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Generalization of Dunkl Dini Lipschitz Functions

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Abstract. Using a generalized spherical mean operator, we obtain a generalization of Younis's Theorem 5.2 in [12] for the Dunkl transform for functions satisfying the d-Dunkl Dini Lipschitz condition in the space $L^p(\mathbb{R}^d, w_l(x)dx)$, $1 < p \leq 2$, where w_l is a weight function invariant under the action of an associated reflection group.

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1. Introduction and Preliminaries

Younis's Theorem 5.2 [12] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

Theorem 1. *[12] Let $f \in L^2(\mathbb{R})$. Then the following are equivalents*

(i) $\|f(x+h) - f(x)\|_2 = O\left(\frac{h^\eta}{(\log \frac{1}{h})^\delta}\right)$, *as* $h \to 0, 0 < \eta < 1, \delta \geq 0$

(ii) $\displaystyle\int_{|\lambda|\geq s} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{s^{-2\eta}}{(\log s)^{2\delta}}\right)$, *as* $s \to \infty$,

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we obtain a generalization of Theorem 1.1 for the Dunkl transform on $\mathbb{R}^d$ in the space $L^p(\mathbb{R}^d, w_l(x)dx)$, $1 < p \leq 2$. For this purpose, we use a generalized spherical mean operator.

We consider the Dunkl operators $D_j, 1 \leq j \leq d$, on $\mathbb{R}^d$ which are the differential- difference operators introduced by Dunkl in [3]. These operators are very important in pure mathematics and in physics. The theory of Dunkl operators provides generalizations of various multivariable analytic structures, among others we cite the exponential function,

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the Fourier transform and the translation operator. For more details about these operators see [6, 5]. The Dunkl Kernel E_l has been introduced by Dunkl in [4]. This Kernel is used to define the Dunkl transform.
Let R be a root system in $\mathbb{R}^d$, W the corresponding reflection group, R_+ a positive subsystem of R (see [6, 5, 1, 8, 9]) and l a non-negative and W-invariant function defined on R. The Dunkl operator is defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} l(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{< \alpha, x >}, x \in \mathbb{R}^d (1 \leq j \leq d).$$

Here $<,>$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and σ_α the reflection with respect to the hyperplane H_α orthogonal to α, and $\alpha_j =< \alpha, e_j >$, $(e_1, e_2, ..., e_d)$ being the canonical basis of $\mathbb{R}^d$.
We consider the weight function

$$w_l(x) = \prod_{\zeta \in R_+} | < \zeta, x > |^{2l(\alpha)}, x \in \mathbb{R}^d,$$

where w_l is W-invariant and homogeneous of degree 2γ where

$$\gamma = \gamma(R) = \sum_{\zeta \in R_+} l(\zeta) \geq 0.$$

The Dunkl kernel E_l on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C. F. Dunkl in [4]. For $y \in \mathbb{R}^d$, the function $x \mapsto E_l(x, y)$ is the unique solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases} D_j u(x,y) = y_j u(x,y) & \text{si } 1 \leq j \leq d \\ u(0,y) = 0 & \text{for all } y \in \mathbb{R}^d \end{cases}$$

E_l is called the Dunkl kernel.

Lemma 1. *[6] Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$*
1. $E_l(z, 0) = 1, \quad E_l(z, w) = E_l(w, z), \quad E_l(\lambda z, w) = E_l(z, \lambda w).$
2. For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}^d$,$x \in \mathbb{R}^d$,$z \in \mathbb{C}^d$, we have

$$|\partial_z^\nu E_l(x; z)| \leq |x|^{|\nu|} exp(|x||Rez|,$$

where

$$\partial_z^\nu = \frac{\partial^{|\nu|}}{\partial_{z_1}^{\nu_1} .... \partial_{z_d}^{\nu_2}}, |\nu| = \nu_1 + ... + \nu_d.$$

In particular $|\partial_z^\nu E_l(ix; z)| \leq |x|^{|\nu|}$ for all $x, z \in \mathbb{R}^d$.

