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Narrowing Cohomology for Complex S^6

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Abstract. We compute Bott Chern, and Aeppli cohomology for a complex structure on the six sphere, S^6. We also give a table for the hodge numbers for the Bott-Chern (and thus also Aeppli) cohomology where hodge numbers are given in terms of whole number parameters $a = h^{2,0}_{\bar{\partial}} - h^{1,0}_{\bar{\partial}}$, $c = h^{0,2}_{\bar{\partial}}$,$d = h^{1,2}_{\bar{\partial}}$, $h^{2,0}_{\bar{\partial}}$, $h^{1,1}_{BC}$, and $h^{2,2}_{BC}$. As an example, we work out the Bott-Chern hodge numbers completely in the hypothetical case that the Dolbeault cohomology has $h^{2,0} = a = c = d = 0$.

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1. Introduction

The existence of a complex structure on S^6 has been a persistent question for many years. In 1954, Hirzebruch[6] showed that if a complex structure on S^6 does exist, then by blowing up a point, one obtains an exotic complex structure on $\mathbb{CP}_3$. In fact, these complex structures on S^6 and $\mathbb{CP}_3$ are non-Kahler. In 1987 Lebrun[8] showed that a complex structure on S^6 cannot be compatible with the standard metric on S^6. In 1998, Campana, Demailly, and Pertenell[3] showed that a complex S^6 has no global non-constant meromorphic functions. In 2000, Huckleberry, Kebekus, and Peternell showed it is not almost homogeneous. Recently in 2015, Etesi[4] has published an article which constructs a complex structure on S^6.

In this paper, we search for the Dolbeault, Bott-Chern and Aeppli cohomology hodge numbers for a complex S^6. In 1997, Gray[5] showed that for the Dolbeault hodge numbers, we have $h^{3,0} = h^{0,3} = 0$ and $h^{0,1} \geq 1$. In 2000, Ugarte essentially gave the following for the Dolbeault cohomology on S^6 which we shall summarise shortly in a table. Let $a = h^{2,0}_2$ where $h^{2,0}_2 = dim_{\mathbb{C}}\, E^{2,0}_2$ from the Frohlicher spectral sequence. Ugarte shows that $h^{2,0}_2 = h^{2,0} - h^{1,0}$. Now, let $c = h^{0,2}$, and $d = h^{2,1}$. We have Ugarte's results in the following:

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Table 1: Ugarte: $h^{p,q}$ for a Complex Structure on S^6

0	$h^{1,0}+a$	$h^{1,0}$	1
c	d	$d-a+1$	$c+1$
$c+1$	$d-a+1$	d	c
1	$h^{1,0}$	$h^{1,0}+a$	0

where $0 \leq a \leq c+1$, and $c \leq d$.

2. Some results on the Dolbeault cohomology of compact complex manifolds and of complex S^6.

We begin with the result of Gray[5], (see also Ugarte[11] and Brown[2]):

Theorem 1. *(Gray) Let X be a compact complex manifold of complex dimension n such that $b^n(X) = 0$. Any complex structure on X has the property $h^{n,0} = h^{0,n} = 0$.*

We will be supposing that X is a complex manifold with $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = H^n(X,\mathbb{C}) = 0$. (For example S^6 with a complex structure). By above we have of course, $h^{n,0} = h^{0,n} = 0$. Note that this implies that the associated canonical bundle to the complex structure, K, is not holomorphically trivial. We also note that because $H^2(X,\mathbb{Z}) = 0$, we have that the first Chern class of K (and for that matter any complex line bundle on X) is 0.

It is straight forward to show for such complex n-fold X, a result of Gray on complex S^6 (See also Brown[2].)

Theorem 2. $h^{0,1} \geq 1$ *(Gray)*

Proof. This can be seen by considering the short exact sequence of sheaves:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}_* \to 0$$

and (the portion of) the resulting long exact sequence

$$\ldots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}_*) \to H^1(X,\mathbf{Z}) \to \ldots$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions on X and $\mathcal{O}_*$ denotes the sheaf of nowhere zero holomorphic functions on X . Also, the map $\mathbb{Z} \to \mathcal{O}$ is the map, $k \mapsto ik$ and the map $\mathcal{O} \to \mathcal{O}_*$ is the exponential map, $f \mapsto exp(f)$. Since $H^1(X,\mathbf{Z}) = H^2(X,\mathbf{Z}) = 0$ we have $H^1(X,\mathcal{O}) = H^1(X,\mathcal{O}_*)$. Note that $1 \neq K \in H^1(X,\mathcal{O}_*)$ and thus $h^{0,1} \neq 0$.

