightarrow m = m'$ is called a **trivial congruence relation** on M.*
- *The congruence relation ρ defined on M by $m\rho m'$ $\forall m, m' \in M$ is called **universal congruence relation** on M.*
- *M is a R-simple semimodule if any congruence relation defined over M is trivial or universal.*

Remark 1. *The set of all R-congruence relation on M, $R-cong(M)$, is partially-ordered by the relation $\leq$ defined by $\rho \leq \rho'$ if and only if $m\rho m' \Longrightarrow m\rho' m'$ $\forall m, m' \in M$. For $m, m' \in M$, $\rho_{(m,m')}$ is the unique smallest element ρ of $R-cong(M)$ satisfying $m\rho m'$.*

Definition 2.

- *A subsemimodule N of a semimodule M is called subtractive if for all $m, m' \in M$, $m+m' \in N$ and $m \in N$ implies $m' \in N$.*
- *The subtractive closure of a subsemimodule N of a semimodule M is the smallest subtractive subsemimodule of M containing N.*
- *A semimodule M is said to be cancellative (additively cancellative) if for all $m, m', m'' \in M$, $m + m' = m + m'' \Longrightarrow m' = m''$.*

Definition 3. *Let N be a subsemimodule of a left R-semimodule M. N induces on M an R-congruence relation $\equiv_N$, know as the Bourne relation: $\forall m, m' \in M$; $m \equiv_N m' \Longleftrightarrow \exists n, n' \in N$ such that: $m + n = m' + n'$.*

- *M/N denotes the factor R-semimodule $M/\equiv_N$, and m/N denotes an element of M/N for some $m \in M$.*
- *$0/N = \overline{N} = \{m \in M/\exists n \in N/m + n \in N\}$ is the subtractive closure of N.*

Definition 4. *Let M_1 and M_2 be subsemimodules of a left R-semimodule M. If M_1 and M_2 span M (i.e $M = M_1 + M_2$), and the restriction of $\equiv_{M_2}$ to M_1 and the restriction of $\equiv_{M_1}$ to M_2 are trivial, then M is the direct sum of its subsemimodules M_1 and M_2. And we write $M = M_1 \oplus M_2$. In this case for each $m \in M$, there exists unique pair $(m_1, m_2) \in M_1 \times M_2$ such that: $m = m_1 + m_2$.*

In [6], we have the following characterization of essential subsemimodules.

Notation: The class of essential subsemimodules of a left R-semimodule M is denoted by $\mathfrak{E}_{_R M}$.

Lemma 1. *If N is a subsemimodule of a left R-semimodule, M then the following conditions are equivalent:*

(i) N is essential (or large) in M;

(ii) If ρ is a nontrivial R-congruence relation on M then the restriction of ρ to N is also nontrivial;

(iii) If m and m' are distinct elements of M then there exist distinct elements n and n' of N satisfying $n\rho_{(m,m')}n'$.

In [5], we have the following characterisation of semi-essential subsemimodules.

Notation: The class of semi-essential subsemimodules of a left R-semimodule M is denoted by $\overline{\mathfrak{C}}_{_RM}$.

Lemma 2. *A subsemimodule K of a left R-semimodule M is semi-essential if, and only if, for all $x \neq 0$, elements of M, there exists $r \in R$ such that: $0 \neq rx \in K$.*

3. New Notions of Essential: On Semi-Weakly-Essential Subsemimodules

In [5] the class $\mathfrak{C}_{_RM}$ of essential subsemimodules and the class $\overline{\mathfrak{C}}_{_RM}$ of semi-essential subsemimodules were studied and neither one of these two classes is contained in the other. In this section these two classes are embedded in a new class namely the class $w\overline{\mathfrak{C}}_{_RM}$ of semi-weakly-essential subsemimodules.

Definition 5. *A subsemimodule N of a semimodule M is said to be semi-weakly-essential in M (denoted by $N \lhd_{swe} M$), if $\forall h : M \longrightarrow M'$ R-homomorphism, the restriction of $\equiv_{Kerh}$ to N is trivial $\Longrightarrow Kerh = 0$.*

Proposition 1. *If N is a subsemimodule of a left R-semimodule M then the following conditions are equivalent:*

(i) N is semi-weakly-essential in M;

(ii) $\forall K \leq M$, the restriction of $\equiv_K$ to N is trivial $\Rightarrow K = 0$.

Proof. $(i) \Longrightarrow (ii)$. Let $K \leq M$ such that the restriction of $\equiv_K$ to N is trivial. Let $h : M \longrightarrow M/_K$ be the surjection map. So $Kerh = \overline{K}$. Since $\equiv_{\overline{K}}$ and $\equiv_K$ are equivalent and the restriction of $\equiv_K$ to N is trivial then restriction of $\equiv_{Kerh}$ to N is trivial.

N is semi-weakly-essential in M by assumption, so the restriction of $\equiv_{Kerh}$ to N is trivial $\Longrightarrow Kerh = 0$ i.e. $\overline{K} = 0$ whence $K = 0$ is trivial.

