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A Note on Prüfer $\star$-multiplication Domains II

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Abstract. We bring some corrections to Corollary 1 of [3]. In [3], we attempted to show that for an arbitrary star operation $\star$ on a domain R, the domain R is a Prüfer $\star$-multiplication domain if and only if $(a) \cap (b)$ is $\star_f$-invertible for all $a, b \in R \setminus \{0\}$. We show in this paper that the characterization does not hold in general and we restate [3, Corollary 1] with justification and proof as follows: if a domain R is a Prüfer $\star$-multiplication domain, then $(a) \cap (b)$ is $\star_f$-invertible for all $a, b \in R \setminus \{0\}$. The converse holds only if $\star_f = t$.

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In [3, Corollary 1], we tried to show that a Prüfer $\star$-multiplication domain (for short P$\star$MD) R is characterized by $(a) \cap (b)$ being $\star_f$-invertible for all nonzero $a, b \in R$. However, it turns out that [3, Corollary 1] is not completely true and needs to be adjusted. We hereby provide an adjustment with proof of [3, Corollary 1].

Theorem 1. *If R is a P$\star$MD, then $aR \cap bR$ is $\star_f$-invertible for every pair of nonzero elements $a, b \in R$. The converse holds only if $\star_f = t$.*

Proof. Suppose R is a P$\star$MD. Note that we have $(ab)^{-1}[(a) \cap (b)] = (a, b)^{-1}$. So $(ab)^{-1}[(a) \cap (b)](a, b) = (a, b)^{-1}(a, b)$ and $\left((ab)^{-1}[(a) \cap (b)](a, b)\right)^{\star_f} = \left((a, b)^{-1}(a, b)\right)^{\star_f}$. Since R is a P$\star$MD, (a, b) is $\star_f$-invertible and thus if $a, b \in R \setminus \{0\}$, $(a) \cap (b)$ is $\star_f$-invertible. Now suppose that $(a) \cap (b)$ is $\star_f$-invertible for every pair of nonzero elements $a, b \in R$. Then there is a fractional ideal A such that $(A(aR \cap bR))^{\star_f} = R$. That is, $A^{\star_f} = (aR \cap bR)^{-1}$ is a divisorial ideal and because A is of finite type, we deduce from discussion in [4, pp. 433-434] that $A^{\star_f} = A_v = A_t$. So R is a P$\star$MD only if $\star_f = t$. □

Now let us proceed to show that there is a pathology in [3, Corollary 1]. First recall that in [1] a Generalized GCD domain (for short GGCD domain) is defined as a domain for which the v-image $(a, b)_v$ of the ideal generated by each pair of nonzero elements is invertible. Note that $(\frac{1}{ab}(a, b))^{-1} = aR \cap bR$. But then we also have $(\frac{1}{ab}(a, b))^{-1} = (\frac{1}{ab}(a, b)_v)^{-1} = aR \cap bR$.

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Now the above two equations work in both Prüfer domains (domains for which every two generated nonzero ideal is invertible) and GGCD domains. In fact, if (a,b) is invertible then (a,b) is divisorial and so $(a,b)=(a,b)_v$ in the Prüfer domain case. On the other hand in the GGCD domain case $aR\cap bR$ being invertible works fine because $\frac{1}{ab}(a,b)_v$ is the inverse of $aR\cap bR$ and $\frac{1}{ab}(a,b)_v$ is invertible.

So, by [3, Corollary 1], GGCD domains are PdMDs. But then we have the following observation: R is a P$\star$MD if and only if every finitely generated nonzero ideal of R is $\star_f$-invertible. That means for every finitely generated ideal A we have $A^{\star_f}=A_v=A_t$. So $\star_f=t$ in a P$\star$MD (see [4, pp. 433-434] and [5]). So this means that in a PdMD, $d=t$. That is a PdMD is a Prüfer domain. Of course $d\neq t$ in a GGCD domain, generally, as the example below shows.

Example 1. *Let R be a Dedekind domain (note that a Dedekind domain is a GGCD domain) that is not a field. According to [1], the polynomial ring $R[X]$ is a GGCD domain. So in $D=R[X]$ for every pair $f,g\in D\setminus\{0\}$ we have $fD\cap gD$ invertible and hence d-invertible. So D is a PdMD by [3, Corollary 1]. But there are maximal d-ideals such as $M=P+XR[X]$, with P a nonzero prime of R for which D_M is not a valuation domain.*

Now PvMDs do not suffer from the malady P$\star$MDs suffer from because in the PvMDs case $aR\cap bR$ being t-invertible gives $(a,b)_v$ being t-invertible which is equivalent to (a,b) being t-invertible because $(\frac{1}{ab}(a,b)(aR\cap bR))_t=(\frac{1}{ab}(a,b)_t(aR\cap bR))_t=(\frac{1}{ab}(a,b)_v(aR\cap bR))_t$, because $(a,b)_t=(a,b)_v$. Similarly one may note that the v-domains do not suffer from this problem because (a,b) is v-invertible if and only if $(a,b)_v$ is v-invertible.

Finally the GGCD domains fall under mixed invertibility as (d,v)-Prüfer i.e. domains in which A_v is invertible for each nonzero finitely generated ideal A. These may serve as PvMDs that are not P$\star$MDs for any $\star\neq v,t,w$ (see section on $\star$-Prüfer domains in [2]).

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