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A Note on Positivity of One-Dimensional Elliptic Differential Operators

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Abstract. We consider the structure of fractional spaces $E_\alpha(C(\mathbb{R}_+),A)$ generated by the positive differential operator A defined by the formula $Au(t) = -u_{tt}(t) + u(t)$ with domain

$$D(A) = \{u : u_{tt}, u \in C(\mathbb{R}_+), u(0) = 0, u(\infty) = 0\},$$

where $\mathbb{R}_+ = [0, \infty)$. It is established that for any $0 < \alpha < 1/2$, the norms in the spaces $E_\alpha(C(\mathbb{R}_+),A)$ and $C^{2\alpha}(\mathbb{R}_+)$ are equivalent. The positivity of the differential operator A in $C^{2\alpha}(\mathbb{R}_+)$ is established.

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1. Introduction

It is well-known that various local and nonlocal boundary value problems for partial differential equation can be considered as an abstract boundary value problem for ordinary differential equation in a Banach space E with a densely defined unbounded operator A. Therefore, the study of various properties of partial differential equations is based on the positivity property of the differential operator in a Banach space [6–8]. Many researcher have studied the positivity of wider class of differential operators (see [12] through [23]).

An differential operator A densely defined in a Banach space E with domain $D(A)$ is called *positive* in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0 < \varphi < \pi$, symmetric with respect to the real axis, and moreover on the edges of this sector

$$S_1(\varphi) = \{\rho e^{i\varphi} : 0 \le \rho \le \infty\}$$

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$$S_2\left(\varphi\right) = \{\rho e^{-i\varphi} : 0 \leq \rho \leq \infty\},$$

and outside of the sector the resolvent $(A-\lambda)^{-1}$ is a subject to the bound (see, [6])

$$\left\|(A-\lambda)^{-1}\right\|_{E\to E} \leq \frac{M}{1+|\lambda|}.$$

The infimum of all such angles φ is called the *spectral angle* of the positive operator A and is denoted by $\varphi(A) = \varphi(E,A)$. The operator A is said to be *strongly positive* in a Banach space E, if $\varphi(E,A) < \frac{\pi}{2}$.

Throughout the paper, M will denote positive constants which can be different from time to time and we are not interested to precise. To stress the fact that the constant depends only on $\alpha, \beta, \ldots$, we will write $M(\alpha, \beta, \ldots)$.

For a positive operator A in the Banach space E, let us define the fractional spaces $E_\alpha = E_\alpha(E,A)(0 < \alpha < 1)$ consisting of those $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \|A(\lambda + A)^{-1}v\|_E + \|v\|_E$$

is finite.

It is well-known that from the positivity of operator A in the Banach space E it follows the positivity of this operator in fractional spaces $E_\alpha = E_\alpha(E,A)(0 < \alpha < 1)$.

In this study, we consider the second order differential operator

$$Au(t) = -u_{tt}(t) + u(t) \tag{1}$$

with domain

$$D(A) = \{u : u_{tt}, u \in C\left(\mathbb{R}_+\right), u(0) = 0, u(\infty) = 0\},$$

where $\mathbb{R}_+ = [0, \infty)$.

The Green's function of A is constructed. The positivity of the operator A in the Banach space $E = C\left(\mathbb{R}_+\right)$ with norm

$$\left\|\varphi\right\|_{C\left(\mathbb{R}_+\right)} = \sup_{t\geq 0} |\varphi(t)|$$

is proved. Moreover, the structure of the fractional spaces $E_\alpha(E,A)$, $\alpha \in (0, 1/2)$ are established and the positivity of A in the Hölder spaces $C^{2\alpha}\left(\mathbb{R}_+\right)$, $\alpha \in (0, 1/2)$ is established.

