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On g-Statistical Convergence in Paranormed Spaces

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Abstract. In this paper we construct some spaces of lacunary almost convergent sequences and lacunary strongly almost convergent sequences via sequence of Orlicz functions over n-normed spaces and established some inclusion relations between these spaces. We also make an effort to define a new concept called g-statistical convergence in paranormed spaces where the base space is a n-normed spaces.

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1. Introduction and Preliminaries

In [13] Gähler introduced an attractive theory of 2-normed spaces. The notion was further generalized by Misiak [21] by introducing n-normed spaces. Since then these spaces were studied by Gunawan [14, 15]. In [16] Gunawan and Mashadi gave a simple way to derive an $(n-1)$-norm from the n-norm and realized that n-normed space is an $(n-1)$-normed space.

Definition 1. *Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of real of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \ldots, \cdot||$ on X^n satisfying the following conditions:*

(i) $||x_1, x_2, \ldots, x_n|| = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent in* X*;*

(ii) $||x_1, x_2, \ldots, x_n||$ *is invariant under permutation;*

(iii) $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha| ||x_1, x_2, \ldots, x_n||$ *for any* $\alpha \in \mathbb{R}$*, and*

(iv) $||x + x', x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x', x_2, \ldots, x_n||$

*is called a n-norm on X and the pair $(X, ||\ldots||)$ a **n-normed space** over the field $\mathbb{R}$.*

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Example 1. *Let $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \ldots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula*

$$||x_1, x_2, \ldots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

Let $(X, ||\ldots||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $||\ldots||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \ldots, x_{n-1}||_\infty = \max\{||x_1, x_2, \ldots, x_{n-1}, a_i|| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\ldots||)$ is said to **converge** to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \ldots, z_{n-1}|| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\ldots||)$ is said to be **Cauchy** if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \ldots, z_{n-1}|| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be **complete** with respect to the n-norm. Any complete n-normed space is said to be **n-Banach space**.

Definition 2. *Let K be a subset of the set of natural number $\mathbb{N}$. Then the* ***asymptotic density*** *of K denoted by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{j \leq n : j \in K\}|$, where vertical bars denote the cardinality of the enclosed set.*

Definition 3. *A sequence $x = (x_j)$ is said to be* ***statistically convergent*** *to a number λ if for every $\varepsilon > 0$, the set $K(\varepsilon) = \{j \leq n : |x_j - \lambda| \geq \varepsilon\}$ has asymptotic density zero, i.e,*

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |x_j - \lambda| \geq \varepsilon\}| = 0,$$

in case we write $S - \lim x = \lambda$.

Definition 4. *Let X be a linear metric space. A function $g : X \to \mathbb{R}$ is called* ***paranorm****, if*

(i) $g(x) \geq 0$ *for all* $x \in X$,

(ii) $g(-x) = g(x)$ *for all* $x \in X$,

(iii) $g(x + y) \leq g(x) + g(y)$ *for all* $x, y \in X$,

(iv) *if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ as $n \to \infty$, then $g(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.*

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called ***total paranorm*** *and the pair (X, g) is called a total paranormed space.*

Note that each seminorm (norm) g on X is a paranorm (total) but converse need not be true. It is well known that the metric of any linear metric space is given by some total paranorm (see [28, Theorem 10.4.2, pp. 183]). For more details about sequence spaces see [2, 6, 7, 22, 24–26] and references therein.

Definition 5. *A sequence $x = (x_j)$ in (X, g) paranormed space is said to be* ***convergent (or g−convergent)*** *to a number λ in (X, g) if for every $\varepsilon > 0$ there exists a positive integer j_0 such that $g(x_j - \lambda) < \varepsilon$ whenever $j \geq j_0$. In case we write $g - \lim x = \lambda$ and λ is called the g−limit of x (see [1]).*

Definition 6. *An* ***Orlicz function*** *M is a function, which is continuous, non-decreasing and convex on $[0, +\infty)$ with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.*

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \Big\{x \in \omega : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{for some } \rho > 0\Big\}$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [23], Mursaleen [22] and many others.

Definition 7. *By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of* ***lacunary strongly convergent sequence*** *was defined by Freedman et al. [11] as follows:*

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

Lorentz [18] and Duran [9] studied the spaces of almost convergent sequences. The concept of strongly almost convergent sequences was introduced by Maddox [19]. In [20], Maddox defined a generalization of strong almost convergence. Related articles with the topic almost convergence and strong almost convergence can be seen in [3, 18–20]. In order to extend convergence of sequences, the notion of statistical convergence has been introduced by Fast [10] in 1951 and Schoenberg [27] independently for real sequences. Later on developed by Fridy [12]. Recently, Alotaibi and Alroqi [1] extended this notion in paranormed space. We may refer to [4, 5] which are related with this topic.