We denote by $L_l^p(\mathbb{R}^d) = L^p(\mathbb{R}^d, w_l(x)dx)$, $1 < p \leq 2$, the space of measurable functions on $\mathbb{R}^d$ with the norm

$$\|f\|_{p,l} = \left(\int_{\mathbb{R}^d} |f(x)|^p w_l(x) dx \right)^{\frac{1}{p}} < \infty.$$

The Dunkl transform is defined for $f \in L^1_l(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_l(x)dx)$ by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = c_l^{-1} \int_{\mathbb{R}^d} f(x) E_l(-i\xi, x) w_l(x) dx,$$

where the constant c_l is given by

$$c_l = \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}} w_l(z) dz.$$

The Dunkl transform shares several properties with its counterpart in the classical case, we mention here in particular that Plancherel's Theorem holds in $L^2_l(\mathbb{R}^d)$, when both f and $\widehat{f}$ are in $L^1_l(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) E_l(ix, \xi) w_l(\xi) d\xi, x \in \mathbb{R}^d.$$

By Plancherel's Theorem and the Marcinkiewicz interpolation theorem (see [10]), we get for $f \in L^p_l(\mathbb{R}^d)$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}(f)\|_{q,l} \leq K \|f\|_{p,l}, \tag{1}$$

where K is a positive constant.
The generalized spherical mean value of $f \in L^p_l(\mathbb{R}^d)$ is defined by

$$M_h f(x) = \frac{1}{d_l} \int_{\mathbb{S}^{d-1}} \tau_x f(hy) d\mu_l(y), x \in \mathbb{R}^d, h > 0.$$

where τ_x Dunkl translation operator (see [9, 11]), μ be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set $d\mu_l(y) = w_l(y) d\mu(y)$, μ_l is a W-invariant measure on $\mathbb{S}^{d-1}$ and $d_l = \mu_l(\mathbb{S}^{d-1})$.
We see that $M_h f \in L^p_l(\mathbb{R}^d)$ whenever $f \in L^p_l(\mathbb{R}^d)$ and

$$\|M_h f\|_{p,l} \leq \|f\|_{p,l}.$$

for all $h > 0$.
For $\beta \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_β defined by

$$j_\beta(z) = \Gamma(\beta + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \beta + 1)}, \quad z \in \mathbb{C}. \tag{2}$$

Lemma 2. *(Analog of lemma 2.9 in [2]) The following inequality is true*

$$|1 - j_\beta(x)| \geq c,$$

with $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depend only on* β.

Moreover, from (1) we see that

$$\lim_{z\to 0}\frac{j_{\gamma+\frac{d}{2}-1}(z)-1}{z^2}\neq 0. \tag{3}$$

Lemma 3. *[7] Let $f\in L_l^p(\mathbb{R}^d)$. Then*

$$\widehat{M_h f}(\xi)=j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi).$$

The first and higher order finite differences of f (x) are defined as follows

$$Z_h f(x)=(M_h-I)f(x),$$

where I is the identity operator $L_l^p(\mathbb{R}^d)$.

$$Z_h^k f(x)=Z_h(Z_h^{k-1}f(x))=(M_h-I)^k f(x)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}M_h^i f(x),$$

where $M_h^0 f(x)=f(x)$, $M_h^i f(x)=M_h(M_h^{i-1}f(x))$, $i=1,2,..$ and $k=1,2,...$
From Lemma 3, we obtain

$$\widehat{Z_h^k f}(\xi)=(j_{\gamma+\frac{d}{2}-1}(h|\xi|)-1)^k\widehat{f}(\xi).$$

By (1), we have

$$\int_{\mathbb{R}^d}|1-j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk}|\widehat{f}(\xi)|^q w_l(\xi)d\xi\leq K^q\|Z_h^k f(x)\|_{p,l}^q, \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

2. Dunkl Dini Lipschitz Condition

Definition 1. *Let $f\in L_l^p(\mathbb{R}^d)$, and define*

$$\|Z_h^k f(x)\|_{p,l}\leq C\frac{h^\eta}{(\log\frac{1}{h})^\delta},\quad \delta\geq 0,$$

i.e.,

$$\|Z_h^k f(x)\|_{p,l}=O\left(\frac{h^\eta}{(\log\frac{1}{h})^\delta}\right),$$

for all x in $\mathbb{R}^d$ and for all sufficiently small h, C being a positive constant. Then we say that f satisfies a d-Dunkl Dini Lipschitz of order η, or f belongs to $Lip(\eta,\delta)$.

Definition 2. *If however*

$$\frac{\|Z_h^k f(x)\|_{p,l}}{\frac{h^\eta}{(\log\frac{1}{h})^\delta}} \to 0, \quad as \quad h\to 0,$$

i.e.,

$$\|Z_h^k f(x)\|_{p,l} = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\delta}\right), \quad as \quad h\to 0, \delta\geq 0,$$

then f is said to be belong to the little d-Dunkl Dini Lipschitz class $lip(\eta,\delta)$.