2. Green's Function of A and Positivity of A in $C\left(\mathbb{R}_+\right)$

To find the Green's function of operator A we need to solve the resolvent equation

$$Au(t) + \lambda u(t) = \varphi(t),\ 0 < t < \infty$$

or

$$\begin{cases} -u_{tt}(t) + (1+\lambda)u(t) = \varphi(t),\ 0 < t < \infty, \\ u(0) = 0,\ u(\infty) = 0 \end{cases} . \tag{2}$$

Let us give a lemma that will be needed below.

Lemma 1. *For $\lambda \geq 0$, equation* (2) *is uniquely solvable and the following formula holds:*

$$u(t) = (A+\lambda)^{-1}\varphi(t) = \int_0^\infty G(t,s)\varphi(s)ds \tag{3}$$

where

$$G(t,s) = \frac{e^{-\sqrt{1+\lambda}|t-s|} - e^{-\sqrt{1+\lambda}(t+s)}}{2\sqrt{1+\lambda}},\ t,s \geq 0.$$

Now, we will prove the positivity of A in the Banach space $C\left(\mathbb{R}_+\right)$.

Theorem 1. *For λ in the sector $\Sigma_{\varphi_0} = \{\lambda = \varrho e^{i\theta}; |\theta| \leq \varphi_0 < \pi/2\}$, the following estimate holds:*

$$\left\|(A+\lambda)^{-1}\right\|_{C(\mathbb{R}_+)\to C(\mathbb{R}_+)} \leq \frac{M(\varphi_0)}{1+|\lambda|} \tag{4}$$

where the resolvent $(A+\lambda)^{-1}$ defined by formula (3).

Proof. For $\lambda = |\lambda| e^{i\varphi} \in \Sigma_{\varphi_0}$, we have $1+\lambda = |1+\lambda| e^{i\psi}$, $\psi \leq \varphi < \varphi_0 < \frac{\pi}{2}$. Then, $\sqrt{1+\lambda} = |1+\lambda|^{1/2} e^{i\frac{\psi}{2}}$ with $\psi < \frac{\pi}{4}$. Clearly, we have

$$\left|\sqrt{1+\lambda}\right| = \sqrt[4]{1+2|\lambda|\cos\varphi + |\lambda|^2} \geq M(\varphi_0)\sqrt{1+|\lambda|}. \tag{5}$$

Using formula (3), estimate (5) and the triangle inequality, we get

$$\begin{aligned}
\left|(A+\lambda)^{-1}f(t)\right| &\leq \frac{\left\|f\right\|_{C(\mathbb{R}_+)}}{M(\varphi_0)\sqrt{1+|\lambda|}} \int_0^\infty e^{-M(\varphi_0)\sqrt{1+|\lambda|}|t-s|}ds \\
&\leq \frac{\left\|f\right\|_{C(\mathbb{R}_+)}}{M(\varphi_0)\sqrt{1+|\lambda|}} \left(\int_0^t e^{-M(\varphi_0)\sqrt{1+|\lambda|}(t-s)}ds + \int_t^\infty e^{-M(\varphi_0)\sqrt{1+|\lambda|}(s-t)}ds \right) \\
&\leq \frac{M(\varphi_0)}{1+|\lambda|}.
\end{aligned}$$

This finishes the proof of Theorem 1. □

Now, we will introduce the Banach space $C^{2\alpha}\left(\mathbb{R}_+\right)$ $(0 < \alpha < 1)$ of all continuous functions $\varphi(x)$ defined on $\mathbb{R}_+$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{C^{2\alpha}(\mathbb{R}_+)} = \|\varphi\|_{C(\mathbb{R}_+)} + \sup_{\substack{t_1 \neq t_2 \\ t_1, t_2 \in \mathbb{R}_+}} \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|^{2\alpha}}.$$

3. The Structure of Fractional Spaces $E_\alpha(C\left(\mathbb{R}_+\right),A)$

Theorem 2. *For $\alpha \in (0,1/2)$, the Banach spaces $E_\alpha(C\left(\mathbb{R}_+\right),A)$ and $C^{2\alpha}\left(\mathbb{R}_+\right)$ are equivalent.*