$(ii) \Longrightarrow (i)$. Indeed let $h : M \longrightarrow M'$ an R-homomorphism such that the restriction of $\equiv_{Kerh}$ to N is trivial. So by assumption we have the restriction of $\equiv_{Kerh}$ to N is trivial $\Longrightarrow Kerh = 0$. □

Proposition 2. *If N is a subsemimodule of a left R-semimodule M, then we have:*

(i) $N \lhd M \Longrightarrow N \lhd_{swe} M$.

(ii) $N \lhd_s M \Longrightarrow N \lhd_{swe} M$.

Proof.

(i) $N \lhd M \Longrightarrow N \lhd_{swe} M$?

Let $h : M \longrightarrow M'$ an R-homomorphism such that the restriction of $\equiv_{Kerh}$ to N is trivial. Suppose that $\equiv_{Kerh}$ is nontrivial on M. So, we have an R-congruence relation on M namely $\rho = \equiv_{Kerh}$ which is nontrivial on M and whose restriction on N is trivial; contradiction because $N \lhd M$. So $\equiv_{Kerh}$ is trivial on M and we deduce that $Kerh = 0$, then $N \lhd_{swe} M$.

(ii) $N \lhd_s M \Longrightarrow N \lhd_{swe} M$?

Let $h : M \longrightarrow M'$ an R-homomorphism such that the restriction of $\equiv_{Kerh}$ to N is trivial. So $Kerh \cap N = 0$. Since $N \lhd_s M$, then $Kerh \cap N = 0 \Longrightarrow Kerh = 0$, then $N \lhd_{swe} M$.

□

The previous proposition shows that the class of semi-weakly-essential subsemimodules of a semimodule M contains the two classes constituted by essential subsemimodules and semi-essential subsemimodules of M.

Since R-congruence relations in modules theory are characterized by submodules, then the three notions of essential in semimodules theory, defined in this paper, are the same for modules theory.

Recall that:

Essential: $\forall \rho \in R$-Cong(M), ρ trivial on N $\Longrightarrow$ ρ trivial on M

Semi-essential: $\forall L \leq M,\ L \cap N = 0 \Longrightarrow L = 0$

Semi-weakly essential: $\forall L \leq M,\ \equiv_L$ trivial on $N \Longrightarrow L = 0$

Here we give some examples for the different notions of "essential" in semimodules theory.

Example 1. *In this example, we propose a finite semi-weakly-essential subsemimodule which is neither essential nor semi-essential.*

Set $R = \{0, 1, \ldots, n\}$ *(with* $n \in \mathbb{N}$ *and* $n \geq 2$*) and define on R the two commutative operations* $(\oplus, \otimes)$ *as follows:*

(i) $\forall x, y \in R : x \oplus y = \min(x, y)$

(ii) $\forall x, y \in R : x \otimes y = \max(x, y)$.

$(R, \oplus, \otimes)$ *is a semiring with* $0_R = n, 1_R = 0$.

Let

$$M = \{0, \frac{1}{n}, \frac{1}{n-1}, \frac{1}{n-2}, \ldots \frac{1}{2}, 1, 1 + \frac{1}{n}, 1 + \frac{1}{n-1},$$

$$\dots 1+\frac{1}{2}, 2, 2+\frac{1}{n}, \dots n-1, (n-1)+\frac{1}{n}, \dots (n-1)+\frac{1}{2}, n\}.(M, \oplus)$$

is a commutative monoid with $0_M = n$. *Let "$\star$" be the external operation defined by :*

$$\begin{aligned} \star : & R \times M \longrightarrow M \\ & (r, m) \longmapsto r \star m = \max(r, m) \end{aligned}$$

It is clear that $(M, \oplus, \star)$ *is an R-semimodule.*

Let $a \in \{0, 1, \dots, n-1\}$. *Set* $N_a = \{a, \dots, n-\frac{1}{2}, n\}$; *then* N_a *is a subsemimodule of* M.

Now let us show that N_a *is semi-weakly-essential but is neither essential nor semi-essential.*

- *Let us prove that* N_a *is semi-weakly-essential.*

  Let $K \leq M$. *Let us prove that the restriction of* $\equiv_K$ *to* N_a *is trivial if and only if* $K = \{0_M\}$.

  ($\Longrightarrow$) If $K \neq \{0_M\}$ *then let* k *be an element of* K *such that* $k \neq 0_M$.

  Since $k \neq n$, *we have* $[n-\frac{1}{2}] \oplus k = n \oplus k$, *so* $n \equiv_K n-\frac{1}{2}$ *with* $n \neq n-\frac{1}{2}$, *contradiction. Thus* $K = \{0_M\}$.

  ($\Longleftarrow$) Trivial.

- *Let us prove that* N_a *is not semi-essential.*

  Set $L = \{(n-1)+\frac{1}{3}, n\}$. *So* L *is a subsemimodule of* M *such that* $L \neq 0$ *and* $L \cap N_a = 0$, *then* N_a *is not semi-essential.*

- *Let us prove that* N_a *is not semi-essential.*

  Let ρ *be the relation defined on* M *by:* $\forall (x, y) \in M \times M$, $x \rho y \Longleftrightarrow \max(x, 1) = \max(y, 1)$.