We now specialize to X being a three dimensional complex manifold with $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = H^3(X,\mathbb{C}) = 0$, i.e. topologically equivalent to S^6.

Lemma 1. $h^{1,0} \leq h^{2,0}$

Proof. We consider the portion of the Frohlicher spectral sequence:

$$\partial \ : \ H^{1,0}(X) \to H^{2,0}(X)$$

We shall show that this is an injective map of vector spaces. Indeed, let ϕ be a $\bar{\partial}$ closed $1,0$ form, such that $\partial[\phi] = 0$. Thus, $[\partial\phi] = 0$ and by type we have $\partial\phi = 0$. Therefore, $d\phi = (\partial + \bar{\partial})\phi = 0$. Since, $b^1 = 0$, we have $\phi = df$ for some complex valued, C^∞ function, f. Considering type, we have $\bar{\partial} f = 0$ and thus f as a global holomorphic function is a constant. Thus $\phi = 0$. This shows ∂ induces an injective map from $H^{1,0}(X) \to H^{2,0}(X)$.

This can also be directly deduced from the result of Ugarte[11] that $E_2^{1,0} = 0$ and $E_2^{0,0} = 1$. More specifically, Brown[2] gives the following table derived by Ugarte for $E_2^{p,q}$ of the Frohlicher spectral sequence for a complex structure on S^6:

Table 2: $E_2^{p,q}$ for a Complex Structure on S^6

0	a	0	1
b	b	0	a
a	0	b	b
1	0	a	0

The bottom row of the table corresponds to the portion of the Frohlicher sequence

$$H^{0,0} \to H^{1,0} \to H^{2,0} \to H^{3,0} \quad .$$

Since $H^{0,0} = \mathbb{C}$, $E^{0,0} = 1$, and $E_2^{1,0} = 0$ the sequence reduces to

$$0 \to H^{1,0} \to H^{2,0} \to 0 \quad .$$

It is exact at $H^{1,0}$ and thus

$$\partial \ : \ H^{1,0}(X) \to H^{2,0}(X)$$

is injective. Note that $H^{1,0} = H^{2,0}$ if and only if $a = 0$ and if $H^{1,0} = 0$ then $H^{2,0} = a$.

Huckleberry, Kebekus and Peternell [7] gave a proof pointed out to them by M. Toma that $h^{1,0} \leq 1$. We give a somewhat different but related proof here. The present author is indebted to Daniel Angella for pointing out the correct statement of Huckleberry, Kebekus, Peternell and Toma's result.

Lemma 2. *(Huckleberry, Kebekus, Peternell, Toma)* $h^{1,0} \leq 1$*, i.e.* $h^{1,0} = 0$ *or* 1

Proof. Indeed, if $h^{1,0} = 2$, then $h^{2,0} \geq 2$. Let ϕ_1 and ϕ_2 be two linearly independent $\bar{\partial}$-closed global $1,0$-forms. Let $\Phi_1 = \phi_1 \wedge \phi_2$ This is a global $\bar{\partial}$-closed $2,0$-form on X that is not identically zero. Since $h^{2,0} \geq 2$ we can select Φ_2 another $\bar{\partial}$-closed global $2,0$-form that is linearly independent of Φ_1.

We may choose a point, $p \in X$ such that Φ_1 and Φ_2 are non-zero and linearly independent at p. Note that ϕ_1 and ϕ_2 are also non-zero and linearly independent of each other at p since $\Phi_1(p) = \phi_1(p) \wedge \phi_2(p)$ is not zero.

Let $\eta_p \in T_p^{1,0}$ be linearly independent of $\phi_1(p)$ and $\phi_2(2)$, completing a basis for $T_p^{1,0}$. Thus for $\Phi_(p)$ and some complex numbers, a, b_1, b_2, we have,

$$\Phi_2(p) = a\phi_1 \wedge \phi_2 + b_1\eta \wedge \phi_1 + b_2\eta \wedge \phi_2 \quad .$$

Now b_1 and b_2 are not both zero. Without loss of generality, assume $b_1 \neq 0$. Hence $\Phi_2(p) \wedge \phi_2 \neq 0$ and $\Phi_2 \wedge \phi_2$ is a non-zero holomorphic $3,0$-form on X. This contradicts $h^{3,0} = 0$. We must then have $h^{1,0} \leq 1$.

We summarize with two tables of the plausible hodge numbers (with $h^{0,0}$ in the bottom lefthand corner) for Dolbeault cohomology for a complex structure on S^6:

Table 3: ($h^{1,0} = 1$): $h^{p,q}$ for a Complex Structure on S^6

0	$a+1$	1	1
c	d	$d-a+1$	$c+1$
$c+1$	$d-a+1$	d	c
1	1	$a+1$	0

where $0 \leq a \leq c+1$, and $c \leq d$.