Proof. Let $\lambda > 0$ and $t \geq 0$. From formula (3) it follows that

$$\begin{aligned} A(A+\lambda)^{-1}f(t) =& \lambda\left[\frac{1}{\lambda}f(t)-(A+\lambda)^{-1}f(t)\right] \\ =& \frac{1}{\lambda+1}f(t)+\lambda\left[\frac{1}{\lambda+1}f(t)-(A+\lambda)^{-1}f(t)\right] \\ =& \frac{1}{\lambda+1}f(t)+\lambda\frac{1}{2\sqrt{1+\lambda}}\int\limits_0^\infty\left(e^{-\sqrt{1+\lambda}|t-s|}-e^{-\sqrt{1+\lambda}(t+s)}\right)\left(f(t)-f(s)\right)ds. \end{aligned}$$

Then, by this formula, the triangle inequality, and the definition of $C^{2\alpha}\left(\mathbb{R}_+\right)$—norm, we have

$$\begin{aligned} \left|\lambda^\alpha A(A+\lambda)^{-1}f(t)\right| \leq& M\left\|f\right\|_{C^{2\alpha}}\left[\frac{\lambda^\alpha}{1+\lambda}+\frac{\lambda^{\alpha+1}}{2\sqrt{1+\lambda}}\int\limits_0^\infty\left|e^{-\sqrt{1+\lambda}|t-s|}-e^{-\sqrt{1+\lambda}(t+s)}\right||t-s|^{2\alpha}ds\right] \\ \leq& M\left\|f\right\|_{C^{2\alpha}}\left[\frac{\lambda^\alpha}{1+\lambda}+\lambda^{\alpha+1}\frac{1}{\sqrt{1+\lambda}}\int\limits_0^\infty e^{-\sqrt{1+\lambda}|t-s|}|t-s|^{2\alpha}ds\right]. \end{aligned} \tag{6}$$

The substitution $\sqrt{1+\lambda}|t-s| = p$ yields that

$$\begin{aligned} \int\limits_0^\infty e^{-\sqrt{1+\lambda}|t-s|}|t-s|^{2\alpha}ds =& \int\limits_0^t e^{-\sqrt{1+\lambda}|t-s|}|t-s|^{2\alpha}ds+\int\limits_t^\infty\frac{e^{-\sqrt{1+\lambda}(s-t)}}{2\sqrt{1+\lambda}}|s-t|^{2\alpha}ds \\ =& -\int\limits_{\sqrt{1+\lambda}t}^0 e^{-p}\frac{p^{2\alpha}}{(1+\lambda)^{\alpha+\frac{1}{2}}}dp+\int\limits_0^\infty e^{-p}\frac{p^{2\alpha}}{(1+\lambda)^{\alpha+\frac{1}{2}}}dp \\ \leq& \frac{2}{(1+\lambda)^{\alpha+\frac{1}{2}}}\Gamma(2\alpha+1) \end{aligned} \tag{7}$$

where $\Gamma(\cdot)$ is the gamma function.

Thus, from estimate (7) it follows that estimate (6) becomes

$$\left|\lambda^\alpha A(A+\lambda)^{-1}f(t)\right| \leq M\left\|f\right\|_{C^{2\alpha}}\left[\frac{\lambda^\alpha}{1+\lambda}+\lambda^{\alpha+1}\frac{1}{2(1+\lambda)^{1+\alpha}}M(\alpha)\right] \leq M(\alpha)\left\|f\right\|_{C^{2\alpha}}.$$

Hence, we get

$$\sup_{\lambda>0}\sup_{t\in[0,\infty)}\left|\lambda^\alpha A(A+\lambda)^{-1}f(t)\right| \leq M(\alpha)\left\|f\right\|_{C^{2\alpha}}$$

or

$$\left\|f\right\|_{E_\alpha(A,C)} \leq M(\alpha)\left\|f\right\|_{C^{2\alpha}}.$$