Lorentz [18] proved that x is almost convergent to a number λ if and only if

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=0}^{n-1}(x_{i+q}-\lambda)\right| = 0, \text{uniformly in } q \geq 1.$$

In other words, he showed that x is almost convergent to a number λ if and only if $t_{nq}(x) \to \lambda$ as $n \to \infty$, uniformly in $q \geq 1$, where

$$t_{nq}(x) = \frac{x_q + x_{q+1} + \ldots + x_{q+n-1}}{n} (n \in \mathbb{N} = \{1, 2, 3, \ldots, \}).$$

Let f be a set of all almost convergent sequences. We write $f - \lim x = \lambda$ if x is almost convergent to λ. Maddox [20] has defined that x is strongly almost convergent to a number λ if and only if

$$t_{nq}(|x-\lambda|) = \frac{1}{n}\sum_{i=0}^{n-1}\left|x_{i+q}-\lambda\right| \to 0 \text{ as } n \to \infty, \text{ uniformly in } q \geq 1.$$

By $[f]$ we denote the set of all strongly almost convergent sequences. If x is strongly almost convergent to λ we write $[f] - \lim x = \lambda$. Let l_∞ be the set of all bounded sequences, it is easy to see that $[f] \subset f \subset l_\infty$ and each inclusion is proper.

In [8] Konca and Başarir defined the almost convergent sequences F and strongly almost convergent sequences $[F]$, in 2-normed spaces for every $z \in X$. They have also introduced the space of lacunary almost convergent sequences F_θ and lacunary strongly almost convergent sequences $[F_\theta]$, respectively in 2-normed spaces.

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $(X, ||\cdot, \ldots, \cdot||)$ is a n-normed space and $p = (p_k)$ be a bounded sequence of positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \ldots, \cdot||)$. In this paper we define the following sequence spaces:

$$\left[\mathcal{M}, F, p, ||\cdot, \ldots, \cdot||\right] = \Big\{x \in S(n-X) : \lim_{n\to\infty}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{t_{nq}(x-\lambda)}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_k} = 0,$$

uniformly in $q \geq 1$, for some $\rho > 0$ and for every nonzero $z_1, \ldots, z_{n-1} \in X$ $\Big\}$ and

$$\left[\mathcal{M}, [F], p, ||\cdot, \ldots, \cdot||\right] = \Big\{x \in S(n-X) : \lim_{n\to\infty}\sum_{k=1}^{\infty}\Big[M_k\Big(t_{nq}\Big(\Big\|\frac{x-\lambda}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big)\Big]^{p_k} = 0,$$

uniformly in $q \geq 1$, for some $\rho > 0$ and for every nonzero $z_1, \ldots, z_{n-1} \in X$ $\Big\}$.

We write $\left[\mathcal{M}, F, p, ||\cdot, \ldots, \cdot||\right] - \lim x = \lambda$ if x is almost convergent to λ and

$$\left[\mathcal{M}, [F], p, ||\cdot, \ldots, \cdot||\right] - \lim x = \lambda$$

if x is strongly almost convergent to λ. Taking advantage to *(iii)* and *(iv)* conditions of n—norm and definitions of $\big[\mathcal{M}, F, p, \|\cdot,\ldots,\cdot\|\big]$ and $\big[\mathcal{M}, [F], p, \|\cdot,\ldots,\cdot\|\big]$, we have the inclusion

$$\big[\mathcal{M}, [F], p, \|\cdot,\ldots,\cdot\|\big] \subset \big[\mathcal{M}, F, p, \|\cdot,\ldots,\cdot\|\big] \subset \big[M, l^{\infty}, p, \|\cdot,\ldots,\cdot\|\big]$$

holds from the following inequality:

$$\Big\|\frac{t_{nq}(x-\lambda)}{\rho}, z_1,\ldots,z_{n-1}\Big\| = \Big\|\frac{\frac{1}{n}\sum_{i=0}^{n-1}(x_{i+q}-\lambda)}{\rho}, z_1,\ldots,z_{n-1}\Big\|$$
$$\leq \frac{1}{n}\sum_{i=0}^{n-1}\Big\|\frac{x_{i+q}-\lambda}{\rho}, z_1,\ldots,z_{n-1}\Big\| = t_{nq}\Big(\Big\|\frac{x-\lambda}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big).$$