Table 4: ($h^{1,0} = 0$): $h^{p,q}$ for a Complex Structure on S^6

0	a	0	1
c	d	$d-a+1$	$c+1$
$c+1$	$d-a+1$	d	c
1	0	a	0

where $0 \leq a \leq c+1$, and $c \leq d$.

Note that we have in both cases, $a = h^{2,0} - h^{1,0}$.

3. Aeppli and Bott-Chern Cohomology on complex S^6.

The Aeppli cohomology of a complex manifold is defined by the vector spaces (see Popovici [9]) :

$$H_A^{p,q} = \frac{ker(\partial\bar{\partial} : C^{\infty\ p,q} \to C^{\infty\ p+1,q+1})}{im(\partial : C^{\infty\ p-1,q} \to C^{\infty\ p,q}) + im(\bar{\partial} : C^{\infty\ p,q-1} \to C^{\infty\ p,q})}$$

The Bott-Chern cohomology of a complex manifold is defined by the vector spaces (again see Popovici [9]) :

$$H^{p,q}_{BC} = \frac{ker(\partial : C^{\infty\ p,q} \to C^{\infty\ p+1,q}) \cap ker(\bar{\partial} : C^{\infty\ p,q} \to C^{\infty\ p,q+1})}{im(\partial\bar{\partial} : C^{\infty\ p-1,q-1} \to C^{\infty\ p,q})}$$

On compact complex manifolds, there is a harmonic theory for each of these cohomologies which ensures that they are finite dimensional complex vector spaces. Let $h^{p,q}_A = dim(H^{p,q}_A)$ and $h^{p,q}_{BC} = dim(H^{p,q}_{BC})$ .We note (see Popovici[9]) that $h^{p,q}_A = h^{q,p}_A$, $h^{p,q}_{BC} = h^{q,p}_{BC}$ and $h^{p,q}_A = h^{n-p,n-q}_{BC}$.

The Serre duality of Bott-Chern and Aeppli cohomology is due to Schweitzer[10]

We try to narrow down as much as possible the Aeppli and Bott-Chern Cohomology on complex S^6.

3.1. Some long exact sequences of cohomology

Consider the following sequence of maps of cohomology on a complex manifold X:

$$0 \to H^{p,0}_{BC} \xrightarrow{/im(\bar{\partial})} H^{p,0}_{\bar{\partial}} \xrightarrow{/(im(\bar{\partial})+im(\partial))} H^{p,0}_{A} \xrightarrow{\bar{\partial}} H^{p,1}_{BC} \xrightarrow{/im(\bar{\partial})} \cdots$$

$$\cdots \xrightarrow{/(im(\bar{\partial})+im(\partial))} H^{p,n-1}_{A} \xrightarrow{\bar{\partial}} H^{p,n}_{BC} \xrightarrow{/im(\bar{\partial})} H^{p,n}_{\bar{\partial}} \xrightarrow{/(im(\bar{\partial})+im(\partial))} H^{p,n}_{A} \to 0 \quad .$$

We prove some claims and lemmas below about this sequence of maps. The sophisticated readers may just read the claims and lemmas skipping over their proofs if they appear to be straight forward or obvious.

Lemma 3. *The sequence of maps above is exact at* $H^{p,q}_A$. *Namely,*

$$\mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H^{p,q}_{\bar{\partial}} \to H^{p,q}_A) = \ker(\bar{\partial} : H^{p,q}_A \to H^{p,q+1}_{BC}) \quad .$$

Proof. Let

$$[\phi]_A \in \ker(\bar{\partial} : H^{p,q}_A \to H^{p,q+1}_{BC})$$

where ϕ is some smooth p,q-form representative of $[\phi]_A$. We have then that

$$\bar{\partial}([\phi]_A) = [\bar{\partial}(\phi)]_{BC} = 0$$

in $H^{p,q+1}_{BC}$, i.e.

$$\bar{\partial}(\phi) = \bar{\partial}\partial\theta$$

for some $p-1,q$-form θ. Thus

$$\bar{\partial}(\phi - \partial\theta) = 0$$

and $\phi - \partial\theta$ is a $\bar{\partial}$-closed p,q-form. We conclude then that

$$[\phi]_A = [\phi - \partial\theta]_A \in \mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H^{p,q}_{\bar{\partial}} \to H^{p,q}_A)$$

and

$$\ker(\bar{\partial} : H_A^{p,q} \to H_{BC}^{p,q+1}) \subseteq \mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \quad .$$

