Fixed Point Results for $\alpha - \psi - \varphi-$ Contractive Type Mappings in b-Metric-Like Spaces

M. A. Akturk[1,*], M. Kır[2] and E. Yolacan[3]

[1] *Department of Engineering Sciences, Istanbul University, 34320 Istanbul, Turkey*
[2] *Department of Civil Engineering, Faculty of Engineering Şırnak University, Şırnak, Turkey*
[3] *Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey*

Abstract. In this paper, we introduce the concept of $\alpha - \psi - \varphi$- contractive type mappings in b-metric-like spaces and state some related fixed point theorems. Our results generalize related results in the literature. Furthermore, an example and an application to integral equations are provided to illustrate the usability of obtained results.

2010 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: b-Metric-Like, α-Admissible Mappings, Fixed Point, Integral Equations

1. Introduction

There are a lot of generalizations of the concept of metric space in the literature. The notion of b-metric-like space was initiated by Alghamdi [1] in 2013 as a new generalization of metric-like space. Recently, Hussain *et al.* [4] examined topological structure of these spaces and presented some fixed point results in b-metric-like space. Very recently, Chen *et al.* [3] established some fixed point theorems in b-metric-like space and showed existence of a solution for an integral equation. In this paper we introduce the concept of $\alpha - \psi - \varphi$-contractive type mappings in b-metric-like spaces and state some related fixed point theorems. Our results generalize related results in the literature. Furthermore, an example and an application to integral equations are provided to illustrate the usability of obtained results.

2. b-Metric-Like Spaces

Definition 1 ([1])**.** *Let X be a nonempty set and $\kappa \geq 1$ a given real number. A function $\varsigma : X \times X \to \mathbb{R}^+$ is b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:*

---
*Corresponding author.

Email addresses: mehmetaliakturk@yandex.com (M. Akturk), mehmetkir04@gmail.com (M. Kır),yolacanesra@gmail.com (E. Yolacan)

(A1) if $\varsigma(x,y)=0 \Rightarrow x=y$;

(A2) $\varsigma(x,y)=\varsigma(y,x)$;

(A3) $\varsigma(x,y) \leq \kappa\left[\varsigma(x,z)+\varsigma(y,z)\right]$.

A b-metric-like space is a pair (X,ς) such that X is nonempty set and ς is b-metric-like on X. The number κ is called the coefficient of (X,ς).

Each *b*-metric-like ς on X generates a topology τ_ς on X whose base is the family of all open $\varsigma-$ balls $\{D_\varsigma(x,\varepsilon):x\in X,\ \varepsilon>0\}$, where $D_\varsigma(x,\varepsilon)=\{a\in X:|\varsigma(x,a)-\varsigma(x,x)|<\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

Definition 2 ([1]). *Let (X,ς) be a b-metric-like space with coefficient κ, and let $\{x_n\}$ be any sequence in X and $x\in X$. Then*

(a) a sequence $\{x_n\}$ is convergent to x with respect to τ_ς, if $\lim_{n\to\infty}\varsigma(x_n,x)=\varsigma(x,x)$;

(b) a sequence $\{x_n\}$ is a Cauchy sequence in (X,ς) if $\lim_{n,m\to\infty}\varsigma(x_n,x_m)$ exists and is finite;

(c) (X,ς) is a complete b-metric-like space if for every Cauchy sequence $\{x_n\}$ in X there exists $x\in X$ such that $\lim_{n,m\to\infty}\varsigma(x_n,x_m)=\lim_{n\to\infty}\varsigma(x_n,x)=\varsigma(x,x)$.

It is obvious that the limit of a sequence in *b*-metric-like space is usually not unique (see [3, Remark 1.1]).

Lemma 1 ([4]). *Let (X,ς) be a b-metric-like space with coefficient κ, and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then one has*

$$\frac{1}{\kappa^2}\varsigma(x,y)-\frac{1}{\kappa}\varsigma(x,x)-\varsigma(y,y) \leq \lim_{n\to\infty}\inf \varsigma(x_n,y_n) \leq \lim_{n\to\infty}\sup \varsigma(x_n,y_n)$$
$$\leq \kappa\varsigma(x,x)+\kappa^2\varsigma(y,y)+\kappa^2\varsigma(x,y).$$

In particular, if $\varsigma(x,y)=0$, then one has $\lim_{n\to\infty}\varsigma(x_n,y_n)=0$.
Moreover, for each $z\in X$ one has

$$\frac{1}{\kappa}\varsigma(x,z)-\varsigma(x,x) \leq \lim_{n\to\infty}\inf \varsigma(x_n,z) \leq \lim_{n\to\infty}\sup \varsigma(x_n,z)$$
$$\leq \kappa\varsigma(x,z)+\kappa\varsigma(x,x).$$

3. Preliminaries

Let Ψ be the family of function $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(i) ψ is continuous and nondecreasing;

(ii) $\psi(t)=0$ if and only if $t=0$.