Now we define the spaces of lacunary almost convergent sequences $\big[\mathcal{M}, F_\theta, p, \|\cdot,\ldots,\cdot\|\big]$ and lacunary strongly almost convergent sequences $\big[\mathcal{M}, [F_\theta], p, \|\cdot,\ldots,\cdot\|\big]$ in n-normed spaces as follows:

$$\big[\mathcal{M}, F_\theta, p, \|\cdot,\ldots,\cdot\|\big] = \Big\{x \in S(n-X) : \lim_{r\to\infty}\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{h_r}\sum_{i\in I_r}\Big(\frac{x_{i+q}-\lambda}{\rho}, z_1,\ldots,z_{n-1}\Big)\Big\|\Big]^{p_k} = 0,$$

uniformly in $q \geq 1$, for some $\rho > 0$ and for every nonzero $z_1,\ldots,z_{n-1} \in X$ $\Big\}$ and

$$\big[\mathcal{M}, [F_\theta], p, \|\cdot,\ldots,\cdot\|\big] = \Big\{x \in S(n-X) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_{i+q}-\lambda}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} = 0,$$

uniformly in $q \geq 1$, for some $\rho > 0$ and for every nonzero $z_1,\ldots,z_{n-1} \in X$ $\Big\}$.

The main purpose of this paper is to study some generalized spaces of lacunary almost convergent sequences and lacunary strongly almost convergent sequences via sequence of Orlicz functions over n-normed spaces. We also established some topological properties and prove some inclusion relations between these spaces. Further we introduced a new concept of statistical convergence which will be called g-statistical convergence in a paranormed spaces where the base space is a n-normed spaces. We define and study the notion of statistical convergence and statistical Cauchy.

2. Main Results

Lemma 1. *Let (x_j) be a strongly almost convergent sequence, for a given $\varepsilon > 0$ there exist n_0 and q_0 such that*

$$\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} < \varepsilon$$

for all $p_k \geq 1$, $n \geq n_0$, $q \geq q_0$, for every nonzero $z_1,\ldots,z_{n-1} \in X$ and for some $\rho > 0$. Then $x \in \big[\mathcal{M}, [F], p, \|\cdot,\ldots,\cdot\|\big]$.

Proof. Let $\varepsilon > 0$ be given. Choose n_0', q_0 such that

$$\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} < \frac{\varepsilon}{2} \tag{1}$$

for all $n \geq n_0'$, $q \geq q_0$, It is enough to prove that there exists n_0'' such that for $n > n_0''$, $0 \leq q \leq q_0$

$$\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} < \varepsilon. \tag{2}$$

By taking $n_0 = \max(n_0', n_0'')$, (2) will holds for $n \geq n_0$ and for all q, which gives the result. Once q_0 has been chosen fixed, so

$$\sum_{j=0}^{q_0-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} = K, \tag{3}$$

for some K. Now taking $0 \leq q \leq q_0$ and $n > q_0$, we have

$$\begin{aligned}\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} &= \frac{1}{n}\Big(\sum_{j=q}^{q_0-1}+\sum_{j=q_0}^{q+n-1}\Big)\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\ &\leq \frac{K}{n}+\frac{1}{n}\sum_{j=q_0}^{q_0+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\ &\leq \frac{K}{n}+\frac{\varepsilon}{2}.\end{aligned}$$

The penultimate inequality is from (3), with the last following (1). Taking n sufficiently large, we can make

$$\frac{K}{n}+\frac{\varepsilon}{2} < \varepsilon$$

which gives (2) and hence the result. □

Theorem 1. *Suppose $p_k \geq 1$ for all k and for every θ, we have*

$$\big[\mathscr{M},[F_\theta],p,\|\cdot,\ldots,\cdot\|\big] = \big[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\big].$$

Proof. Let $\{x_j\} \in \big[\mathscr{M},[F_\theta],p,\|\cdot,\ldots,\cdot\|\big]$, then for given $\varepsilon > 0$, there exist r_0 and λ such that

$$\frac{1}{h_r}\sum_{j=q}^{q+h_r-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} < \varepsilon \tag{4}$$

for $r \geq r_0$ and $q = Q_{r-1}+1+i$, $i \geq 0$. Let $n \geq h_r$, write $n = mh_r + \theta$, where m is an integer. Since $h \geq h_r$, $m \geq 1$. Now

$$\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k} \leq \frac{1}{n}\sum_{j=q}^{q+(m+1)h_r-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}$$

$$=\frac{1}{n}+\sum_{u=0}^{m}\sum_{j=q+uh_r}^{q+(u+1)h_r-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_k}$$
$$\leq\frac{m+1}{n}h_r\varepsilon$$
$$\leq\frac{2mh_r\varepsilon}{n}(m\geq 1).$$