Now let

$$[\phi]_A \in \mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q})$$

where ϕ is some smooth p, q-form representative of $[\phi]_A$. We may assume that ϕ is $\bar{\partial}$-closed since

$$[\phi]_A \in \mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \quad .$$

Clearly, $\bar{\partial}([\phi]_A) = [\bar{\partial}\phi]_{BC} = 0$ in $H_{BC}^{p,q+1}$. Thus

$$[\phi]_A \in \ker(\bar{\partial} : H_A^{p,q} \to H_{BC}^{p,q+1})$$

and

$$\mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \subseteq \ker(\bar{\partial} : H_A^{p,q} \to H_{BC}^{p,q+1}) \quad .$$

The two inclusions give

$$\mathrm{im}(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) = \ker(\bar{\partial} : H_A^{p,q} \to H_{BC}^{p,q+1}) \quad .$$

Lemma 4. *The sequence of maps above is exact at $H_{BC}^{p,q}$. Namely,*

$$\ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) = \mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q+1}) \quad .$$

Proof. Let

$$[\gamma]_{BC} \in \mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q})$$

where γ is some smooth p, q-form representative of $[\gamma]_{BC}$. Since

$$[\gamma]_{BC} \in \mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q})$$

we may assume $\gamma = \bar{\partial}\mu$ for some $p, q-1$-form μ such that $\partial\bar{\partial}\mu = 0$. Clearly,

$$\gamma/im(\bar{\partial}) = \bar{\partial}\mu/im(\bar{\partial}) = 0/im(\bar{\partial})$$

and

$$[\gamma] \in \ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \quad .$$

Thus

$$\mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q}) \subseteq \ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \quad .$$

Now let

$$[\gamma]_{BC} \in \ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q})$$

where γ is some smooth p, q-form representative of $[\gamma]_{BC}$. Since

$$[\gamma]_{BC} \in \ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q})$$

we have $\gamma/im(\bar{\partial}) = 0/im(\bar{\partial})$. Thus $\gamma = \bar{\partial}\mu$ for some $p, q-1$-form μ. Now, γ is $\bar{\partial}$-closed and also ∂-closed. Thus,

$$\partial\gamma = \partial\bar{\partial}\mu = 0 \quad .$$

This shows $[\gamma]_{BC} = \bar{\partial}([\mu]_A)$ and

$$[\gamma]_{BC} \in \mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q}) \quad .$$

Hence,

$$\ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \subseteq \mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q})$$

and the two inclusions give the equality,

$$\ker(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \subseteq \mathrm{im}(\bar{\partial} : H_A^{p,q-1} \to H_{BC}^{p,q}) \quad .$$

Lemma 5. *If the Betti number, $b^{p+q} = 0$ on our complex manifold, X, then the sequence above is exact at $H_{\bar{\partial}}^{p,q}$. More specifically,*

$$\mathrm{im}(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) = \ker(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \quad .$$

Proof. Let

$$[\phi]_{\bar{\partial}} \in \mathrm{im}(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \quad .$$

We have $d\phi = \partial\phi + \bar{\partial}\phi = 0$. Since $b^{p+q} = 0$, we have that $\phi = d\lambda$ for some $p+q-1$-form,λ on X. Thus $\phi = \partial\lambda^{p-1,q} + \bar{\partial}\lambda^{p,q-1}$ where $\partial\lambda^{p-1,q}$ and $\bar{\partial}\lambda^{p,q-1}$ are the projections of λ to its $p-1, q$ and $p, q-1$ parts respectively. Thus $[\phi]_A = 0$ in $H_A^{p,q}$, and

$$[\phi]_{\bar{\partial}} \in \ker(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q})$$

and we have

$$\mathrm{im}(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \subseteq \ker(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \quad .$$

Now let

$$[\phi]_{\bar{\partial}} \in \ker(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \quad .$$

Note that ϕ is $\bar{\partial}$-closed. Since $[\phi]_A = 0$, we have

$$\phi = \partial\mu + \bar{\partial}\nu$$

for some $p-1, q$-form μ and some $p, q-1$-form, ν. Since $\bar{\partial}\phi = 0$ we have $\bar{\partial}(\partial\mu) = 0$ and $[\phi]_{\bar{\partial}} = [\partial\mu]$ in $H_{\bar{\partial}}^{p,q}$. We also have obviously that $\partial(\partial\mu) = 0$ and thus

$$[\phi]_{\bar{\partial}} = (/im(\bar{\partial}))([\partial\mu]_{BC}) \in \mathrm{im}(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \quad .$$

and hence

$$\ker(/(im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_A^{p,q}) \subseteq \mathrm{im}(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \quad .$$