  ρ *is a congruence relation on* M. *Clearly* ρ *is not trivial on* M *and* ρ *is trivial on* N_a, *hence* N_a *is not essential.*

Example 2. *In this example, we give an infinite semi-weakly-essential subsemimodule which is neither semi-essential nor essential.*

Recall the semiring R *of the previous example. So* $M = [0, n]$ *is a R-semimodule. Set* $N_a = \mathbb{Q} \cap [a, n]$ *where* $a \in \mathbb{Q}^*_+$. *By a same way as in previous example, we show that* N_a *is semi-weakly-essential but is neither essential nor semi-essential.*

Example 3 ([5]). *In this example, we propose a semi-essential subsemimodule which is a semi-weakly essential subsemimodule but not an essential subsemimodule.*

Let $n \geq 1$ *be an integer. Consider the set* $R = \{r \in \mathbb{Q}^+ / r \leq n\} \cup \{-\infty\}$ *in which* $\mathbb{Q}^+$ *is the set of all nonnegative rational numbers,* $-\infty$ *is assumed to satisfy the conditions that* $-\infty \leq i$ *and* $-\infty + i = -\infty$, $\forall i \in R$. *Define on* R *the operations* $\oplus$ *and* $\otimes$ *as following:* $\forall i, j \in R$; $i \oplus j = \max(i; j)$ *and* $i \otimes j = \min(i + j; n)$.

We easily verify that $(R; \oplus; \otimes)$ *is a commutative semiring having* $-\infty$ *as additive identity.*

R is also a left R-semimodule. Put $R^* = R \setminus \{0\}$, *then* R^* *is an ideal of R.* R^* *is a semi-essential subsemimodule of R, hence* R^* *is a semi-weakly essential subsemimodule, but* R^* *is not essential.*

Example 4 ([5]). *In this example, we propose an essential subsemimodule which is a semi-weakly essential subsemimodule but not an semi-essential subsemimodule.*

Set $R = \{0, 1, a\}$ *and define on* R *the two commutative operations* $(+, \times)$ *as following:*

(i) $0_R = 0$; $1_R = 1$

(ii) $1 + 1 = 1 + a = 1$; $a + 0 = a + a = a$

(iii) $0 \times 0 = 0 \times 1 = 0 \times a = 0$; $1 \times 1 = 1$; $1 \times a = a \times a = a$.

Then $(R, +, \times, 0, 1)$ *is a commutative semiring.*

Let $M = \{0, 1, a, b\}$ *with the same operations defined in* R *and* $1_M = 1_R = 1$, $0_M = 0_R = 0$, $b + 0 = b + b = b$, $b + 1 = b + a = a$, $0 \times b = b \times a = 0$, $b \times 1 = b \times b = b$. *It's easy to see that* $(M, +, \times, 0, 1)$ *is a commutative R-semimodule.*

Now, put $N = R = \{0; 1; a\}$, *then* N *is a semi-weakly essential subsemimodule of* M, *but* N *is not semi-essential.*

In the class of semi-weakly-essential subsemimodules, we have the following result:

Proposition 3. *Let* M *be a left R-semimodule,* K *and* N *be subsemimodules of* M *such that:* $K \leq N \leq M$. *Then we have:*

(i) $K \lhd_s M \Longleftrightarrow (K \lhd_s N$ *and* $N \lhd_s M)$.

(ii) $K \lhd M \Longleftrightarrow (K \lhd N$ *and* $N \lhd M)$.

(iii) $K \lhd_{swe} M \Longrightarrow (K \lhd_{swe} N$ *and* $N \lhd_{swe} M)$.

Proof.

(i) Similar methods to [5].

(ii) $\Longrightarrow$ Suppose that $K \lhd M$.

- Let $n \neq n' \in N$. Since $N \leq M$ and $K \lhd M$, there exist $k \neq k' \in K$ such that $k\rho_{(n,n')}k'$. Therefore $K \lhd N$.
- Let $m \neq m' \in M$. Since $K \lhd M$, there exist $k \neq k' \in K$ such that $k\rho_{(m,m')}k'$. But $K \leq N$, so $k, k' \in N$, therefore $N \lhd M$.

$\Longleftarrow$ Suppose that $K \lhd N$ and $N \lhd M$. Let ρ be an R-congruence of M such that ρ is not trivial on M. Since $N \lhd M$, we have the restriction of ρ to N is not trivial and since $K \lhd N$, we have the restriction of ρ to K is not trivial, hence $K \lhd M$.

(iii) Trivial.

$\square$

Remark 2. *We have the following remark.*

Let $N \leq K \leq M$ and let ρ be and congruence relation which is trivial on K. So ρ is trivial on N but is not trivial in general on M.

In the following lemma, we prove a particular case where we can enlarge a trivial congruence relation.

Lemma 3. *Let M be a left R-semimodule. Suppose that*

$$K_1 \leq M_1 \leq M; K_2 \leq M_2 \leq M; L_1 \leq M_1; L_2 \leq M_2$$

and $M = M_1 \oplus M_2$. Then:

(i) $\equiv_{L_1}$ *trivial on* $K_1 \Longrightarrow \equiv_{L_1}$ *trivial on* $K_1 \oplus K_2$.