For $\frac{h_r}{n}\leq 1$, since $\frac{mh_r}{n}\leq 1$

$$\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_k}\leq 2\varepsilon.$$

Then by Lemma 1, $\big[\mathscr{M},[F_\theta],p,\|\cdot,\dots,\cdot\|\big]\subseteq\big[\mathscr{M},[F],p,\|\cdot,\dots,\cdot\|\big]$. It is trivial to show that

$$\big[\mathscr{M},[F],p,\|\cdot,\dots,\cdot\|\big]\subseteq\big[\mathscr{M},[F_\theta],p,\|\cdot,\dots,\cdot\|\big]$$

for every θ. Hence we have the result. □

Lemma 2. *Suppose for a given $\varepsilon>0$ there exist n_0 and q_0 such that*

$$\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q}^{q+n-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}<\varepsilon$$

for all $n\geq n_0$, $q\geq q_0$, for every nonzero $z_1,\dots,z_{n-1}\in X$ and for some $\rho>0$. Then $x\in\big[\mathscr{M},F,p,\|\cdot,\dots,\cdot\|\big]$.

Proof. Let $\varepsilon>0$ be given. Choose n_0', q_0 such that

$$\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q}^{q+n-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}<\frac{\varepsilon}{2}\tag{5}$$

for all $n\geq n_0'$, $q\geq q_0$. As in Lemma 1, it is enough to prove that there exists n_0'' such that for $n\geq n_0''$, $0\leq q\leq q_0$

$$\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{i=0}^{q+n-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}<\varepsilon.\tag{6}$$

Since q_0 is fixed, let

$$\sum_{j=0}^{q_0-1}\sum_{k=1}^{\infty}\Big[M_k\Big\|\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}=K',\tag{7}$$

for some K'. Now taking $0\leq q\leq q_0$ and $n>q_0$, we have

$$\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q}^{q+n-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}\leq\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q}^{q_0-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}$$

$$+\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q_0}^{q+n-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}$$

$$\leq\frac{K'}{n}+\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q_0}^{q_0+n+q-q_0-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}. \tag{8}$$

Let $n-q_0>n_0'$. Then for $0\leq q<q_0$, we have $n+q-q_0\geq n_0'$. From (5) we have

$$\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n+q+q_0}\sum_{j=q_0}^{q_0+n+q-q_0}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}<\frac{\varepsilon}{2}. \tag{9}$$

From equation (8) and (9) we have

$$\begin{aligned}\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q}^{q+n-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}&\leq\frac{K'}{n}+\frac{n+q-q_0}{n}\frac{\varepsilon}{2}\\&\leq\frac{K'}{n}+\frac{\varepsilon}{2}\\&<\varepsilon,\end{aligned}$$

for sufficiently large n. Hence the result. □

Theorem 2.

(i) For every θ, we have

$$\big[\mathcal{M},F_\theta,p,\|\cdot,\dots,\cdot\|\big]\cap\big[\mathcal{M},l^\infty p,\|\cdot,\dots,\cdot\|\big]=\big[\mathcal{M},F,p,\|\cdot,\dots,\cdot\|\big].$$

(ii) For every θ, we have $\big[\mathcal{M},F_\theta,p,\|\cdot,\dots,\cdot\|\big]\not\subset\big[\mathcal{M},l^\infty,p,\|\cdot,\dots,\cdot\|\big]$.

Proof. (i) Let $\{x_j\}\in\big[\mathcal{M},F_\theta,p,\|\cdot,\dots,\cdot\|\big]\cap\big[\mathcal{M},l^\infty,p,\|\cdot,\dots,\cdot\|\big]$ for every $\varepsilon>0$, there exist r_0 and q_0 such that

$$\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{h_r}\sum_{j=q}^{q+h_r-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}<\frac{\varepsilon}{2} \tag{10}$$

for $r\geq r_0$, $q\geq q_0$, $q=Q_{r-1}+1+i$, $i\geq 0$. Now let $n\geq h_r$, m is an integer greater than equal to 1. Then

$$\begin{aligned}\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}M_k\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}&\leq\sum_{k=1}^{\infty}\Big[M_k\Big\|\frac{1}{n}\sum_{\mu=0}^{m-1}\sum_{j=q+\mu h_r}^{q+(\mu+1)h_r-1}\Big(\frac{x_j-\lambda}{\rho},z_1,\dots,z_{n-1}\Big)\Big\|\Big]^{p_k}\\&+\frac{1}{n}\end{aligned}$$

$$= \sum_{k=1}^{\infty} \Big[M_k \sum_{j=q+mh_r}^{q+n-1} \Big\| \Big(\frac{x_j - \lambda}{\rho}, z_1, \dots, z_{n-1} \Big) \Big\| \Big]^{p_k}. \tag{11}$$