The two inclusions allow us to conclude that

$$\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A})=\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}}) \quad .$$

Notice above that we nowhere used that $b^{p+q}=0$ in proving the inclusion,

$$[\phi]_{\bar{\partial}}\in\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A})\subseteq\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}}) \quad .$$

This suggests that we define what we shall call BCA-cohomology:

$$E^{p,q}_{BCA}=\frac{ker(\partial)\cap ker(\bar{\partial})}{ker(\bar{\partial})\cap im(\partial)+ker(\partial)\cap im(\bar{\partial})} \quad .$$

This definition is along the lines of Varouchas[12] who defines similar in spirit vector spaces to create long exact sequences involving Bott-Chern, Dolbeault, and Aeppli cohomology. We refer the reader also to Angella[1] for more details.

It is easy to show that if $b^{p+q}=0$ then $E^{p,q}_{BCA}=\{0\}$. We have the following claim:

Lemma 6. *For a compact complex manifold, X, the sequence above is exact at $H^{p,q}_{\bar{\partial}}$ if and only if $E^{p,q}_{BCA}=\{0\}$.*

Proof. Let us first assume that $E^{p,q}_{BCA}=\{0\}$. We need to only to show

$$\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}})\subseteq\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A})$$

since the reverse inclusion has already been shown true in general. Let

$$[\phi]_{\bar{\partial}}\in\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}}) \quad .$$

We may assume $\phi\in ker(\partial)\cap ker(\bar{\partial})$. Since $E^{p,q}_{BCA}=\{0\}$, we have

$$\phi\in(ker(\bar{\partial})\cap im(\partial)+ker(\partial)\cap im(\bar{\partial}))\subseteq(im(\partial)+im(\bar{\partial})) \quad .$$

Hence

$$[\phi]_{\bar{\partial}}\in\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A})$$

and

$$\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}})\subseteq\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A}) \quad .$$

Thus

$$\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}})=\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A}) \quad .$$

and the sequence above is exact at $H^{p,q}_{\bar{\partial}}$.

In the other direction we assume the sequence above is exact at $H^{p,q}_{\bar{\partial}}$ and

$$\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}})=\ker(/(im(\bar{\partial})+im(\partial)):H^{p,q}_{\bar{\partial}}\to H^{p,q}_{A}) \quad .$$

Let $\phi\in ker(\partial)\cap ker(\bar{\partial})$. Then

$$[\phi]_{\bar{\partial}}\in\operatorname{im}(/im(\bar{\partial}):H^{p,q}_{BC}\to H^{p,q}_{\bar{\partial}})$$

and thus

$$[\phi]_{\bar{\partial}} \in \ker(/(im(\bar{\partial}) + im(\partial)) : H^{p,q}_{\bar{\partial}} \to H^{p,q}_{A}) \quad .$$

Specifically, $[\phi]_A = 0$. Hence

$$\phi = \partial\mu + \bar{\partial}\nu$$

for some $p-1, q$-form, μ and $p, q-1$-form ν. Since $\bar{\partial}\phi = 0$ we have $\bar{\partial}(\partial\mu) = 0$ and $\partial\mu \in ker(\bar{\partial}) \cap im(\partial)$. Similiarly, Since $\partial\phi = 0$ we have $\partial(\bar{\partial}\nu) = 0$ and $\bar{\partial}\nu \in ker(\partial) \cap im(\bar{\partial})$. Thus

$$\phi \in (ker(\bar{\partial}) \cap im(\partial) + ker(\partial) \cap im(\bar{\partial}))$$

and $E^{p,q}_{BCA} = \{0\}$.

Lemma 7. *Let X be a compact complex manifold of complex dimension n. If we have the first Betti number is zero ($b^1 = 0$), then*

$$h^{1,0}_{BC} = h^{0,1}_{BC} = h^{n,n-1}_{A} = h^{n-1,n}_{A} = 0 \quad .$$

Proof. Consider a $1,0$-form, μ such that $\mu \in ker(\partial) \cap ker(\bar{\partial})$. We have then $d\mu = 0$. Since $b^1 = 0$, we have $\mu = df$ form some function f. Thus $\mu = \partial f + \bar{\partial} f$. We must have $\bar{\partial} f = 0$ since μ is a $1,0$-form. Hence f is a holomorphic function on a compact complex manifold and thus must be a constant function. Finally, we have $\mu = \partial f = 0$ since f is constant. Thus $ker(\partial) \cap ker(\bar{\partial}) = \{0\}$ for $1,0$-forms and $h^{1,0}_{BC} = 0$. The other equalities follow from "complex conjugation" and Serre duality mentioned above.