Therefore, we prove

$$C^{2\alpha}\left(\mathbb{R}_+\right) \subset E_\alpha(C\left(\mathbb{R}_+\right),A).$$

Next, let us prove that $E_\alpha(C\left(\mathbb{R}_+\right),A) \subset C^{2\alpha}\left(\mathbb{R}_+\right)$. Clearly, for a positive operator A in a Banach space E, we have

$$v = \int_0^\infty A(\lambda + A)^{-2} v d\lambda.$$

By this fact, for $t+\tau > t \geq 0$, we have

$$\begin{aligned}
f(t) &= \int_0^\infty A(\lambda + A)^{-2} f(t)d\lambda = \int_0^\infty (\lambda + A)^{-1}A(\lambda + A)^{-1} f(t)d\lambda \\
&= \int_0^\infty \int_0^\infty \frac{1}{2\sqrt{1+\lambda}}\left(e^{-\sqrt{1+\lambda}|t-s|} - e^{-\sqrt{1+\lambda}(t+s)}\right)A(\lambda + A)^{-1} f(s)dsd\lambda, \qquad (8)
\end{aligned}$$

and

$$f(t+\tau) = \int_0^\infty \int_0^\infty \frac{1}{2\sqrt{1+\lambda}}\left(e^{-\sqrt{1+\lambda}|t+\tau-s|} - e^{-\sqrt{1+\lambda}(t+\tau+s)}\right)A(\lambda + A)^{-1} f(s)dsd\lambda. \qquad (9)$$

Clearly,

$$\|f\|_{C(\mathbb{R}_+)} \leq M(\alpha). \qquad (10)$$

From equations (8) and (9) it follows that

$$\begin{aligned}
\frac{f(t+\tau)-f(t)}{\tau^{2\alpha}} &= \int_0^\infty \frac{\lambda^{-\alpha}}{2\sqrt{1+\lambda}}\frac{1}{\tau^{2\alpha}}\int_0^t \left(e^{-\sqrt{1+\lambda}|t+\tau-s|} - e^{-\sqrt{1+\lambda}(t+\tau+s)} - e^{-\sqrt{1+\lambda}|t-s|} - e^{-\sqrt{1+\lambda}(t+s)}\right) \\
&\times \lambda^\alpha A(\lambda + A)^{-1} f(s)dsd\lambda \\
&+ \int_0^\infty \frac{\lambda^{-\alpha}}{2\sqrt{1+\lambda}}\frac{1}{\tau^{2\alpha}}\int_t^{t+\tau} \left(e^{-\sqrt{1+\lambda}|t+\tau-s|} - e^{-\sqrt{1+\lambda}(t+\tau+s)} - e^{-\sqrt{1+\lambda}(s-t)} - e^{-\sqrt{1+\lambda}(s+t)}\right) \\
&\times \lambda^\alpha A(\lambda + A)^{-1} f(s)dsd\lambda \\
&+ \int_0^\infty \frac{\lambda^{-\alpha}}{2\sqrt{1+\lambda}}\frac{1}{\tau^{2\alpha}}\int_{t+\tau}^\infty \left(e^{-\sqrt{1+\lambda}|t+\tau-s|} - e^{-\sqrt{1+\lambda}(t+\tau+s)} - e^{-\sqrt{1+\lambda}(s-t)} - e^{-\sqrt{1+\lambda}(s+t)}\right)
\end{aligned}$$

$$\times \lambda^{\alpha} A(\lambda+A)^{-1} f(s) ds d\lambda$$
$$= J_1 + J_2 + J_3.$$

Clearly, we have

$$1 - e^{-\sqrt{1+\lambda}\tau} \le (1+\lambda)^{\alpha} \tau^{2\alpha}. \tag{11}$$