(ii) $\equiv_{L_1 \oplus L_2}$ *trivial on* $K_1 \oplus K_2 \Longrightarrow \equiv_{L_1}$ *trivial on* K_1 *and* $\equiv_{L_2}$ *trivial on* K_2.

Proof.

(i) Let $k_1 + k_2$, $k_1' + k_2'$ be two elements of $K_1 \oplus K_2$.

We have $k_1 + k_2 \equiv_{L_1} k_1' + k_2' \Longrightarrow k_1 + k_2 + l_1 = k_1' + k_2' + l_1' \Longrightarrow k_1 + l_1 \equiv_{M_2} k_1' + l_1'$ and $k_2 \equiv_{M_1} k_2'$. Now by definition of $M = M_1 \oplus M_2$ and by hypothesis we have: $k_1 + l_1 \equiv_{M_2} k_1' + l_1' \Longrightarrow k_1 = k_1'$ and $k_2 \equiv_{M_1} k_2' \Longrightarrow k_2 = k_2'$. So $k_1 + k_2 = k_1' + k_2'$ and therefore $\equiv_{L_1}$ is trivial on $K_1 \oplus K_2$.

(ii) Suppose that $\equiv_{L_1 \oplus L_2}$ trivial on $K_1 \oplus K_2$.

Let k_1, k_1' be two elements of K_1 such that $k_1 \equiv_{L_1} k_1'$.

So there exist $l_1, l_1' \in L_1$ such that $k_1 + l_1 = k_1' + l_1'$. But $k_1 = k_1 + 0 \in K_1 \oplus K_2$, $l_1 = l_1 + 0 \in L_1 \oplus L_2$, idem we have $k_1' \in K_1 \oplus K_2$, $l_1 \in L_1 \oplus L_2$.

Since $\equiv_{L_1 \oplus L_2}$ trivial on $K_1 \oplus K_2$, we deduce that $k_1 = k_1'$. Thus $\equiv_{L_1}$ is trivial on K_1 and a same way shows that $\equiv_{L_2}$ is trivial on K_2.

□

Proposition 4. *Let M be a left R-semimodule. Suppose that $K_1 \leq M_1 \leq M$; $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$.*

Then $(K_1 \oplus K_2) \lhd_{swe} (M_1 \oplus M_2) \Longrightarrow (K_1 \lhd_{swe} M_1$ and $K_2 \lhd_{swe} M_2)$

Proof. Let us show that $K_1 \lhd_{swe} M_1$. Let $L_1 \leq M_1$ such that $\equiv_{L_1}$ is trivial on K_1. By Lemma 3*(i)* we have $\equiv_{L_1}$ trivial on $K_1 \oplus K_2$. Now $\equiv_{L_1}$ is trivial on $K_1 \oplus K_2$ and $K_1 \oplus K_2 \lhd_{swe} M_1 \oplus M_2$ so $L_1 = 0$. □

Proposition 5. *Let M be a left R-semimodule. Suppose that $K_1 \leq M_1 \leq M$; $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$.*

Then $(K_1 \oplus K_2) \lhd_s (M_1 \oplus M_2) \Longleftrightarrow (K_1 \lhd_s M_1$ and $K_2 \lhd_s M_2)$.

Proof. $\Longrightarrow$. Suppose for example $K_1\ _s\ M_1$. Then there exists a subsemimodule $L_1 \neq 0$ of M_1 such that: $L_1 \cap K_1 = 0$. So let us prove that $L_1 \cap (K_1 + K_2) = 0$.

Let $l_1 \in L_1 \cap (K_1 + K_2)$. There exists $(k_1; k_2) \in K_1 \times K_2$ such that: $l_1 = k_1 + k_2$. We have: $l_1 \in L_1 \leq M_1$; $k_1 \in K_1 \leq M_1$ and $k_2 \in K_2 \leq M_2$, so by the direct sum $M_1 \oplus M_2$ we deduce that $k_2 = 0$ and $l_1 = k_1$; hence $l_1 \in L_1 \cap K_1 = 0$, whence $l_1 = 0$. Consequently $L_1 \cap (K_1 \oplus K_2) = 0$ which contradicts the fact that $(K_1 \oplus K_2) \lhd_s (M_1 \oplus M_2)$. So $K_1 \lhd_s M_1$. A same argument prove that $K_2 \lhd_s M_2$.

$\Longleftarrow$. Suppose that $K_i \lhd_s M_i$ for all $i \in \{1,2\}$.

Let $0 \neq x \in M_1 \oplus M_2$. Then, there exists $(0,0) \neq (x_1, x_2) \in M_1 \times M_2$ such that:

$0 \neq x = x_1 + x_2$. Without lost of generality we can suppose that: $0 \neq x_1 \in M_1$. Since $K_1 \lhd_s M_1$ then from Lemma 1, there exists a $r_1 \in R$ such that: $r_1 x_1 \in K_1$ and $r_1 x_1 \neq 0$.