Lemma 8. *The map of vector spaces,*

$$H^{p,0}_{BC} \quad \stackrel{/im(\bar{\partial})}{\longrightarrow} H^{p,0}_{\bar{\partial}} \quad ,$$

is injective.

Proof. A $p,0$-form cannot be in the image of $\bar{\partial}\partial$ or $\bar{\partial}$. Thus, $(\ /im(\bar{\partial})\ )([\mu]_{BC}) = [0]_{\bar{\partial}}$ if and only if $\mu = 0$.

We also note on the end of the sequence we have

$$H^{p,n}_{A} \stackrel{\bar{\partial}}{\to} H^{p,n+1}_{BC} = \{0\} \quad .$$

Now we focus again on our compact complex manifold, X, being topologically equivalent to S^6. Noting, that $b^0 = b^6 = 1$,and $b^j = 0$ for $1 \le j \le 5$, we have, using the above lemmas, the following:

Theorem 3. *For a complex structure on a manifold X topologically equivalent to S^6, we have the following long exact sequences of vector spaces:*
0. ($p = 0$)

$$\begin{array}{ccccccc}
0 & \to & H^{0,1}_{\bar{\partial}} & \stackrel{/(im(\bar{\partial})+im(\partial))}{\longrightarrow} & H^{0,1}_{A} & \stackrel{\bar{\partial}}{\to} & \\
H^{0,2}_{BC} & \stackrel{/im(\bar{\partial})}{\longrightarrow} & H^{0,2}_{\bar{\partial}} & \stackrel{/(im(\bar{\partial})+im(\partial))}{\longrightarrow} & H^{0,2}_{A} & \stackrel{\bar{\partial}}{\to} & \\
H^{0,3}_{BC} & \stackrel{/im(\bar{\partial})}{\longrightarrow} & H^{0,3}_{\bar{\partial}} & \stackrel{/(im(\bar{\partial})+im(\partial))}{\longrightarrow} & H^{0,3}_{A} & \stackrel{\bar{\partial}}{\to} & 0
\end{array}$$

1. (p = 1)

$$\begin{array}{ccccccc} 0 & \rightarrow & H_{\bar\partial}^{1,0} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{1,0} & \overset{\bar\partial}{\rightarrow} & \\ H_{BC}^{1,1} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{1,1} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{1,1} & \overset{\bar\partial}{\rightarrow} & \\ H_{BC}^{1,2} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{1,2} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{1,2} & \overset{\bar\partial}{\rightarrow} & \\ H_{BC}^{1,3} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{1,3} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{1,3} & \overset{\bar\partial}{\rightarrow} & 0 \end{array}$$

2. (p = 2)

$$\begin{array}{cccccccc} 0 & \rightarrow & H_{BC}^{2,0} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{2,0} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{2,0} & \overset{\bar\partial}{\rightarrow} \\ & & H_{BC}^{2,1} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{2,1} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{2,1} & \overset{\bar\partial}{\rightarrow} \\ & & H_{BC}^{2,2} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{2,2} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{2,2} & \overset{\bar\partial}{\rightarrow} \\ & & H_{BC}^{2,3} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{2,3} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{2,3} & \overset{\bar\partial}{\rightarrow} \; 0 \end{array}$$

3. (p = 3)

$$\begin{array}{cccccccc} 0 & \rightarrow & H_{BC}^{3,0} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{3,0} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{3,0} & \overset{\bar\partial}{\rightarrow} \\ & & H_{BC}^{3,1} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{3,1} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & H_A^{3,1} & \overset{\bar\partial}{\rightarrow} \\ & & H_{BC}^{3,2} & \overset{/im(\bar\partial)}{\longrightarrow} & H_{\bar\partial}^{3,2} & \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} & 0 & \end{array}$$

One of our goals is to complete as much as possible a table of hodge numbers for Bott-Chern cohomology on a complex S^6. The table of hodge numbers for Aeppli cohomology is, of course, given by the Serre duality with Bott-Chern cohomology. Since $h_{BC}^{p,q} = h_{BC}^{q,p}$, we may concern ourselves with just the bottom triangle of the table.