Samet *et al.* [5] introduced the class of $\alpha-$admissable mappings.

Definition 3 ([5])**.** *For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be given mappings. We say that T is α-admissible if for all $x, y \in X$, we have*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha((Tx, Ty)) \geq 1.$$

Now, we establish the $\alpha - \psi - \varphi$-contractive type mapping on b-metric-like space.

Definition 4. *Let (X, ς) be a b-metric-like space with coefficient $\kappa \geq 1$. We say that $T : X \to X$ is an $\alpha - \psi - \varphi$- contractive type mapping if there exists three functions $\alpha : X \times X \to [0, \infty)$ and $\psi, \varphi \in \Psi$ such that*

$$\alpha(x, y)\psi(\kappa\varsigma(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{1}$$

where

$$M(x, y) = \max\left\{\varsigma(x, y), \varsigma(x, Tx), \varsigma(y, Ty), \frac{\varsigma(x, Ty) + \varsigma(y, Tx)}{2\kappa}\right\} \tag{2}$$

for all $x, y \in X$.

4. Main Results

Theorem 1. *Let (X, ς) be a complete b-metric-like space with the constant $\kappa \geq 1$ and $T : X \to X$ be an $\alpha - \psi - \varphi$-contractive mapping. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous and if $\varsigma(x, x) = 0$ for some $x \in X$, then $\alpha(\omega, \omega) \geq 1$.*

Then, such ω is a fixed point of T, that is $T\omega = \omega$.

Proof. From condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then it is clear that x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all n. Observe that

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1,$$

since T is α-admissible. By repeating the process above, we derive

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{3}$$

Using (1) and (3) for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\psi(\kappa\varsigma(x_{n+1}, x_{n+2})) &= \psi(\kappa\varsigma(Tx_n, Tx_{n+1})) \\
&\leq \alpha(x_n, x_{n+1})\psi(\kappa\varsigma(Tx_n, Tx_{n+1}))
\end{aligned}$$

$$\leq \psi\left(M\left(x_n, x_{n+1}\right)\right) - \varphi\left(M\left(x_n, x_{n+1}\right)\right) \tag{4}$$

where

$$M\left(x_n, x_{n+1}\right) = \max\left\{\varsigma\left(x_n, x_{n+1}\right), \varsigma\left(x_n, Tx_n\right), \varsigma\left(x_{n+1}, Tx_{n+1}\right), \frac{\varsigma\left(x_n, Tx_{n+1}\right) + \varsigma\left(x_{n+1}, Tx_n\right)}{2\kappa}\right\}$$

$$= \max\left\{\varsigma\left(x_n, x_{n+1}\right), \varsigma\left(x_{n+1}, x_{n+2}\right), \frac{\varsigma\left(x_n, x_{n+2}\right) + \varsigma\left(x_{n+1}, x_{n+1}\right)}{2\kappa}\right\}$$

$$\leq \max\left\{\varsigma\left(x_n, x_{n+1}\right), \varsigma\left(x_{n+1}, x_{n+2}\right), \frac{\kappa\varsigma\left(x_n, x_{n+1}\right) + \kappa\varsigma\left(x_{n+1}, x_{n+2}\right) + \varsigma\left(x_{n+1}, x_{n+1}\right)}{2\kappa}\right\}.$$

Since $\varsigma\left(x, x\right) \leq \varsigma\left(x, y\right) \leq k\varsigma\left(x, y\right)$ for each $x, y \in X$, we arrive at

$$M\left(x_n, x_{n+1}\right) = \max\left\{\varsigma\left(x_n, x_{n+1}\right), \varsigma\left(x_{n+1}, x_{n+2}\right), \frac{2\varsigma\left(x_n, x_{n+1}\right) + \varsigma\left(x_{n+1}, x_{n+2}\right)}{2}\right\}$$

$$= \max\left\{\varsigma\left(x_{n+1}, x_{n+2}\right), \frac{2\varsigma\left(x_n, x_{n+1}\right) + \varsigma\left(x_{n+1}, x_{n+2}\right)}{2}\right\}. \tag{5}$$