- If $r_1 x_2 \in K_2$ then $r_1 x_1 + r_1 x_2 \in K_1 + K_2$ therefore $r_1(x_1 + x_2) \in K_1 \oplus K_2$ with $r_1(x_1 + x_2) \neq 0$ because if $r_1 x_1 + r_1 x_2 = 0$, then by the sum direct we have $r_1 x_1 = 0$, which is absurd. Consequently $(K_1 \oplus K_2) \lhd_s (M_1 \oplus M_2)$.

- If $r_1 x_2$ isn't in K_2 then there exists $r_2 \in R$ such that: $0 \neq r_2 r_1 x_2 \in K_2$. We have $r_2 r_1 x_1 \in K_1$ then $r_2 r_1(x_1 + x_2) \in K_1 \oplus K_2$. If we put $r = r_2 r_1$ then there exists $r \in R$ such that: $r(x_1 + x_2) \in K_1 \oplus K_2$ with $r(x_1 + x_2) \neq 0$ because if $r x_1 + r x_2 = 0$, then by the direct sum we have $r x_1 = 0$, which is absurd. Therefore $(K_1 \oplus K_2) \lhd_s (M_1 \oplus M_2)$. Thus $K_i \lhd_s M_i$ for all $i \in \{1; 2\} \Longrightarrow (K_1 \oplus K_2) \lhd_s (M_1 \oplus M_2)$.

□

Definition 6. *Let N be a subsemimodule of a left R-semimodule M. A subsemimodule N' of M is called M-w-complement of N if the restriction of $\equiv_{N'}$ to N is trivial and N' is maximal with this property.*

Proposition 6.

(i) Every subsemimodule N of a left R-semimodule M has a M-w-complement.

(ii) If N' is a M-w-complement of a subsemimodule N of M and if $N \oplus N'$ exists then: $N \oplus N' \lhd_{swe} M$.

Proof.

(i) Let $\$ = \{A \leq M /$ the restriction of $\equiv_A$ to N, is trivial$\}$. $0 \in \$$, then $\$ \neq \emptyset$. $(\$; \subseteq)$ is an ordered poset. It is easy to show that $\$$ is a non-empty inductive poset. Therefore $\$$ has at least one maximal element N'. And N' is a M-ω-complement of N.

(ii)
- If $N = 0$ then $N' = M$, and so $N \oplus N' \lhd_{swe} M$.
- If $N \neq 0$, then let $0 \neq L \leq M$ such that the restriction of $\equiv_L$ to $N \oplus N'$ is trivial. Let us show that $\equiv_L$ is trivial on M.
  First of all we show that the restriction of $\equiv_{N'+L}$ to N is trivial. Let $n_1, n_2 \in N$ such that $n_1 \equiv_{N'+L} n_2$.

$n_1 \equiv_{N'+L} n_2 \Longrightarrow \exists n_1', n_2' \in N, l_1, l_2 \in L$ such that $n_1 + n_1' + l_1 = n_2 + n_2' + l_2$. So by hypothesis and by the direct sum $N \oplus N'$ we deduce that $n_1 = n_2$. Therefore the restriction of $\equiv_{N'+L}$ to N is trivial. We deduce that $N' + L \in \$$. Since N' is maximal, then $N' + L = N'$ or $N' + L = M$. We have $N' + L \neq M$, otherwise, since the restriction of $\equiv_{N'+L}$ to N is trivial, we obtain $(N' + L) \cap N = M \cap N = N = 0$ which is a contradiction. So $N' + L = N'$ and consequently $L \subseteq N'$. Moreover we have $(N \oplus N') \cap L = 0$ because the restriction of $\equiv_L$ to $N \oplus N'$ is trivial; so $N' \cap L = 0$. Thus we have $L \subseteq N'$ and $N' \cap L = 0$, so $L = 0$.

We conclude that $N \oplus N' \lhd_{swe} M$

□

Definition 7. *Let N be a subsemimodule of a left R-semimodule M. A subsemimodule N' of M is called M-s-complement of N if the restriction of $\equiv_N$ to N' is trivial, the restriction of $\equiv_{N'}$ to N is trivial and N' is maximal with this property.*

Proposition 7.

(i) Every subsemimodule N of a left R-semimodule M has a M-s-complement.

(ii) If M is a cancellative left R-semimodule and N a subsemimodule of M, then:

(a) Every M-w-complement of N is a M-s-complement of N

(b) If N' is a M-s-complement of N then: $N \oplus N' \lhd_{swe} M$.

Proof. (i). Similar to the above proof.

For *(2)(a)* it suffices to see that, if M is cancellative and if the restriction of $\equiv_{N'}$ to N is trivial, then the restriction of $\equiv_N$ to N' is trivial.

(2)(b). Similar to the above proof by using the fact that M is cancellative. □

4. Applications: On Weakly Co-Hopfian Semimodules

We have three differents notions of essential subsemimodules, so we can define three differents types of weakly co-hopfian semimodules

4.1. On Weakly Co-Hopfian Semimodules of Type 1

Definition 8. *A nonzero left R-semimodule $_RM$ is said to be weakly co-hopfian-1 (denoted by wch-1) if every monomorphism $f : M \to M$ is semi-weakly-essential i.e $f(M) \lhd_{swe} M$.*

Proposition 8. *The following are equivalent conditions on a left R-semimodule M.*

(i) M is weakly co-hopfian-1.