Since $\{x_j\} \in \big[\mathcal{M}, l^{\infty}, p, \|\cdot, \dots, \cdot\|\big]$ for all j, we have

$$\sum_{k=1}^{\infty} \Big[M_k \Big\| \Big(\frac{x_j - \lambda}{\rho}, z_1, \dots, z_{n-1} \Big) \Big\| \Big]^{p_k} < K,$$

for some K. So from (10) and (11)

$$\sum_{k=1}^{\infty} \Big[M_k \Big\| \frac{1}{n} \sum_{j=q}^{q+n-1} \Big(\frac{x_j - \lambda}{\rho}, z_1, \dots, z_{n-1} \Big) \Big\| \Big]^{p_k} \le \frac{1}{n} m.h_r \frac{\varepsilon}{2} + \frac{Kh_r}{n},$$

for $\frac{h_r}{n} \le 1$, since $\frac{mh_r}{n} \le 1$ and $\frac{Kh_r}{n}$ can be made less than $\frac{\varepsilon}{2}$, taking n sufficiently large so

$$\sum_{k=1}^{\infty} \Big[M_k \Big\| \frac{1}{n} \sum_{j=q}^{q+n-1} \Big(\frac{x_j - \lambda}{\rho}, z_1, \dots, z_{n-1} \Big) \Big\| \Big]^{p_k} < \varepsilon \text{ for } r \ge r_0, q \ge q_0.$$

Hence, by Lemma 2, $\big[\mathcal{M}, F_\theta, p, \|\cdot, \dots, \cdot\|\big] \cap \big[\mathcal{M}, l_\infty, p, \|\cdot, \dots, \cdot\|\big] \subseteq \big[\mathcal{M}, F, p, \|\cdot, \dots, \cdot\|\big]$. It is trivial to show that $\big[\mathcal{M}, F, p, \|\cdot, \dots, \cdot\|\big] \subseteq \big[\mathcal{M}, F_\theta, p, \|\cdot, \dots, \cdot\|\big] \cap \big[\mathcal{M}, l_\infty, p, \|\cdot, \dots, \cdot\|\big]$.

(ii) It is enough to show $\big[\mathcal{M}, F_\theta, p, \|\cdot, \dots, \cdot\|\big] \not\subset \big[\mathcal{M}, l_\infty, p, \|\cdot, \dots, \cdot\|\big]$. Let $\{x_j\} = (-1)^j j^{\mu}$ where μ is constant with $0 < \mu < 1$. Then

$$\sum_{j=q}^{q+h_r-1} x_j, q \ge 0$$

will contains an even number of terms. Let us take $X = \mathbb{R}^n$. It is a straightforward matter to verify that $\{x_j\} \in \big[\mathcal{M}, F_\theta, p, \|\cdot, \dots, \cdot\|\big]$ with $\lambda = 0$. But $\{x_j\}$ is not bounded. □

Now, we define the paranorm $g(x)$ on the sequence space $\big[\mathcal{M}, [F], p, \|\cdot, \dots, \cdot\|\big]$ and shown that the sequence space $\big[\mathcal{M}, [F], p, \|\cdot, \dots, \cdot\|\big]$ is total paranormed space. We also define a new concept of statistical convergence which will be called g-statistical convergence on the paranormed space $\big(\big[\mathcal{M}, [F], p, |\cdot, \dots, \cdot\|\big], g\big)$.

Theorem 3. *The sequence space* $\big[\mathcal{M}, [F], p, \|\cdot, \dots, \cdot\|\big]$ *is a linear topological space total parnormed by*

$$g(x) = \sup_{\substack{n\ge1,\, q\ge1 \\ 0\ne z_1,\dots,z_{n-1}\in X}} \Big(\frac{1}{n} \sum_{j=q}^{q+n-1} \sum_{k=1}^{\infty} \Big[M_k \Big(\Big\| \frac{x_j}{\rho}, z_1, \dots, z_{n-1} \Big\| \Big) \Big]^{p_k} \Big)$$
$$= \sup_{\substack{n\ge1,\, q\ge1 \\ 0\ne z_1,\dots,z_{n-1}\in X}} \sum_{k=1}^{\infty} M_k \Big[\Big(t_{nq} \Big(\Big\| \frac{x_j}{\rho}, z_1, \dots, z_{n-1} \Big\| \Big) \Big) \Big]^{p_k}.$$

