EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 4, 2016, 443-451
ISSN 1307-5543 – www.ejpam.com

Generalized Fixed Point Theorems in Partial Metric Spaces

Mehmet Kir [1,*] and Hukmi Kiziltunc [2]

[1] *Department of Civil Engineering, Faculty of Engineering Şırnak University, Şırnak, Turkey*
[2] *Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey*

Abstract. This paper consist of some generalized fixed point theorems in partial metric spaces. The concept of T_F -contractive mappings are introduced in partial metric space and thus, a generalization of Banach's, Kannan's, Chatterjea's, Bianciari's fixed point theorems are established for concept of partial metric space.

2010 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Fixed point, Kannan fixed point theorem, Chatterjea fixed point theorem, Contraction mappings, T_F -contraction, Partial metric space

1. Introduction and Preliminaries

The notion of partial metric space was introduced by Matthews in 1992 [2]. A partial metric is a extension of metric by replacing the condition $d(x,x)=0$ of the (usual) metric with the inequality $d(x,x) \leq d\left(x,y\right)$ for all x,y. Also, this concept provide to study denotational semantics of dataflow networks [1–4].

Matthews gave some basic definitions and properties on partial metric space such as Cauchy sequence, Convergent sequence etc. One of the most interesting properties of this space is the self-distance $(p(x,x))$ for any point may not be zero. He also introduced the first fixed point theorem that re-named partial contraction mapping theorem.

Due to importance of the fixed point theory it is very interesting to study fixed point theorems on different concepts. Recently, many mathematicians have studied generalized fixed point theorems that arising from concept of partial metric space and the authors obtained some useful results [8, 9].

Now, we give some basic structures and results on the concept of partial metric space.

Definition 1 ([1]). *Let X be a nonempty set and $p: X \times X \to [0,\infty)$ be such that for all $x,y,z \in X$ the followings are satisfied:*

*Corresponding author.

Email addresses: `mehmetkir04@gmail.com` (M. Kir), `hukmu@atauni.edu.tr` (H. Kiziltunc)

P1) $x = y$ *if and only if* $p(x,x) = p(y,y) = p(x,y)$,

P2) $p(x,x) \leq p(x,y)$,

P3) $p(x,y) = p(y,x)$,

P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then, p *is called partial metric on* X *and the pair* (X,p) *is called partial metric space.*

Remark 1. *It is clear that if* $p(x,x) = 0$, *then* $x = y$. *But, on the contrary* $p(x,x)$ *need not be zero.*

Example 1. *[[8]]*

a) *Let* $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$ *and define* $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. *Then,* (X,p) *is a partial metric space.*

b) *Let* $X = [0,\infty)$ *and define* $p(x,y) = \max\{x,y\}$. *Then,* (X,p) *is a partial metric space.*

Note that each partial metric p that defined on X generates a T_0 topology τ that has a base the family of open balls $\{B(x,\epsilon) : x \in X, \epsilon > 0\}$ such that

$$B(x,\epsilon) = \{y \in X : p(x,y) < \epsilon + p(x,x)\}$$

Remark 2. *[9] Let* (X,p) *be a partial metric space.*

M1) *The function* $d^s : X \to X$ *defined by*

$$d^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a (usual) metric on X *and* (X,d^s) *is a (usual) metric space.*

M2) *The function* $d^M : X \to X$ *defined by*

$$d^M(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}$$

is a (usual) metric on X *and* (X,d^M) *is a (usual) metric space.*

Corollary 1 ([9])**.** *Let* (X,p) *be a partial metric space. Then,* d^s *and* d^M *are equivalent metric on* X. *Furthermore, if we take* (X,p) *as in Example 1, part* ***b)****, we obtain the following equality;*

$$d^M(x,y) = d^s(x,y) = |x - y|.$$

Definition 2 ([8])**.** *Let* (X,p) *be a partial metric space.*

(i) *A sequence* $\{x_n\}$ *in* (X,p) *converges to* $x \in X$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

(ii) *A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite).*

(iii) *A partial metric space is called complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ, to a point $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x,x)$.*

Lemma 1. *[[8]] Let (X,p) be a partial metric space.*

a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is Cauchy sequence in (X,d^s).*

b) *(X,p) is complete if and only if (X,d^s) is complete. Moreover, $\lim_{n\to\infty} d^s(x_n,x) = 0$ if and only if*

$$\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_m) = p(x,x).$$

Moradi and Beiranvand [12] introduced T_F - type contraction mappings as follows:

Definition 3 ([12])**.** *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be graph closed if for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = a$ then for some $b \in X, Tb = a$.*

Definition 4 ([12])**.** *Let (X,d) be a metric space and $f, T : X \to X$ be two functions. The mapping f is said to be a T_F-contraction if there exists $\alpha \in [0,1)$ such that for all $x, y \in X$*

$$F(d(Tfx, Tfy)) \leq \alpha F(d(Tx, Ty)) \tag{1}$$

where

1) *$F : [0,\infty) \to [0,\infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.*

2) *T is one to one and graph closed.*

Theorem 1 ([12])**.** *Let (X,d) be a complete metric space. If $f : X \to X$ is a T_F-contraction mapping then f has a unique fixed point in complete metric space (X,d).*

Kir and Kiziltunc [13] introduced T_F -contractive conditions for Kannan fixed point theorem and Chatterjea fixed point theorem.

Definition 5 ([10, 11])**.** *Let (X,d) be a metric space.*

(i) *A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ is also convergent.*

(ii) *T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.*

For instance, the mappings $Tx = x, Tx = \ln x (x > 0)$ are sequentially convergent on the metric space $(\mathbb{R}, |\cdot|)$. The mapping $Tx = x^2$ is not sequentially convergent on the metric space $(\mathbb{R}, |\cdot|)$ but it is subsequentially convergent.

In this study, we aim to introduce T_F type fixed point theorems in partial metric space.

2. T_F- Type Contractive Conditions for Banach's, Kannan's and Chatterjea's Fixed Point Theorems

Theorem 2. *Let (X,p) be a complete partial metric space and $T,f : X \to X$ be mappings such that T is one to one and subsequentially convergent. If for all $k \in [0,1)$ and $x,y \in X$*

$$F\left(p\left(Tfx,Tfy\right)\right) \le kF\left(p\left(Tx,Ty\right)\right) \tag{2}$$

where $F : [0,\infty) \to [0,\infty)$ is nondecreasing continuous and $F(t)=0$ if and only if $t=0$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0, n = 1,2,3,\cdots$

$$\begin{aligned} F\left(p\left(Tx_n,Tx_{n+1}\right)\right) =& F\left(p\left(Tfx_{n-1},Tfx_n\right)\right)\\ \le& kF\left(p\left(Tx_{n-1},Tx_n\right)\right)\\ &\vdots\\ \le& k^{n-1}F\left(p\left(Tx_0,Tx_1\right)\right). \end{aligned} \tag{3}$$

Also, for all $m,n \in \mathbb{N}$, for $m > n$, we have

$$\begin{aligned} F\left(p\left(Tx_n,Tx_m\right)\right) =& F\left(p\left(Tf^n x_0,Tf^m x_0\right)\right)\\ \le& k^n F\left(p\left(Tx_0,Tf^{m-n}x_0\right)\right). \end{aligned} \tag{4}$$

Let $m,n \to \infty$ in (4), we obtain

$$F\left(p\left(Tx_n,Tx_m\right)\right) \to 0^+ \text{ as } m,n \to \infty.$$

As F is continuous, we obtain

$$\lim_{m,n\to\infty} p\left(Tx_n,Tx_m\right) = 0. \tag{5}$$

Thus, we see that $\{Tx_n\}$ is a Cauchy sequence in (X,p). From Lemma 1, we get that $\{Tx_n\}$ is Cauchy sequence in (X,d^s). Since (X,p) is a complete partial metric space then (X,d^s) is also complete metric space and there exists $v \in X$ such that $\{Tx_n\}$ converges to $v \in X$.

Note that T is subsequentially convergent, then there exists an $u \in X$ such that

$$\lim_{k\to\infty} p\left(x_{n(k)},u\right) = \lim_{k\to\infty} p(u,u).$$

Also, T is continuous and $x_{n(k)} \to u$, therefore

$$\lim_{k\to\infty} Tx_{n(k)} = Tu \text{ and } \lim_{k\to\infty} p\left(Tx_{n(k)},Tu\right) = p(Tu,Tu).$$

Since, $\{Tx_{n(k)}\}$ is a subsequence of $\{Tx_n\}$, so we obtain $Tu = v$.