If for some n, $M\left(x_n, x_{n+1}\right) = \varsigma\left(x_{n+1}, x_{n+2}\right)(\neq 0)$ then (4) and (5) turn into

$$\psi\left(\kappa\varsigma\left(x_{n+1}, x_{n+2}\right)\right) \leq \psi\left(\varsigma\left(x_{n+1}, x_{n+2}\right)\right) - \varphi\left(\varsigma\left(x_{n+1}, x_{n+2}\right)\right) < \psi\left(\varsigma\left(x_{n+1}, x_{n+2}\right)\right),$$

which is a contraction. Hence, $M\left(x_n, x_{n+1}\right) = \varsigma\left(x_n, x_{n+1}\right)$ for all $n \in \mathbb{N}$ and (4) with (5) we obtain

$$\psi\left(\kappa\varsigma\left(x_{n+1}, x_{n+2}\right)\right) \leq \psi\left(\varsigma\left(x_n, x_{n+1}\right)\right) - \varphi\left(\varsigma\left(x_n, x_{n+1}\right)\right). \tag{6}$$

Consequently, the sequence $\left\{\varsigma\left(x_{n+1}, x_{n+2}\right)\right\}$ is non-increasing for all $n \in \mathbb{N}$. Hence, there exists $a \geq 0$ such that $\lim_{n\to\infty} \varsigma\left(x_{n+1}, x_{n+2}\right) = a$.

Taking $n \to \infty$ in (6), the continuity of ψ and φ and $\lim_{n\to\infty} \varsigma\left(x_{n+1}, x_{n+2}\right) = a$ show that $\psi\left(\kappa a\right) \leq \psi\left(a\right) - \varphi\left(a\right)$, yielding $a = 0$. So, we have

$$\lim_{n\to\infty} \varsigma\left(x_{n+1}, x_{n+2}\right) = 0. \tag{7}$$

Next, we show that $\left\{x_n\right\}$ is a Cauchy sequence. If it is not, then there exists $\varepsilon > 0$ for which we can find subsequences $\left\{x_{m_k}\right\}$ and $\left\{x_{n_k}\right\}$ of sequence $\left\{x_n\right\}$ where n_k is the smallest index for which $n_k > m_k > k$ with

$$\varsigma\left(x_{m_k}, x_{n_k}\right) \geq \varepsilon. \tag{8}$$

Then

$$\varsigma\left(x_{m_k}, x_{n_k-1}\right) < \varepsilon. \tag{9}$$

Using (8) and (9), we obtain

$$\varepsilon \leq \varsigma\left(x_{m_k}, x_{n_k}\right) \leq \kappa\left[\varsigma\left(x_{m_k}, x_{n_k-1}\right) + \varsigma\left(x_{n_k-1}, x_{n_k}\right)\right] < \kappa\varepsilon + \kappa\varsigma\left(x_{n_k-1}, x_{n_k}\right). \tag{10}$$

Taking the upper and lower limits as $k \to \infty$, we conclude

$$\varepsilon \leq \lim_{k \to \infty} \inf \varsigma\left(x_{m_k}, x_{n_k}\right) \leq \lim_{k \to \infty} \sup \varsigma\left(x_{m_k}, x_{n_k}\right) \leq \kappa\varepsilon. \tag{11}$$

By using *(A3)* and we deduce

$$\varsigma\left(x_{m_k+1}, x_{n_k}\right) \leq \kappa\varsigma\left(x_{m_k+1}, x_{m_k}\right) + \kappa^2\varsigma\left(x_{m_k}, x_{n_k-1}\right) + \kappa^2\varsigma\left(x_{n_k-1}, x_{n_k}\right), \tag{12}$$

with taking the upper limit as $k \to \infty$ in (12), we obtain

$$\lim_{k \to \infty} \sup \varsigma\left(x_{m_k+1}, x_{n_k}\right) \leq \kappa^2\varepsilon. \tag{13}$$