Proof. It is easy to see that $\left[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right]$ is a linear space with coordinate-wise addition and scalar multiplication. Clearly $g(x)=0 \Longleftrightarrow x=0$, $g(x)=g(-x)$ and g is subadditive. To prove the continuity of scalar multiplication, assume that $(x^{(k)})$ be any sequence of the points in $\left[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right]$ such that $g(x^{(k)}-x)\to 0$ as $k\to\infty$ and (μ_k) be any sequence of scalars such that $\mu_k\to\mu$ as $k\to\infty$. Since the inequality

$$g(x^{(k)})\leq g(x)+g(x^{(k)}-x)$$

holds by subadditivity of g, $g(x^{(k)})$ is bounded. Thus, we have

$$\begin{aligned}
g(\mu_k x^{(k)}-\mu x)=&\sup_{\substack{n\geq 1,\, q\geq 1\\ 0\neq z_1,\ldots,z_{n-1}\in X}}\left(\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\mu_k x_j^{(k)}-\mu x_j}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}\right)\\
\leq&|\mu_k-\mu|\sup_{\substack{n\geq 1,\, q\geq 1\\ 0\neq z_1,\ldots,z_{n-1}\in X}}\left(\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j^{(k)}}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}\right)\\
&+|\mu|\sup_{\substack{n\geq 1,\, q\geq 1\\ 0\neq z_1,\ldots,z_{n-1}\in X}}\left(\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j^{(k)}-x_j}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}\right)\\
=&|\mu_k-\mu|g(x^{(k)})+|\mu|g(x^{(k)}-x),
\end{aligned}$$

which tends to zero as $k\to\infty$. This proves the fact that g is a paranorm on $\left[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right]$. □

Definition 8. *A sequence $x=(x_j)$ is said to be* ***strongly** p**−Cesaro summable*** $(0<p<\infty)$ *to a limit λ in $\left(\left[(\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right],g\right)$ if $\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k}(g(x_j-\lambda e))^p=0$ and we write it as $x_j\to\lambda[C,g]_p$. In this case λ is called the $[C,g]_p$-limit of x. We denote the set of all strongly p-Cesaro summable sequences in $(\left[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right],g)$ as*

$$[C,g]_p=\{x:\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k}(g(x_j-\lambda e))^p=0\}.$$

Definition 9. *A sequence $x=(x_j)$ is said to be* ***statistically convergent (or** g**-statistically convergent)*** *to a number λ in $\left(\left[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right],g\right)$ if for each $\varepsilon>0$*

$$\lim_{k\to\infty}\frac{1}{k}|\{j\leq k: g(x_j-\lambda e)\geq\varepsilon\}|=0$$

where

$$g(x_j-\lambda e)=\sup_{\substack{n\geq 1,\, q\geq 1\\ 0\neq z_1,\ldots,z_{n-1}\in X}}\left(\frac{1}{n}\sum_{j=q}^{q+n-1}\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{x_j-\lambda e}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_k}\right).$$

In this case we write $g(stat)-\lim x=\lambda$. We denote the set of all g-statistically convergent sequences in $\left(\left[\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|\right],g\right)$ by $S_{\left(\left[\mathscr{M},[F]p,\|\cdot,\ldots,\cdot\|\right],g\right)}$.

Definition 10. *A sequence $x = (x_j)$ is said to be a* ***statistically Cauchy sequence*** *in $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$ (or $g(stat) - Cauchy$) if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} |\{j \leq n : g(x_j - x_N) \geq \varepsilon\}| = 0.$$

Theorem 4. *If a sequence $x = (x_j)$ is statistically convergent in $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$, then $g(stat) - \lim x$ is unique.*

Proof. Suppose that $g(stat) - \lim x = \lambda_1$ and $g(stat) - \lim x = \lambda_2$. Given $\varepsilon > 0$, define the following set as:

$$J_1(\varepsilon) = \Big\{ j \in \mathbb{N} : g(x_j - \lambda_1) \geq \frac{\varepsilon}{2} \Big\}$$

and

$$J_2(\varepsilon) = \Big\{ j \in \mathbb{N} : g(x_j - \lambda_2) \geq \frac{\varepsilon}{2} \Big\}.$$

Since $g(stat) - \lim x = \lambda_1$ we have $\delta(J_1(\varepsilon)) = 0$. Similarly $g(stat) - \lim x = \lambda_2$ we have $\delta(J_2(\varepsilon)) = 0$, now let $J(\varepsilon) = J_1(\varepsilon) \cup J_2(\varepsilon)$. Then $\delta(J(\varepsilon)) = 0$ and hence the compliment $J^c(\varepsilon)$ is a non-empty set and $\delta(J^c(\varepsilon)) = 1$. Now if $j \in \mathbb{N} - J(\varepsilon)$, then we have

$$g(\lambda_1 - \lambda_2) \leq g(x_j - \lambda_1) + g(x_j - \lambda_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $g(\lambda_1 - \lambda_2) = 0$ and hence $\lambda_1 = \lambda_2$. □

Theorem 5. *Let $g(stat) - \lim x = \lambda_1$ and $g(stat) - \lim y = \lambda_2$. Then*

(i) $g(stat) - \lim(x \pm y) = \lambda_1 \pm \lambda_2$

(ii) $g(stat) - \lim(\alpha x) = \alpha\lambda_1, \alpha \in \mathbb{R}$.