Also,

$$d^s\left(Tx_n, Tu\right) = 2p\left(Tx_n, Tu\right) - p\left(Tx_n, Tx_n\right) - p\left(Tu, Tu\right). \tag{6}$$

Let $n \to \infty$ in (6), we have

$$\lim_{n\to\infty} d^s\left(Tx_n, Tu\right) = 0.$$

Consider Lemma 1/part b) and (5) we hold

$$\lim_{n\to\infty} p\left(Tx_n, Tu\right) = \lim_{m,n\to\infty} p\left(Tx_n, Tx_m\right) = p\left(Tu, Tu\right) = 0.$$

Now, we will show that $u \in X$ is a fixed point of f. Indeed, as F is continuous

$$\begin{aligned} F\left(p\left(Tfu, Tx_{n+1}\right)\right) =& F\left(p\left(Tfu, Tfx_n\right)\right) \\ \leq& kF\left(p\left(Tu, Tx_n\right)\right). \end{aligned} \tag{7}$$

Let $n \to \infty$ in (7), we obtain

$$F(p\left(Tfu, Tu\right)) \leq 0$$

this implies that $p\left(Tu, Tfu\right) = 0$ and hence $Tu = Tfu$. Also, T is one to one, we obtain $fu = u$.

Now, we show that the fixed point is unique. Assume u' is an other fixed point of f then, we have $fu' = u'$ and

$$\begin{aligned} F(p\left(Tu, Tu'\right)) =& F(p\left(Tfu, Tfu'\right)) \\ \leq& kF\left(p\left(Tu, Tu'\right)\right) \end{aligned} \tag{8}$$

The inequality (8) is contradiction unless $p\left(Tu, Tu'\right) = 0$. Thus, $Tu = Tu'$ with consideration T is one to one, we obtain the fixed point is unique.

Also, if we take T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we conclude that

$$\lim_{n\to\infty} x_n = u$$

this shows that $\left\{x_n\right\}$ converges to the fixed point of f. □

In Theorem 2, if we consider F and T as identity, we get the following result given by Matthews [2].

Corollary 2. *Let $\left(X, p\right)$ be a complete partial metric space and $f : X \to X$ be mapping. If $\alpha \in [0,1)$ and $x, y \in X$,*

$$p\left(fx, fy\right) \leq \alpha p\left(x, y\right) \tag{9}$$

then, f has a unique fixed point.

Theorem 3. *Let (X,p) be a complete partial metric space and $T,f : X \to X$ be mappings such that T is one to one, continuous and subsequentially convergent (or graph closed). If for each $\beta \in \left[0,\frac{1}{2}\right)$ and $x,y \in X$, we have*

$$F\left(p\left(Tfx,Tfy\right)\right) \leq \beta\left[F\left(p\left(Tx,Tfx\right)\right)+F\left(p\left(Ty,Tfy\right)\right)\right] \tag{10}$$

where $F:[0,\infty)\to[0,\infty)$ is nondecreasing continuous and $F(t)=0$ if and only if $t=0$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0, n = 1,2,3,\ldots$

$$\begin{aligned} F\left(p\left(Tx_n,Tx_{n+1}\right)\right) =& F\left(p\left(Tfx_{n-1},Tfx_n\right)\right) \\ &\leq \beta\left[F\left(p\left(Tx_{n-1},Tx_n\right)\right)+F\left(p\left(Tx_n,Tx_{n+1}\right)\right)\right] \end{aligned}$$

therefore, we have

$$F\left(p\left(Tx_n,Tx_{n+1}\right)\right) \leq \frac{\beta}{1-\beta}F\left(p\left(Tx_{n-1},Tx_n\right)\right).$$

Also, we obtain that

$$F\left(p\left(Tx_n,Tx_{n+1}\right)\right) \leq \left(\frac{\beta}{1-\beta}\right)^n F\left(p\left(Tx_0,Tx_1\right)\right). \tag{11}$$

Let $n\to\infty$ in (11), we obtain that

$$F\left(p\left(Tx_n,Tx_{n+1}\right)\right) \to 0^+ \text{ as } n\to\infty.$$

Again using (11), for all $m,n \in \mathbb{N}$, taking $m > n$, we have

$$F\left(p\left(Tx_n,Tx_m\right)\right) \leq \left(\frac{\beta}{1-\beta}\right)^n F\left(p\left(Tx_0,Tf^{m-n}x_0\right)\right) \tag{12}$$

Letting $m,n\to\infty$ in (12), we have

$$F\left(p\left(Tx_n,Tx_m\right)\right) \to 0^+ \text{ as } m,n\to\infty.$$

So, we have $p\left(Tx_n,Tx_m\right)\to 0$ as $m,n\to\infty$.