(ii) $\forall \{0\} \neq N \leq M$, if $g \in End(M)$ is injective, then the restriction of $\equiv_N$ to $g(M)$ is not trivial.

Proof. *(i)⇒(ii)* Let $\{0\} \neq N \leq M$ and let g be an injective endomorphism of M. By assumption $g(M)$ is semi-weakly-essential in M, so, since $N \leq M$ and $N \neq \{0\}$, we deduce that the restriction of $\equiv_N$ to $g(M)$ is not trivial.

(ii)⇒(i) Let $g : M \to M$ be an injective endomorphism of M and let N be a subsemimodule of M such that the restriction of $\equiv_N$ to $g(M)$ is trivial. Suppose that $N \neq \{0\}$. So by hypothesis the restriction of $\equiv_N$ to $g(M)$ is not trivial which contradicts the assumption. So $N = \{0\}$ whence $g(M)$ is semi-weakly-essential and so M is weakly co-hopfian-1. □

Proposition 9. *For a left R-semimodule M, consider the following statements.*

(i) *M is weakly co-hopfian-1.*

(ii) *For any left R-semimodule N, if there is an R-monomorphism $M \dot{\oplus} N \to M$ then $N = \{0\}$.*

Then $(i) \Longrightarrow (ii)$ and if M is cancellative we have $(i) \Longleftrightarrow (ii)$.

NB:

- A module M which verifies (ii) is said Dedekind finite.
- A semimodule which is finite verifies (ii).
- In the sequel, a semimodule M which verifies (ii) is said a F-semimodule.

Proof. *(i)⇒(ii)* Suppose that $f : M \dot{\oplus} N \to M$ is a monomorphism where N is a left R-semimodule. Let $M \xrightarrow{i} M \dot{\oplus} N \xrightarrow{f} M$ be an sequence where i is the canonical injection. Then $f \circ i$ is a monomorphism, hence $(f \circ i)(M)$ is semi-weakly-essential in M by assumption.

In an easy way one can show that $f(N) = f(0 \dot{\oplus} N) = 0$.

Since f is monic, $f(N) = 0 \Longrightarrow N = 0$.

(ii)⇒(i) By hypothesis we deduce this property **(P)**: For any left R-semimodule N, if $M \dot{\oplus} N \to M$ is an semi-weakly-essential monomorphism then $N = \{0\}$.

Now let $g : M \to M$ be a monomorphism with non semi-weakly-essential image. Then, by Proposition 7 there exists a nonzero subsemimodule K with $g(M) \oplus K \lhd_{swe} M$.

Define $f : M \dot{\oplus} K \to M : (m, k) \mapsto g(m) + k$. By the direct sum $g(M) \oplus K$ and the fact that M is cancellative, f is monic. We have $g(M) \oplus K \subseteq f(M \dot{\oplus} K) \subseteq M$, hence by Proposition 3 $f(M \dot{\oplus} K) \lhd_{swe} M$ (because $g(M) \oplus K \lhd_{swe} M$). So f is an semi-weakly-essential monomorphism contradicting the property **(P)** . Hence $g(M) \lhd_{swe} M$ as desired. □

Proposition 10. *The following are equivalent conditions on a left R-semimodule M.*

(i) *M is weakly co-hopfian-1.*

(ii) *M is a F-semimodule and the image of any injective endomorphism of M is either semi-weakly-essential or a proper direct summand.*

Proof. *(i)⇒(ii)* By hypothesis and by the previous proposition, we have: for any left R-semimodule N, if there is an R-monomorphism $M \dot{\oplus} N \to M$ then $N = \{0\}$.

Now suppose that $M \dot{\oplus} K \cong M$, then there is a monomorphism $f : M \dot{\oplus} K \to M$ and we have $K = \{0\}$, hence M is a F-semimodule. By hypothesis the image of any injective endomorphism of M is in fact a semi-weakly-essential subsemimodule.

(ii)⇒(i) Let $g : M \to M$ be an injective endomorphism and suppose that $g(M)$ is not semi-weakly-essential in M. Then by hypothesis $g(M)$ is a proper direct summand of M. So there exits a nonzero subsemimodule K of M such that $g(M) \oplus K = M$.

Define $f : M \dot{\oplus} K \to M : (m, k) \mapsto g(m) + k$. Then f is a monomorphism.

We have $M = g(M) \oplus K \subseteq f(M \dot{\oplus} K) \subseteq M$, hence $f(M \dot{\oplus} K) = M$, whence f is surjective. Therefore there is an isomorphism $M \dot{\oplus} K \cong M$ which contradicts the fact that M is a F-semimodule. Thus $g(M) \lhd_{swe} M$. □

Proposition 11.

(i) The following are equivalent conditions on a left R-semimodule M.

(a) M is weakly co-hopfian-1.

(b) There exists a subsemimodule K of M such that $g(K) \lhd_{swe} M$ for all injective $g \in End(M)$.

(ii) A direct summand of a weakly co-hopfian-1 semimodule is weakly co-hopfian-1.

Proof.

(i) *(b)⇒(a)* Trivial by Proposition 3.

(a)⇒(b) Trivial.