Proof. It is easy to prove. □

Theorem 6. *A sequence $x = (x_j)$ in $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$ is statistically convergent to λ if and only if there exists a set $J = \{j_1 < j_2 < \ldots < j_n < \ldots\} \subseteq \mathbb{N}$ with $\delta(J) = 1$ such that $g(x_{j_n} - \lambda) \to 0$ as $n \to \infty$.*

Proof. Suppose that $g(stat) - \lim x = \lambda$. Now write for $r = 1, 2, \ldots$.

$$J_r(\varepsilon) = \Big\{ n \in \mathbb{N} : g(x_{j_n} - \lambda_1) \leq 1 + \frac{1}{r} \Big\}$$

and

$$L_r(\varepsilon) = \Big\{ n \in \mathbb{N} : g(x_{j_n} - \lambda_1) > \frac{1}{r} \Big\}.$$

Then $\delta(J_r) = 0$

$$L_1 \supset L_2 \supset \ldots \supset L_i \supset L_{i+1} \supset \ldots \tag{12}$$

and

$$\delta(L_r) = 1, \ r = 1, 2, \ldots \tag{13}$$

Now we have to show that for $n \in L_r$. Since $\{x_{j_n}\}$ is g-convergent to λ. On contrary suppose that $\{x_{j_n}\}$ is not g—convergent to λ. Therefore, there is $\varepsilon > 0$ such that $g(x_{j_n} - \lambda) \leq \varepsilon$ for infinitely many terms. Let $L_\varepsilon = \left\{n \in \mathbb{N} : g(x_{j_n} - \lambda) > \varepsilon\right\}$ and $\varepsilon > \frac{1}{r}$, $r \in \mathbb{N}$. Then

$$\delta(L_\varepsilon) = 0 \tag{14}$$

and by (12) $L_r \subset L_\varepsilon$. Hence $\delta(L_r) = 0$ which contradicts (13) and we get that $\{x_{j_n}\}$ is g-convergent to λ.

Conversely, suppose that there exists a set $J = \{j_1 < j_2 < \ldots < j_n < \ldots\}$ with $\delta(J) = 1$ such that $g - \lim_{n\to\infty} x_{j_n} = \lambda$ then there exists a positive integer N such that

$$g(x_j - \lambda) < \varepsilon \text{ for } j > N.$$

Put

$$J_\varepsilon(t) = \left\{n \in \mathbb{N} : g(x_j - \lambda) \geq \varepsilon\right\}$$

and $J' = \{J_{N+1}, J_{N+2}, \ldots\}$. Then $\delta(J') = 1$ and $J_\varepsilon \subseteq \mathbb{N} \backslash J'$ which implies that $\delta(L_\varepsilon) = 0$. Hence $g(stat) - \lim x = \lambda$. □

Theorem 7. *Let* $\left(\left[\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|\right], g\right)$ *be a complete paranormed space. Then a sequence* $x = (x_j)$ *of points in* $\left(\left[\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|\right], g\right)$ *is statistically convergent if and only if it is statistically Cauchy.*

Proof. Suppose that $g(stat) - \lim x = \lambda$, then we get $\delta(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \left\{j \in \mathbb{N} : g(x_j - \lambda) \geq \frac{\varepsilon}{2}\right\}.$$

This implies

$$\delta(A^c(\varepsilon)) = \delta(\{j \in \mathbb{N} : g(x_j - \lambda)) < \varepsilon\}) = 1.$$

Let $l \in A^c(\varepsilon)$, then $g(x_l - \lambda) < \frac{\varepsilon}{2}$. Now let

$$B(\varepsilon) = \left\{j \in \mathbb{N} : g(x_l - x_j) \geq \varepsilon\right\}.$$

We need to show that $B(\varepsilon) \subset A(\varepsilon)$. Let $j \in B(\varepsilon)$ then $g(x_l - x_j) \geq \varepsilon$ and hence $g(x_j - \lambda)) \geq \varepsilon$ that $j \in A(\varepsilon)$. Otherwise if $g(x_j - \lambda)) < \varepsilon$ then

$$\varepsilon \leq g(x_j - x_l) \leq g(x_j - \lambda) + g(x_l - \lambda) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is not possible. Hence $B(\varepsilon) \subset A(\varepsilon)$, implies that $x = (x_j)$ is $g(stat)$-convergent.