In the next stage, by using similar methods in Theorem 2, we obtain that $\left\{Tx_n\right\}$ is Cauchy sequence in complete partial metric space (X,p) and there exist $u \in X$ such that $\left\{Tx_n\right\}$ converges to $Tu \in X$ and $x_{n(k)} \to u$, such that

$$\lim_{k\to\infty} Tx_{n(k)} = Tu \text{ and } \lim_{k\to\infty} p\left(Tx_{n(k)},Tu\right) = p(Tu,Tu) = 0.$$

Now, we will show that $u \in X$ is a fixed point of f. Indeed, we have

$$F\left(p\left(Tfu,Tx_{n+1}\right)\right) = F\left(p\left(Tfu,Tfx_n\right)\right)$$

$$\leq \beta \left[F\left(p\left(Tu, Tfu\right)\right) + F\left(p\left(Tx_n, Tx_{n+1}\right)\right)\right]. \tag{13}$$

Let $n \to \infty$ in (13), we have

$$F(p\left(Tfu, Tu\right)) \leq \beta F\left(p\left(Tu, Tfu\right)\right). \tag{14}$$

The inequality (14) is contradiction unless $p\left(Tu, Tfu\right) = 0$. Thus, $Tu = Tfu$. Also, T is one to one, we obtain $fu = u$. Thus we provide $u \in X$ is a fixed point of f. The uniqueness of the fixed point can be shown easily. □

Some results of the Theorem 3 are following.

Corollary 3. *Let $\left(X, p\right)$ be a complete partial metric space and $T, f : X \to X$ be mappings such that T is one to one, continuous and subsequentially convergent (or graph closed). If for $\beta \in \left[0, \frac{1}{2}\right)$ and for $x, y \in X$,*

$$p\left(Tfx, Tfy\right) \leq \beta \left[p\left(Tx, Tfx\right) + p\left(Ty, Tfy\right)\right].$$

Then, f has a unique fixed point in $\left(X, p\right)$.

Corollary 4. *Let $\left(X, p\right)$ be a complete partial metric space and $f : X \to X$ be a mapping. If for $\beta \in \left[0, \frac{1}{2}\right)$ and for $x, y \in X$,*

$$F\left(p\left(fx, fy\right)\right) \leq \beta \left[F\left(p\left(x, fx\right)\right) + F\left(p\left(y, fy\right)\right)\right]$$

where $F : [0, \infty) \to [0, \infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then f has a unique fixed point.

Corollary 5. *Let $\left(X, p\right)$ be a complete partial metric space and $f : X \to X$ be mapping. If $\beta \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$.*

$$p\left(fx, fy\right) \leq \beta \left[p\left(x, fx\right) + p\left(y, fy\right)\right]$$

Then, f has a unique fixed point in $\left(X, p\right)$.

Theorem 4. *Let $\left(X, p\right)$ be a complete partial metric space and $T, f : X \to X$ be mappings such that T is one to one, continuous and subsequentially convergent (or graph closed). If $\lambda \in \left[0, \frac{1}{2}\right)$ and for each $x, y \in X$*

$$F\left(p\left(Tfx, Tfy\right)\right) \leq \lambda \left[F\left(p\left(Tx, Tfy\right)\right) + F\left(p\left(Ty, Tfx\right)\right)\right] \tag{15}$$

where $F : [0, \infty) \to [0, \infty)$ is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then, f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0$. Also consider $p\left(Tx_n, Tx_n\right) \leq p\left(Tx_n, Tx_{n+1}\right)$

$$F(p\left(Tx_n, Tx_{n+1}\right)) = F(p\left(Tfx_{n-1}, Tfx_n\right))$$

$$\leq \lambda F(p\left(Tx_{n-1}, Tx_{n+1}\right)) + \lambda F(p\left(Tx_n, Tx_n\right))$$
$$\leq \lambda F(p\left(Tx_{n-1}, Tx_{n+1}\right)) + \lambda F(p\left(Tx_{n+1}, Tx_n\right))$$

therefore, we have

$$F(p\left(Tx_n, Tx_{n+1}\right)) \leq \frac{\lambda}{1-\lambda} F(p\left(Tx_{n-1}, Tx_{n+1}\right)).$$

Also, for all $m(k), n(k) \in \mathbb{N}$, taking $m(k) > n(k)$, we have

$$F(p\left(Tx_{m(k)}, Tx_{n(k)}\right) \leq \left(\frac{\lambda}{1-\lambda}\right)^{n(k)} F(p\left(Tx_{m(k)-n(k)}, Tx_{n(k)}\right)). \tag{16}$$