Using $h_{\bar\partial}^{3,0} = 0$, we can see straight away from the long exact sequence for $p = 3$ that $h_{BC}^{3,0} = 0$. We also see that

$$h_{BC}^{2,3} = h_{BC}^{3,2} \geq h_{\bar\partial}^{3,2} = h_{\bar\partial}^{0,1} \geq 1 \quad .$$

We also show the following:

Lemma 9. *$h_{BC}^{2,0} = h_{\bar\partial}^{2,0}$ and $h_{BC}^{3,1} = h_{\bar\partial}^{3,1} = c$.*

Proof. We know that if ϕ is a $\bar\partial$-closed 2,0-form, then $\partial\phi = 0$. Thus

$$d\phi = \partial\phi + \bar\partial\phi = 0 \quad .$$

Since $b^1 = 0$, we have

$$\phi = d\eta = \partial\eta + \bar\partial\eta$$

for some 1-form, η. Thus the image of the map,

$$H_{\bar\partial}^{2,0} \overset{/(im(\bar\partial)+im(\partial))}{\longrightarrow} H_A^{2,0}$$

is $\{0\}$. Hence, since the $p = 2$ sequence is exact, the map,

$$H^{2,0}_{BC} \overset{/im(\bar{\partial})}{\longrightarrow} H^{2,0}_{\bar{\partial}}$$

is an isomorphism and we have $h^{2,0}_{BC} = h^{2,0}_{\bar{\partial}}$.

Notice in the above argument we could also have concluded more specifically for our $\bar{\partial}$-closed 2,0-form, ϕ, that

$$\phi = \partial\eta$$

for some 1,0-form, η. In a similar manner, we take μ to be a 0,2-form representative of an element in $H^{0,2}_{BC}$. Since $d\mu = 0$, we may conclude that $\mu = \bar{\partial}\chi$ for some 0,1-form, χ. Thus the image of the map,

$$H^{0,2}_{BC} \overset{/im(\bar{\partial})}{\longrightarrow} H^{0,2}_{\bar{\partial}}$$

is $\{0\}$ and hence, using the fact $h^{0,3}_{BC} = h^{3,0}_{BC} = 0$, the map,

$$H^{0,2}_{\bar{\partial}} \overset{/(im(\bar{\partial})+im(\partial))}{\longrightarrow} H^{0,2}_{A}$$

is an isomorphism. We have then

$$h^{3,1}_{BC} = h^{0,2}_{A} = c \quad .$$

The fact that the image of the map,

$$H^{0,2}_{BC} \overset{/im(\bar{\partial})}{\longrightarrow} H^{0,2}_{\bar{\partial}}$$

is $\{0\}$ also shows from the p=2 sequence that we have the short exact sequence

$$0 \to H^{0,1}_{\bar{\partial}} \overset{/(im(\bar{\partial})+im(\partial))}{\longrightarrow} H^{0,1}_{A} \overset{\bar{\partial}}{\to} H^{0,2}_{BC} \overset{/im(\bar{\partial})}{\longrightarrow} 0$$

and thus that

$$h^{3,2}_{BC} = h^{0,1}_{A} = h^{0,1}_{\bar{\partial}} + h^{2,0}_{\bar{\partial}} = c + 1 + h^{2,0}_{\bar{\partial}} \quad .$$

Please recall that

$$a = h^{2,0}_{\bar{\partial}} - h^{1,0}_{\bar{\partial}} \quad .$$

In trying to be as complete as possible, we also show:

Theorem 4. *If X is a compact complex manifold then $h^{0,0}_{BC} = 1$. Furthermore, if $b^1 = 0$ then $h^{3,3}_{BC} = 1$.*

Proof. A 0,0-form or function, f, on a compact complex manifold, X, such that $df = 0$ is a constant. Thus, $H^{0,0}_{BC} = \mathbb{C}$ and $h^{0,0}_{BC} = 1$.

We consider now $H^{0,0}_A$. We know that $H^{0,0}_{\bar\partial}$ consists of the constant functions. The sequence,

$$0 \to H^{0,0}_{BC} \xrightarrow{/im(\bar\partial)} H^{0,0}_{\bar\partial} \xrightarrow{/(im(\bar\partial)+im(\partial))} H^{0,0}_A \xrightarrow{\bar\partial} H^{0,1}_{BC}$$

is not exact at $H^{0,0}_{\bar\partial}$ but it *is* exact at $H^{0,0}_A$. Thus, since $h^{0,1}_{BC} = 0$, we know that

$$H^{0,0}_A = \mathrm{im}(/(im(\bar\partial) + im(\partial)) : H^{0,0}_{\bar\partial} \to H^{0,0}_A) \quad .$$

Now if f is a constant function, then $f \notin (im(\bar\partial) + im(\partial))$ so

$$\mathrm{im}(/(im(\bar\partial) + im(\partial)) : H^{0,0}_{\bar\partial} \to H^{0,0}_A) = \mathbb{C}$$

and $H^{0,0}_A = \mathbb{C}$. By the Serre duality, we have $h^{3,3}_{BC} = 1$.