Use *(A3)* and we find

$$\varsigma\left(x_{m_k+1}, x_{n_k-1}\right) \leq \kappa\varsigma\left(x_{m_k+1}, x_{m_k}\right) + \kappa\varsigma\left(x_{m_k}, x_{n_k-1}\right) \tag{14}$$

by taking the upper limit as $k \to \infty$ in (14), we get

$$\lim_{k \to \infty} \sup \varsigma\left(x_{m_k+1}, x_{n_k-1}\right) \leq \kappa\varepsilon. \tag{15}$$

On the other hand,

$$\varsigma\left(x_{m_k}, x_{n_k}\right) \leq \kappa\varsigma\left(x_{m_k}, x_{m_k+1}\right) + \kappa^2\varsigma\left(x_{m_k+1}, x_{n_k-1}\right) + \kappa^2\varsigma\left(x_{n_k-1}, x_{n_k}\right). \tag{16}$$

Using (11) and (7), we obtain

$$\frac{\varepsilon}{\kappa^2} \leq \lim_{k \to \infty} \inf \varsigma\left(x_{m_k+1}, x_{n_k-1}\right). \tag{17}$$

Moreover,

$$\varepsilon \leq \varsigma\left(x_{m_k}, x_{n_k}\right) \leq \kappa\varsigma\left(x_{m_k}, x_{m_k+1}\right) + \kappa\varsigma\left(x_{m_k+1}, x_{n_k}\right), \tag{18}$$

with taking the upper limit as $k \to \infty$ in (18), we have

$$\frac{\varepsilon}{\kappa} \leq \lim_{k \to \infty} \sup \varsigma\left(x_{m_k+1}, x_{n_k}\right). \tag{19}$$

By using (1), we have

$$\begin{aligned} \psi\left(\kappa\varsigma\left(x_{m_k+1}, x_{n_k}\right)\right) &\leq \alpha\left(x_{m_k}, x_{n_k-1}\right)\psi\left(\kappa\varsigma\left(Tx_{m_k}, Tx_{n_k-1}\right)\right) \\ &\leq \psi\left(M\left(x_{m_k}, x_{n_k-1}\right)\right) - \varphi\left(M\left(x_{m_k}, x_{n_k-1}\right)\right) \end{aligned} \tag{20}$$

where

$$M\left(x_{m_k}, x_{n_k-1}\right) = \max\left\{\varsigma\left(x_{m_k}, x_{n_k-1}\right), \varsigma\left(x_{m_k}, x_{m_k+1}\right), \varsigma\left(x_{n_k-1}, x_{n_k}\right) \right. \tag{21}$$

$$\left. , \frac{\varsigma\left(x_{m_k}, x_{n_k}\right) + \varsigma\left(x_{n_k-1}, x_{m_k+1}\right)}{2\kappa}\right\}$$

from on taking the upper limit as $k \to \infty$, from (7), (9), (11) and (15) we obtain

$$\lim_{k \to \infty} \sup M\left(x_{m_k}, x_{n_k-1}\right) = \max\left\{\varepsilon, 0, 0, \frac{\kappa\varepsilon + \kappa\varepsilon}{2\kappa}\right\} = \varepsilon. \tag{22}$$

Thus, from (19) and (20), we have

$$\psi\left(\kappa\frac{\varepsilon}{\kappa}\right) \leq \psi(\varepsilon) - \varphi(\varepsilon) \tag{23}$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\omega \in X$ such that

$$0 = \lim_{n,m \to \infty} \varsigma\left(x_n, x_m\right) = \lim_{n \to \infty} \varsigma\left(x_n, \omega\right) = \varsigma(\omega, \omega). \tag{24}$$

By using *(A3)*, we deduce

$$\varsigma(\omega, T\omega) \leq \kappa\varsigma\left(\omega, Tx_n\right) + \kappa\varsigma\left(Tx_n, T\omega\right). \tag{25}$$

Taking the upper limit as $n \to \infty$ in (25) and using the continuity of T we have

$$\varsigma(\omega, T\omega) \leq \kappa\varsigma(T\omega, T\omega). \tag{26}$$

Since $\alpha(\omega, \omega) \geq 1$ and using (1) we have

$$\psi(\kappa\varsigma(T\omega, T\omega)) \leq \alpha(\omega, \omega)\psi(\kappa\varsigma(T\omega, T\omega)) \leq \psi(M(\omega, \omega)) - \varphi(M(\omega, \omega)) \tag{27}$$

where

$$M(\omega, \omega) = \max\left\{\varsigma(\omega, \omega), \varsigma(\omega, T\omega), \varsigma(\omega, T\omega), \frac{\varsigma(\omega, T\omega) + \varsigma(\omega, T\omega)}{2\kappa}\right\} = \varsigma(\omega, T\omega). \tag{28}$$