Note that T is subsequentially convergent, then there exists $u \in X$ such that

$$\lim_{k\to\infty} p\left(x_{n(k)}, u\right) = \lim_{k\to\infty} p(u,u).$$

Let $k \to \infty$ in (16), we obtain that

$$F(p\left(Tx_{m(k)}, Tx_{n(k)}\right) \to 0^+ \text{ as } k \to \infty. \tag{17}$$

The inequality (17) implies that

$$p\left(Tx_{m(k)}, Tx_{n(k)}\right) = 0$$

Hence, we obtain that $\left\{Tx_n\right\}$ is Cauchy sequence in complete partial metric space $\left(X, p\right)$ and there exist a point $u \in X$ such that this point the unique fixed point of f. □

Corollary 6. *Let $\left(X,p\right)$ be a complete partial metric space and $T, f : X \to X$ be mappings such that T is one to one, continuous and subsequentially convergent (or graph closed). If $\lambda \in \left[0, \frac{1}{2}\right)$ and for each $x, y \in X$,*

$$p\left(Tfx, Tfy\right) \leq \lambda\left[p\left(Tx, Tfy\right) + p\left(Ty, Tfx\right)\right]$$

then f has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\left\{f^n x_0\right\}$ converges to the fixed point.

Corollary 7. *Let $\left(X,p\right)$ be a complete partial metric space and $f : X \to X$ be a mapping. If $\lambda \in \left[0, \frac{1}{2}\right)$ and for each $x, y \in X$,*

$$F\left(p\left(fx, fy\right)\right) \leq \lambda\left[F\left(p\left(x, fy\right)\right) + F\left(p\left(y, fx\right)\right)\right]$$

where $F : [0,\infty) \to [0,\infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$. Then f has a unique fixed point.

Corollary 8. *Let $\left(X,p\right)$ be a complete partial metric space and $f : X \to X$ be mapping. If $\lambda \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$,*

$$p\left(fx, fy\right) \leq \lambda\left[p\left(x, fy\right) + p\left(y, fx\right)\right]$$

then, f has a unique fixed point.

References

[1] S.G. Matthews .*Partially metric topology*, Research Report 212, Department of Computer Science, University of Warwick, 1992.

[2] S.G. Matthews. *Partially metric topology, in Proceedings of the 8th Summer Conference, Queen's College, General Topology and its Applications*, Vol. 728 of Annals of the New York Academy of Science, 183-197. 1992.

[3] M. Schellekens. *A characterization of partial metrizebility: domains are quantifiable*, Theoretical Computer Sciences, 305(1-3), 409-432. 2003.

[4] P. Waszkiewicz. *Partial metrizebility of continuous posets*, Mathematical Structures in Computer Science, 16(2), 359-372. 2006.

[5] S. Banach. *Sur les operations dans les ensembles abstraits et leur application aux equations integerales*, Fundamenta Mathematicae, 3, 133-181. 1922.

[6] R. Kannan. *Some results on fixed points*, Bulletin of Calcutta Mathematical Society, 60, 71-76. 1968.

[7] S. K. Chatterjea. *Fixed point theorems*, Comptes Rendus de l'Académie bulgare des Sciences, 25, 727-730. 1972.

[8] Ö. Acar and I. Altun. *Some generalizations of Caristi type fixed point theorem on partial metric space*, Filomat, 26(4), 833–837. 2012.

[9] E. Karapınar and U. Yüksel. *Some common fixed point theorems in partial metric space*, Journal of Applied Mathematics, 2011, Article ID:263621, 2011.

[10] S. Moradi and A. Davood. *New extension of Kannan fixed point theorem on complete metric and generalized metric spaces*, International Journal of Mathematical Analysis, 5(47), 2313-1320. 2011.

[11] Z. Mustafa, J.R. Roshan, V. Parvaneh, and Z. Kadelburg *Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces*, Journal of Inequality and Applications 2014, 46, 2014.

[12] S. Moradi and A. Beiranvand. *Fixed point of T_F-contractive single-valued mappings*, Iranian Journal of Mathematical Sciences and Informatics, 5, 25-32. 2010.

[13] M. Kir and H. Kızıltuc. *T_F-contractive conditions for Kannan and Chatterjea fixed point theorems*, Advances in fixed point theory, 4(1), 140-148. 2014.

[14] A. Branciari. *A fixed point theorem for mapping satisfying a general contractive condition of integral type*, International Journal of Mathematics and Mathematical Sciences, 29(9), 531-536. 2002.