Recall that $h^{p,q}_{BC} = h^{q,p}_{BC}$. We thus have so far, for our essential lower triangle for Bott-Chern cohomology

$$\begin{array}{cccc}
 & & & 1 \\
 & & h^{2,2}_{BC} & c + 1 + h^{2,0}_{\bar\partial} \\
 & h^{1,1}_{BC} & h^{2,1}_{BC} & c \\
1 & 0 & h^{2,0}_{\bar\partial} & 0
\end{array}$$

We still have not computed $h^{1,1}_{BC}, h^{2,2}_{BC}$ and $h^{2,1}_{BC}$. We can determine $h^{2,1}_{BC}$ in terms of the others by plugging our results so far into the following well known result for long exact sequences of vector spaces:

Theorem 5. *If*

$$0 \to A_1 \to A_2 \to \ldots \to A_n \to 0$$

is a long exact sequence of vector spaces with $a_j = \dim(A_j)$ *then*

$$\sum_{j=1}^{n} (-1)^{j+1} a_j \;= 0 \quad .$$

We apply this to the long exact sequence for $p = 1$:

$$h^{1,0} - (c + 1 + h^{2,0}) + h^{1,1}_{BC} - (d - a + 1) + h^{2,2}_{BC} - h^{2,1}_{BC} + d - h^{2,1}_{BC} + c - h^{2,0} + h^{2,0} = 0 \ .$$

This reduces to

$$h_{BC}^{1,1} + h_{BC}^{2,2} = 2h_{BC}^{2,1} + 2 \quad .$$

Thus our essential lower triangle for Bott-Chern cohomology is

$$\begin{array}{cccc} & & & 1 \\ & & h_{BC}^{2,2} & c+1+h_{\bar\partial}^{2,0} \\ & h_{BC}^{1,1} & \frac{h_{BC}^{1,1}+h_{BC}^{2,2}}{2}-1 & c \\ 1 & 0 & h_{\bar\partial}^{2,0} & 0 \end{array}$$

3.2. The Bott-Chern and Aeppli cohomology for a hypothetical possibility of the Dolbeault cohomology on complex S^6

We consider a specific possible scenario of the Dolbeault cohomology on complex S^6. Namely, $h^{2,0} = a = c = d = 0$. In terms of hodge numbers, this is

$$h^{1,0} = h^{2,0} = h^{0,2} = h^{1,2} = 0$$

and

$$h^{0,1} = h^{1,1} = 1 \quad .$$

This is one of the Dolbeault cohomology scenarios suggested at the end of Etesi[4]. In fact, the other cohomology scenario, with

$$h^{1,1} = h^{2,1} = 1$$

and

$$h^{1,0} = h^{2,0} = a = 0$$

is not possible on complex S^6 according to our table for Dolbeault cohomology above since

$$h^{1,1} = h^{1,2} + 1 - a \quad .$$

Etesi does actually in fact also show the incompatibility of this other cohomology scenario.

We look at the following portion of the $p = 2$-long exact sequence:

$$H_A^{2,0} \to H_{BC}^{2,1} \to H_{\bar\partial}^{2,1} \to H_A^{2,1} \to H_{BC}^{2,2} \to H_{\bar\partial}^{2,2} \to H_A^{2,2} \to H_{BC}^{2,3} \to H_{\bar\partial}^{2,3}$$

Using a total abuse of notation where we write just the dimensions of the vector spaces, this is:

$$0 \to h^{2,1}_{BC} \to 0 \to h^{2,1}_{A} \to h^{2,2}_{BC} \to 1 \to h^{2,2}_{A} \to 1 \to 0$$

Since the sequence is exact, we have right away, $h^{2,1}_{BC} = 0$ and thus $h^{2,1}_{A} = 0$ by the Serre duality. By the Frohlicher sequence, we know that $\partial : H^{2,2}_{\bar{\partial}} \to H^{3,2}_{\bar{\partial}}$ is an isomorphism. In particular, we cannot have a non zero $\bar{\partial}$-harmonic 2,2-form being d-closed. Thus we cannot have $H^{2,2}_{BC}$ being isomorphic to $H^{2,2}_{\bar{\partial}}$. Thus $h^{2,2}_{BC} = 0$ and $h^{2,2}_{A} = h^{1,1}_{BC} = 2$. This completes the Bott-Chern and Aeppli cohomology for this possible Dolbeault cohomology.

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