Using estimate (11), the triangle inequality, and the definition of E_α—norm, we obtain

$$|J_1| \le \|f\|_{C(E_\alpha)} \int_0^\infty \frac{\lambda^{-\alpha}}{2\sqrt{1+\lambda}} \times \frac{1}{\tau^{2\alpha}} \int_0^t \left| e^{-\sqrt{1+\lambda}|t+\tau-s|} - e^{-\sqrt{1+\lambda}(t+\tau+s)} - e^{-\sqrt{1+\lambda}|t-s|} - e^{-\sqrt{1+\lambda}(t+s)} \right| ds d\lambda$$

$$\le \|f\|_{C(E_\alpha)} \int_0^\infty \frac{\lambda^{-\alpha}}{2(1+\lambda)} \frac{1}{\tau^{2\alpha}} \left(1 - e^{-\sqrt{1+\lambda}\tau}\right) d\lambda$$

$$= \|f\|_{C(E_\alpha)} \left(\int_0^1 \frac{\lambda^{-\alpha}}{2(1+\lambda)} \frac{1}{\tau^{2\alpha}} \left(1 - e^{-\sqrt{1+\lambda}\tau}\right) d\lambda + \int_1^\infty \frac{\lambda^{-\alpha}}{2(1+\lambda)} \frac{1}{\tau^{2\alpha}} \left(1 - e^{-\sqrt{1+\lambda}\tau}\right) d\lambda \right)$$

$$\le M(\alpha) \|f\|_{C(E_\alpha)}. \tag{12}$$

In the same manner, we get

$$|J_2| \le M(\alpha) \|f\|_{C(E_\alpha)}, \tag{13}$$

$$|J_3| \le M(\alpha) \|f\|_{C(E_\alpha)}. \tag{14}$$

Estimates (12)-(14) yield that

$$\sup_{0 \le t < t+\tau < \infty} \frac{|f(t+\tau) - f(t)|}{\tau^{2\alpha}} \le M(\alpha) \|f\|_{C(E_\alpha)}. \tag{15}$$

Therefore, estimates (10) and (15) finish the proof of Theorem 2. □

From the positivity of an elliptic operator A in the Banach space $C\left(\mathbb{R}_+\right)$ and estimate (9) it follows the positivity of this operator in Banach spaces $C^{2\alpha}\left(\mathbb{R}_+\right)$.

4. Applications

In this section, we will consider some applications of Theorems 1-2.

First, we will consider the boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x),\ 0 < t < T,\ x \in \mathbb{R}_+, \\ u(0,x) = \varphi(x),\ u(T,x) = \psi(x),\ x \in \mathbb{R}_+, \\ u(t,0) = 0,\ 0 \le t \le T \end{cases} . \tag{16}$$

Here, $\varphi(x)$, $\psi(x)$ and $f(t,x)$ are sufficiently smooth functions and they satisfy every compatibility conditions which guarantee the problem (16) has a smooth solution $u(t,x)$. Assume that the assumption of the uniform ellipticity holds.

Theorem 3. *Let* $0 < 2\alpha < 1$. *Then for the solution of boundary value problem* (16)*, we have the following coercive stability inequality*

$$\|u_{tt}\|_{C(C^{2\alpha}(\mathbb{R}_+))} + \|u\|_{C(C^{2+2\alpha}(\mathbb{R}_+))} \leq M(\alpha)\left[\|\varphi\|_{C^{2+2\alpha}(\mathbb{R}_+)} + \|\psi\|_{C^{2+2\alpha}((\mathbb{R}_+)} + \|f\|_{C(C^{2\alpha}(\mathbb{R}_+))}\right].$$

The proof of Theorem 3 is based on Theorem 2 on the structure of the fractional spaces $E_\alpha(C(\mathbb{R}_+),A)$, Theorem 1 on the positivity of the operator A, on the following theorems on coercive stability of boundary value for the abstract elliptic equation and on the structure of the fractional space $E'_\alpha = E_\alpha(E,A^{1/2})$ which is the Banach space consists of those $v \in E$ for which the norm

$$\|v\|_{E'_\alpha} = \sup_{\lambda>0}\lambda^\alpha \left\|A^{1/2}\left(\lambda + A^{1/2}\right)^{-1} v\right\|_E + \|v\|_E$$

is finite.