Hence,

$$\psi(\kappa\varsigma(T\omega, T\omega)) \leq \alpha(\omega, \omega)\psi(\kappa\varsigma(T\omega, T\omega)) \leq \psi(\varsigma(\omega, T\omega)) - \varphi(\varsigma(\omega, T\omega)). \tag{29}$$

The property of ψ, we obtain

$$\kappa\varsigma(T\omega, T\omega) \leq \varsigma(\omega, T\omega). \tag{30}$$

Here we deduce $\varphi(\varsigma(\omega, T\omega)) = 0$. Hold

$$\varsigma(T\omega, \omega) = \varsigma(T\omega, T\omega) = \varsigma(T\omega, \omega) = 0 \text{ and } T\omega = \omega.$$

Hence, ω is a fixed point of T. $\qquad\square$

If we replace the continuity condition (iii), Theorem 1 remains true. This statement is given as follows.

Theorem 2. *Let (X, ς) be a complete b-metric-like space with the constant $\kappa \geq 1$ and let $T : X \to X$ be an $\alpha - \psi - \varphi$-contractive type mapping. Suppose that*

   (i) *T is α-admissible;*

  (ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

 (iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.*

Then, such ω is a fixed point of T, that is $T\omega = \omega$.

Proof. From proof of Theorem 1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ is Cauchy in (X, ς) and converges to some $\omega \in X$. Consider (24),

$$\lim_{k \to \infty} \varsigma(x_{n_k+1}, T\omega) = \varsigma(\omega, T\omega) \tag{31}$$

holds. By the assumption on X, we have

$$\begin{aligned} \psi\left(\kappa\varsigma\left(x_{n_k+1}, T\omega\right)\right) &\leq \alpha\left(x_{n_k}, \omega\right)\psi\left(\kappa\varsigma\left(Tx_{n_k}, T\omega\right)\right) \\ &\leq \psi\left(M\left(x_{n_k}, \omega\right)\right) - \varphi\left(M\left(x_{n_k}, \omega\right)\right) \end{aligned} \tag{32}$$

where

$$\begin{aligned} M\left(x_{n_k}, \omega\right) &= \max\left\{\varsigma\left(x_{n_k}, \omega\right), \varsigma\left(x_{n_k}, Tx_{n_k}\right), \varsigma(\omega, T\omega), \frac{\varsigma\left(x_{n_k}, T\omega\right) + \varsigma\left(\omega, Tx_{n_k}\right)}{2\kappa}\right\} \\ &= \max\left\{\varsigma\left(x_{n_k}, \omega\right), \varsigma\left(x_{n_k}, x_{n_k+1}\right), \varsigma(\omega, T\omega), \frac{\varsigma\left(x_{n_k}, T\omega\right) + \varsigma\left(\omega, x_{n_k+1}\right)}{2\kappa}\right\}. \end{aligned}$$

With (7) and (31), we have

$$\lim_{k \to \infty} M\left(x_{n_k}, \omega\right) = \varsigma(\omega, T\omega). \tag{33}$$

Since $\alpha(x_n, \omega) \geq 1$ we have

$$\begin{aligned} \psi\left(\varsigma(T\omega, \omega)\right) &\leq \psi\left(\kappa\left[\varsigma\left(T\omega, Tx_n\right) + \varsigma\left(Tx_n, \omega\right)\right]\right) \\ &\leq \psi\left(\kappa\varsigma\left(T\omega, Tx_n\right)\right) + \psi\left(\kappa\varsigma\left(Tx_n, \omega\right)\right) \\ &\leq \alpha(\omega, x_n)\psi\left(\kappa\varsigma\left(T\omega, Tx_n\right)\right) + \psi\left(\kappa\varsigma\left(Tx_n, \omega\right)\right) \\ &\leq \psi\left(M\left(\omega, x_n\right)\right) - \varphi\left(M\left(\omega, x_n\right)\right) \end{aligned} \tag{34}$$

Let $n \to \infty$ in (34), we have $\psi\left(\varsigma(T\omega, \omega)\right) \leq 0$. Hence ω is a fixed point of T, or equivalently, $\omega = T\omega$.   □

Corollary 1. *Let (X, ς) be a b-metric-like space with coefficient $\kappa \geq 1$ and $T : X \rightarrow X$ be such that*

$$\kappa\varsigma(Tx, Ty) \leq M(x, y) - \varphi(M(x, y))$$

for all $x, y \in X$ where $M(x, y)$ defined by (2). Then, T has a fixed point.