(ii) Suppose that M is a weakly co-hopfian-1 semimodule.

Let $M = N \oplus K$ and let $f : N \longrightarrow N$ be an injective endomorphism of N. Then the map

$$f \oplus Id_K : M = N \oplus K \to M = N \oplus K$$

defined by $n + k \mapsto (f \oplus Id_K)(n + k) = f(n) + k$ is an injective endomorphism of M and $(f \oplus Id_K)(M) \lhd_{swe} M$. Then $(f \oplus Id_K)(M) \lhd_{swe} M \Rightarrow f(N) \oplus K \lhd_{swe} N \oplus K$ and by Proposition 4 $f(N) \lhd_{swe} N$, therefore N is weakly co-hopfian-1.

□

4.2. On Weakly Co-Hopfian Semimodules of Type 2

Definition 9. *A nonzero left R-semimodule ${}_RM$ is said to be weakly co-hopfian-2 (denoted by wch-2) if every monomorphism $f : M \to M$ is semi-essential i.e $f(M) \lhd_s M$ (see definition in the introduction).*

Proposition 12. *If M is wch-2, then M is wch-1.*

Proof. By the Proposition 2. □

Proposition 13. *The following are equivalent conditions on a left R-semimodule M.*

(i) *M is weakly co-hopfian-2.*

(ii) *For every monomorphism $f : M \to M$, if x is an nonzero element of M, then there exists $r \in R$ such that $0 \neq rx \in f(M)$.*

(iii) *Injective endomorphisms of M map semi-essential subsemimodules to semi-essential subsemimodules.*

Proof. *(i)$\Longleftrightarrow$(ii) By Definition 9 and Lemma 2.*
(iii)$\Longleftrightarrow$(i) is trivial.
(i)$\Longleftrightarrow$(iii) Let $g : M \to M$ be an injective endomorphism of M, and let K be a semi-essential subsemimodule of M. Let us prove that $g(K) \lhd_s M$.

We have $g(K) \leq g(M) \leq M$ and by hypothesis $g(M) \lhd_s M$, so, according to the Proposition 3, to obtain $g(K) \lhd_s M$, it suffices to prove that $g(K) \lhd_s g(M)$.

Let $x \in g(M)$. So there exists $m \in M$ such that $x = g(m)$. Since $K \lhd_s M$, we deduce that there exists an $r \in R$ such that $rm \in K$. So $g(rm) = rg(m) = rx \in g(K)$; thus $g(K) \lhd_s g(M)$. □

Proposition 14.

(i) *A direct summand of a weakly co-hopfian-2 semimodule is weakly co-hopfian-2.*

(ii) *Let $M = M_1 \oplus M_2$ such that each M_i is fully invariant. Then M is weakly co-hopfian-2 if and only if so is each M_i.*

Proof. *(i)* Suppose that M is a weakly co-hopfian-2 semimodule.

Let $M = N \oplus K$ and let $f : N \longrightarrow N$ be an injective endomorphism of N. Then the map $f \oplus Id_K : M = N \oplus K \to M = N \oplus K$ defined by $n + k \mapsto (f \oplus Id_K)(n + k) = f(n) + k$ is an injective endomorphism of M and so $(f \oplus Id_K)(M) \lhd_s M$. Since $(f \oplus Id_K)(M) \leq f(N) \oplus K$, we deduce by Proposition 3 that $f(N) \oplus K \lhd_s N \oplus K$ and by Proposition 5 that $f(N) \lhd_s N$, therefore N is weakly co-hopfian-2.

(2)$\Longrightarrow$) By *(i)*.

$\Longleftarrow$) Let $f : M = M_1 \oplus M_2 \longrightarrow M = M_1 \oplus M_2$ be an injective endomorphism of $M = M_1 \oplus M_2$.

We have $f(M_1) \oplus f(M_2) \subseteq f(M_1 \oplus M_2) \subseteq M_1 \oplus M_2$. By assumption we have $f(M_1) \lhd_s M_1$ and $f(M_2) \lhd_s M_2$, and by the Proposition 5, we obtain $f(M_1) \oplus f(M_2) \lhd_s M_1 \oplus M_2$ and consequently $f(M_1 \oplus M_2) \lhd_s M_1 \oplus M_2$. □

4.3. On Weakly Co-Hopfian Semimodules of Type 3

Definition 10. *A nonzero left R-semimodule ${}_RM$ is said to be weakly co-hopfian-3 (denoted by wch-3) if every monomorphism $f : M \to M$ is essential i.e $f(M) \lhd M$.*

Proposition 15. *The following are equivalent conditions on a left R-semimodule M.*

(i) M is weakly co-hopfian-3.

(ii) If m and m' are distinct elements of M then for every monomorphism $f : M \to M$, there exist distinct elements $f(m_1)$ and $f(m_2)$ of $f(M)$ satisfying $f(m_1)\rho_{(m,m')}f(m_2)$.

Proof. By Definition 10 and Lemma 1. □

Proposition 16. *If M is wch-3, then M is wch-1.*

Proof. By Proposition 2. □

By a same way as above, we show the following results.

Proposition 17.

(i) The following are equivalent conditions on a left R-semimodule M.