Conversely, suppose that $x = (x_j)$ is $g(stat)$-Cauchy but not $g(stat)$-convergent. Then there exists $t \in \mathbb{N}$ such that $\delta(G(\varepsilon)) = 0$. where

$$G(\varepsilon) = \left\{j \in \mathbb{N} : g(x_j - x_t) \geq \varepsilon\right\}$$

and $\delta(D(\varepsilon)) = 0$, where

$$D(\varepsilon) = \left\{ j \in \mathbb{N} : g(x_j - \lambda) < \frac{\varepsilon}{2} \right\}$$

i.e, $\delta(D^c(\varepsilon)) = 1$, since $g(x_j - x_l) \leq 2g(x_j - \lambda) < \varepsilon$. If $g(x_j - \lambda) < \frac{\varepsilon}{2}$ then $\delta(G^c(\varepsilon)) = 0$, i.e, $\delta(G(\varepsilon)) = 1$ which leads to a contradiction since $x = (x_j)$ was $g(stat)$-Cauchy. Hence $x = (x_j)$ must be $g(stat)$-convergent. □

Theorem 8. *If* $0 < p < \infty$ *and* $x_j \to \lambda[C, g]_p$*, then* $x = (x_j)$ *is* g*-statistically convergent to* λ *in* $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$.

Proof. Let $x_j \to \lambda[C, g]_p$, then

$$\begin{aligned} \frac{1}{k}\sum_{j=1}^{k}(g(x_j - \lambda e))^p \geq & \frac{1}{k} \sum_{\substack{j=1 \\ g(x_j - \lambda e) \geq \varepsilon}}^{k} (g(x_j - \lambda e))^p \\ \geq & \frac{\varepsilon^p}{k}|K_\varepsilon|. \end{aligned}$$

Since $\lim_{k\to\infty} \frac{1}{k}|K_\varepsilon| = 0$ and so $\delta(K_\varepsilon) = 0$, where $K_\varepsilon = \{j \leq k : g(x_j - \lambda e) \geq \varepsilon\}$. Hence $x = (x_j)$ is statistically convergent to λ in $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$. □

Theorem 9. *If* $x = (x_j)$ *is* g*-statistically convergent to* λ *in* $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$ *then* $x_j \to \lambda[C, g]_p$.

Proof. Suppose that $x = (x_j)$ is g-statistically convergent to λ in $\big([\mathscr{M}, [F], p, \|\cdot, \ldots, \cdot\|], g\big)$. Then for $\varepsilon > 0$, we have $\delta(K_\varepsilon) = 0$, where $K_\varepsilon = \{j \leq k : g(x_j - \lambda e) \geq \varepsilon\}$. Since $x = (x_j) \in l^\infty(M, p, \|\cdot, \ldots, \cdot\|)$, then there exists $K > 0$ such that

$$\left[M\left\|\left(\frac{x_j - \lambda e}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right]^{p_k} \leq K,$$

for all j. Thus,

$$g(x_j - \lambda e) = \sup_{\substack{n\geq 1,\, q\geq 1 \\ 0\neq z_1,\ldots,z_{n-1}\in X}} \left(\frac{1}{n}\sum_{j=q}^{q+n-1}\left[M\left\|\left(\frac{x_j - \lambda e}{\rho}, z_1, \ldots, z_{n-1}\right)\right\|\right]^{p_k}\right) \leq K.$$

Hence we have result from the following inequality

$$\begin{aligned} \frac{1}{k}\sum_{j=1}^{k}(g(x_j - \lambda e))^p = & \frac{1}{k}\sum_{\substack{j=1 \\ j\notin K_\varepsilon}}^{k}(g(x_j - \lambda e))^p + \frac{1}{k}\sum_{\substack{j=1 \\ j\in K_\varepsilon}}^{k}(g(x_j - \lambda e))^p \\ \leq & \varepsilon^p + \frac{K^p}{k}|K_\varepsilon|. \end{aligned}$$

Let A and B be two sequence spaces. We use the notation $A_{reg} \subset B_{reg}$ to mean if the sequence x converges to a limit λ in A then the sequence x converges to the same limit in B. □

Theorem 10. $(S_{([\mathscr{M},[F],p,\|\cdot,\ldots,\cdot\|],g)})_{reg} = ([C,g]_p)_{reg}$.

Proof. The proof can be done by combining Theorem 8 with Theorem 9 so we omit it. □

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