To prove Corollary 1 it suffices to take $\alpha(x, y) = 1$ and $\psi(t) = t$ in Theorem 2.

Corollary 2. *Let (X, ς) be a b-metric-like space with coefficient $\kappa \geq 1$ and $T : X \rightarrow X$ be such that*

$$\kappa\varsigma(Tx, Ty) \leq sM(x, y)$$

for all $x, y \in X$ where $s \in (0, 1)$ and $M(x, y)$ defined by (2). Then, T has a fixed point.

To prove Corollary 2 it suffices to take $\varphi(t) = (1 - s)t$ in Corollary 1.

Remark 1.

1. *Note that b-metric-like spaces are a proper extension of metric-like and b-metric spaces. Therefore, it is clear that one can easily state the Analog of Theorem 1, Theorem 2, Corollary 1, and Corollary 2 in the setting of metric-like and b-metric spaces.*

2. *Theorem 1, Theorem 2, and Corollary 2 improve and generalized Theorem 2.1 in [2], Theorem 2.2 in [2] and Corollary 3.2 in [2], respectively.*

Example 1. *Let $X = [0, \infty)$ and ς on X be given by $\varsigma(x, y) = x^2 + y^2 + |x - y|^2$ for all $x, y \in X$. (X, ς) is a complete b-metric-like space with coefficient $\kappa = 2$ (see [4, Example 14]). Define the mappings $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t$, $\varphi(t) = \frac{t}{2}$ and $\alpha : X \times X \rightarrow [0, \infty)$ by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Let $T : X \rightarrow X$ be defined by $Tx = \frac{\ln(x+1)}{2}$. *It is easy to see that T is a continuous on X.*
Understandably T is an $\alpha - \psi - \varphi$-contractive type mapping with $\psi(t) = t$ and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$, for $x, y \in X$,

$$
\begin{aligned}
\alpha(x, y)\psi(\kappa\varsigma(Tx, Ty)) &= \psi(2\varsigma(Tx, Ty)) \\
&= 2\varsigma(Tx, Ty) \\
&= 2\left(T^2 x + T^2 y + |Tx - Ty|^2\right) \\
&= 2\left(\left(\frac{\ln(x+1)}{2}\right)^2 + \left(\frac{\ln(y+1)}{2}\right)^2\right) \\
&\quad + 2\left(\left|\frac{\ln(x+1)}{2} - \frac{\ln(y+1)}{2}\right|^2\right)
\end{aligned}
$$

$$\leq 2\left(\frac{x^2}{4}+\frac{y^2}{4}+\left|\frac{x}{2}-\frac{y}{2}\right|^2\right)$$

$$=\frac{1}{2}\varsigma(x,y)$$

$$=\varsigma(x,y)-\frac{1}{2}\varsigma(x,y)$$

$$=\psi(\varsigma(x,y))-\varphi(\varsigma(x,y))$$

$$\leq\psi(M(x,y))-\varphi(M(x,y)).$$

(i) *Now, we claim that T is α-admissible.*

Let $(x,y)\in X\times X$ such that $\alpha(x,y)\geq 1$. From the definition of T and α we have both $Tx=\frac{\ln(x+1)}{2}$ and $Ty=\frac{\ln(y+1)}{2}$ are in $[0,1]$. Therefore, $\alpha(Tx,Ty)=1\geq 1$. Then T is α-admissible.

(ii) *Taking $x_0=0$ and $Tx_0=T0=\frac{\ln(0+1)}{2}=0$, we have*

$$\alpha(x_0,Tx_0)=\alpha(0,T0)=1\geq 1.$$

It is also obvious that hypothesis (iii) of Theorem 1 is satisfied. Thus, we apply Theorem 1 and so T has a fixed point, which is $\omega=0$.

5. Existence of the Solution for Nonlinear Fredholm Integral Equations

In this section we will present an existence theorem for solution of nonlinear Fredholm integral equations.