(a) M is weakly co-hopfian-3.

(b) There exists a subsemimodule K of M such that $g(K) \lhd M$ for all injective $g \in End(M)$.

(ii) A direct summand of a weakly co-hopfian-3 semimodule is weakly co-hopfian-3.

(iii) Let $M = M_1 \oplus M_2$ such that each M_i is fully invariant.Then M is weakly co-hopfian-3 if and only if so is each M_i.

Proposition 18. *Any R—simple semimodule is wch-3*

Proof. Obvious. □

4.4. Examples of Co-Hopfian Semimodules

Here we give examples of co-hopfian semimodules.

Example 5. *Recall the semimodule $(M, \oplus, \star)$ of Example 1. Let us show that M is wch-1, wch-2 and wch-3.*

Let $g : M \longrightarrow M$ be an injective endomorphism of M. So g verifies the following property: $\forall m, m' \in M$: $m \leq m' \Longrightarrow g(m) \leq g(m')$.

This property with the injectivity of g make that there exists an unique injective endomorphism of M, namely the identity map on M. We deduce that the image of any injective endomorphism of M is both semi-weakly-essential,semi-essential and essential and consequently that M is wch-1, wch-2 and wch-3.

Example 6. *Let $\mathbb{N}$ be the set of natural integers and put $(\mathbb{N}, \oplus, \odot)$ the semiring where $\forall a, b \in \mathbb{N}$, $a \oplus b = gcd(a, b)$ is the greatest common divisor of a, b and $a \odot b = lcm(a, b)$ is the least common divisor of a, b.*

So $\mathbb{N}$ is a left $\mathbb{N}$-semimodule. Every subsemimodule of $\mathbb{N}$ is in form $< p >= p\mathbb{N} = \{pn/n \in \mathbb{N}\}$ where $p \in \mathbb{N}$ and pn is the usual product of p and n (see [3]).

Every subsemimodule $p\mathbb{N}$ of $\mathbb{N}$ is semiessential on $\mathbb{N}$ because $\forall n \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $rn \in p\mathbb{N}$.

So the image of any injective endomorphism of $\mathbb{N}$ is of the form $p\mathbb{N}$, and therefore we deduce that $\mathbb{N}$ is weakly-co-hopfian-2.

Example 7. *Let $R = \{0; 1\}$ be a set. $(R; +; \times)$ and $(R; +; \otimes)$ (where for all $i, j \in R$, $i + j = \max(i, j)$, $i \times j = 0$, $i \otimes j = 0$ except $1 \otimes 1 = 1$) are semirings (see [16]). It's easy to see that $({}_R R; +; \times)$ and $({}_R R; +; \otimes)$ are two R-simples semimodules, therefore they are wch-3 by the Proposition 18.*

References

[1] J. Abuhlail. Semicorings and semicomodules. Research Report S-1, Department of Mathematics, SUNY at Albany, 2008.

[2] F. W. Anderson and K. R. Fuller. *Rings and categories of modules*. Springer-Verlag, New York, 1975.

[3] J. N. Chaudhari and K. J. Ingale. A note on strongly euclidean semirings. *International Journal of Algebra*, 6(6):271–275, 2012.

[4] B. Eckman and A. Schpof. Uber injective moduln. *Arch der math*, 4:75–78, 1953.

[5] L. Fall, J. R. Tsiba, and D. Sow. On semi-essential subsemimodules. *JP Journal of Algebra*, 22(2):187–203, 2011.

[6] J. S. Golan. *The Theory of Semirings with Applications to Mathematics and Theoretical Computer Science*. Kluwer Academic Publishers, Dordrecht, 1999.

[7] R. E. Johnson. The extended centralizer of module. *Proc. Amer. Math. Soc.*, 2:891–895, 1951.

[8] T. Y. Lam. *Lectures on Modules and Rings*. Springer-Verlag, New York, 1998.

[9] M. Takahashi. On the bordism categories ii (elementary properties of semimodules). *Mathematics Seminar notes*, 9:445–530, 1981.

[10] M. Takahashi. On the bordism categories iii (fonctors hom and ⊗ for semimodules). *Mathematics Seminar notes*, 10:211–236, 1982.

[11] M. Takahashi and H. X. Wang. On epimorphisms of semimodules. *Kobe J. Math.*, 6(2):297–298, 1989.

[12] J. R. Tsiba, L. Fall, and D. Sow. Introduction to s-hopfian and s-cohopfian semimodules. *JP Journal of Algebra, Number Theory and Applications*, 22(1):81–109, 2011.

[13] J. R. Tsiba and D. Sow. On generator and projective semimodules. *International Journal of Algebra*, 4(2):1153–1167, 2010.

[14] N. X. Tuyen and T. G. Nam. On projective covers of semimodules in the category - cssmod and their applications. *Southeast Asian Bulletin of Mathematics*, 31:363–377, 2007.

[15] R. Wisbauer. *Foundations of Module and Ring Theory*. Gordon and Breach, Philadelphia, 1991.

[16] J. Zumbraegel. Classification of finite congruence-simple semiring with zero. *Journal of Algebra and its applications*, 7(3):363–377, 2008.