Theorem 4 ([5]). *The spaces* $E_\alpha(E,A)$ *and* $E'_{2\alpha}(A^{1/2},E)$ *coincide for any* $0 < \alpha < \frac{1}{2}$, *and their norms are equivalent.*

Theorem 5 ([7]). *Let* A *be positive operator in a Banach space* E *and* $f \in C([0,T],E'_\alpha)$ $(0 < \alpha < 1)$. *Then, for the solution of boundary value problem*

$$\begin{cases} -u''(t) + Au(t) = f(t),\ 0 < t < T, \\ u(0) = \varphi,\ u(T) = \psi \end{cases} \tag{17}$$

in a Banach space E with positive operator A the coercive inequality

$$\|u''\|_{C([0,T],E'_\alpha)} + \|Au\|_{C([0,T],E'_\alpha)} \leq M\left[\|A\varphi\|_{E'_\alpha} + \|A\psi\|_{E'_\alpha} + \frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E'_\alpha)}\right]$$

holds.

Second, we will consider the nonlocal-boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x),\ 0 < t < T,\ x \in \mathbb{R}_+, \\ u(0,x) = u(T,x),\ u_t(0,x) = u_t(T,x),\ x \in \mathbb{R}_+, \\ u(t,0) = 0,\ 0 \leq t \leq T \end{cases}. \tag{18}$$

Here, $f(t,x)$ *is a sufficiently smooth function and they satisfies every compatibility conditions which guarantee the problem* (18) *has a smooth solution* $u(t,x)$. *Assume that the assumption of the uniform ellipticity holds.*

Theorem 6. *Let* $0 < 2m\alpha < 1$. *Then for the solution of boundary value problem* (18)*, we have the following coercive stability inequality*

$$\|u_{tt}\|_{C(C^{2\alpha}(\mathbb{R}_+))} + \|u\|_{C(C^{2+2\alpha}(\mathbb{R}_+))} \leq M(\alpha)\|f\|_{C(C^{2\alpha}(\mathbb{R}_+))}.$$

The proof of Theorem 6 is based on Theorem 2 on the structure of the fractional spaces $E_\alpha(C(\mathbb{R}_+),A)$, Theorem 1 on the positivity of the operator A, Theorem 4 on the structure of the fractional space $E'_\alpha = E_\alpha(E,A^{1/2})$ and on the following theorem on coercive stability of nonlocal boundary value problem for the abstract elliptic equation.

Theorem 7 ([7]). *Let A be positive operator in a Banach space E and $f \in C([0,T],E'_\alpha)$ $(0 < \alpha < 1)$. Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases} -u''(t) + Au(t) = f(t),\ 0 < t < T, \\ u(0) = u(T), u'(0) = u'(T) \end{cases} \tag{19}$$

in a Banach space E with positive operator A the coercive inequality

$$\|u''\|_{C([0,T],E'_\alpha)} + \|Au\|_{C([0,T],E'_\alpha)} \le \frac{M}{\alpha(1-\alpha)} \|f\|_{C([0,T],E'_\alpha)}$$

holds.

5. Conclusion

In the present article, the structure of the fractional spaces $E_\alpha(C(\mathbb{R}_+),A)$ generated by the one-dimensional elliptic differential operator A is investigated. The positivity of this operator A in Banach spaces is established. Of course, the difference operator A_h approximates to the operator A can be presented. The positivity of this operator A_h in Banach spaces can be established.

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