Define the nonlinear Fredholm integral equations by

$$x(s)=\int_0^T G(s,r,x(r))\,dr \text{ where } T>0. \tag{35}$$

We will examine (35) under the following conditions:

(a) $G:[0,T]\times[0,T]\times R\rightarrow R$ is continuous;

(b) for all $(s,r)\in[0,T]^2$ and $x,y\in R$, there exists a continuous $a:[0,T]\times[0,T]\rightarrow R$ such that

$$|G(s,r,x)|+|G(s,r,y)|\leq\left(\frac{1}{\kappa^3}\right)^{\frac{1}{p}}a(s,r)(|x|+|y|) \tag{36}$$

and

$$\sup_{s\in[0,T]}\int_0^T a(s,r)\leq 1. \tag{37}$$

Let $X = C[0, T]$ be the set of continuous real functions defined on $[0, 1]$. We endow X with the b-metric-like

$$\varsigma(u, v) = \max_{s \in [0,1]} (|u(s)| + |v(s)|)^p \text{ for all } u, v \in X$$

where $p > 1$. Also, (X, ς) is complete b-metric-like space with the constant $\kappa = 2^{p-1}$ (see more details [3]).

Theorem 3. *Under conditions (a) and (b), (35) has a unique solution in $C[0, T]$.*

Proof. By (36) and (37), we have

$$
\begin{aligned}
\kappa \varsigma(Tx(s), Ty(s)) &= \kappa \left(|Tx(s)| + |Ty(s)| \right)^p \\
&= \kappa \left(\left| \int_0^T G(s, r, x(r)) \, dr \right| + \left| \int_0^T G(s, r, y(r)) \, dr \right| \right)^p \\
&\leq \kappa \left(\int_0^T |G(s, r, x(r))| \, dr + \int_0^T |G(s, r, y(r))| \, dr \right)^p \\
&\leq \kappa \left(\int_0^T \left(\frac{1}{\kappa^3} \right)^{\frac{1}{p}} a(s, r) \left(\left((|x(r) + y(r)|)^p \right)^{\frac{1}{p}} \right) dr \right)^p \\
&\leq \kappa \left(\int_0^T \left(\frac{1}{\kappa^3} \right)^{\frac{1}{p}} a(s, r) \varsigma^{\frac{1}{p}}(x(r), y(r)) \, dr \right)^p \\
&\leq \frac{1}{\kappa^2} \varsigma(x(r), y(r)) \left(\int_0^T a(s, r) \, dr \right)^p \\
&\leq \frac{1}{\kappa^2} \varsigma(x(r), y(r)) \\
&\leq \frac{1}{\kappa^2} \max \Big\{ \varsigma(x(r), y(r)), \varsigma(x(r), Tx(r)), \varsigma(y(r), Ty(r)), \\
&\qquad \frac{\varsigma(x(r), Ty(r)) + \varsigma(y(r), Tx(r))}{2\kappa} \Big\} \\
&= sM(x(r), y(r))
\end{aligned}
$$

where $s = \frac{1}{\kappa^2} \in (0, 1)$.

Now, all the conditions of Corollary 2 hold and T has a unique fixed point $x \in X$, that is x is the unique solution for the integral equation (35). $\qquad \square$

References

[1] M.A. Alghamdi, N. Hussain, and P. Salimi. *Fixed Point and Coupled Fixed Point Theorems on b-metric-like Spaces*, Journal of Inequalities and Applications 2013, Article ID 402, 2013.

[2] H. Aydi and E. Karapinar. *Fixed Point Results for Generalized $\alpha-\psi-$contractions in metric-like Spaces and Applications*, Electronic Journal of Differential Equations, 2015(133), 1-15, 2015.

[3] C. Chen, J. Dong, and C. Zhu. *Some Fixed Point Theorems in b-metric-like Spaces*, Fixed Point Theory and Applications, 2015(122), 2015.

[4] N. Hussain, J.R. Roshan, V. Parvaneh, and Z. Kadelburg. *Fixed Points of Contractive Mappings in b-metric-like Spaces*, The Scientific World Journal 2014, Article ID 471827, 2014.

[5] B. Samet, C. Vetro, and P. Vetro. *Fixed Point Theorems for $\alpha-\psi-$contractive Type Mappings*, Nonlinear Analysis: Theory, Methods and Applications, 75 (4), 2154-2165, 2012.