Remark. It follows immediately from these definitions that

$$lip(\eta,\delta)\subset Lip(\eta,\delta).$$

Theorem 2. *Let $\eta > 1$. If $f\in Lip(\eta,\delta)$, then $f\in lip(1,\delta)$.*

Proof. For $x\in\mathbb{R}^d$, h small and $f\in Lip(\eta,\delta)$ we have

$$\|Z_h^k f(x)\|_{p,l} \leq C\frac{h^\eta}{(\log\frac{1}{h})^\delta}.$$

Then

$$(\log\frac{1}{h})^\delta \|Z_h^k f(x)\|_{p,l} \leq Ch^\eta.$$

Therefore

$$\frac{(\log\frac{1}{h})^\delta}{h}\|Z_h^k f(x)\|_{p,l} \leq Ch^{\eta-1},$$

which tends to zero with $h\to 0$. Thus

$$\frac{(\log\frac{1}{h})^\delta}{h}\|Z_h^k f(x)\|_{p,l} \to 0, \quad h\to 0.$$

Then $f\in lip(1,\delta)$.

Theorem 3. *If $\eta<\nu$, then $Lip(\eta,0)\supset Lip(\nu,0)$ and $lip(\eta,0)\supset lip(\nu,0)$.*

Proof. We have $0\leq h\leq 1$ and $\eta<\nu$, then $h^\nu\leq h^\eta$.
Then the proof of the theorem is immediate.

3. New Results on Dunkl Dini Lipschitz Class

Theorem 4. *Let $\eta > 2k$. If f belong to the d-Dunkl Dini Lipschitz class, i.e.,*

$$f\in Lip(\eta,\delta), \quad \eta>2k, \delta\geq 0.$$

Then f is equal to the null function in $\mathbb{R}^d$.

Proof. Assume that $f \in Lip(\eta, \delta)$. Then

$$\|Z_h^k f(x)\|_{p,l} \leq C \frac{h^\eta}{(\log \frac{1}{h})^\delta}.$$

From (4), we have

$$\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \leq K^q C^q \frac{h^{q\eta}}{(\log \frac{1}{h})^{q\delta}}.$$

Then

$$\frac{\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi}{h^{2qk}} \leq K^q C^q \frac{h^{q\eta - 2qk}}{(\log \frac{1}{h})^{q\delta}},$$

Since $\eta > 2k$ we have

$$\lim_{h \to 0} \frac{h^{q\eta - 2qk}}{(\log \frac{1}{h})^{q\delta}} = 0.$$

Thus

$$\lim_{h \to 0} \int_{\mathbb{R}^d} \left(\frac{|1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|}{|\xi|^2 h^2} \right)^{qk} |\xi|^{2qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi = 0.$$

and also from the formula (3) and Fatou's theorem, we obtain

$$\int_{\mathbb{R}^d} |\xi|^{2qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi = 0.$$

Hence $|\xi|^{2k} \widehat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$, then $f(x)$ is the null function.

Analog of the Theorem 4, we obtain this theorem.

Theorem 5. *Let $f \in L_l^p(\mathbb{R}^d)$. If f belong to $lip(2, 0)$, i.e.,*

$$\|Z_h^k f(x)\|_{p,l} = O(h^2), \quad as \quad h \to 0.$$

Then f is equal to null function in $\mathbb{R}^d$.

Now, we give another the main result of this paper analog of Theorem 1.

Theorem 6. *Let $f \in L_l^p(\mathbb{R}^d)$. If $f(x)$ belong to $Lip(\eta, \delta)$, then*

$$\int_{|\xi| \geq s} |\widehat{f}(\xi)|^q w_l(\xi) d\xi = O\left(\frac{s^{-q\eta}}{(\log s)^{q\delta}} \right), \quad s \to \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $f \in Lip(\eta,\delta)$. Then

$$\|Z_h^k f(x)\|_{p,l} = O\left(\frac{h^\eta}{(\log \frac{1}{h})^\delta}\right), \quad h \to 0.$$

From (4), we have

$$\int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \leq K^q \|Z_h^k f(x)\|_{p,l}^q.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$ then $h|\xi| \geq 1$ and Lemma 2 implies that

$$1 \leq \frac{1}{c^{qk}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk}.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |\widehat{f}(\xi)|^q w_l(\xi) d\xi &\leq \frac{1}{c^{qk}} \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \\
&\leq \frac{1}{c^{qk}} \int_{\mathbb{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \\
&\leq \frac{K^q}{c^{qk}} \|Z_h^k f(x)\|_{p,l}^q \\
&= O\left(\frac{h^{q\eta}}{(\log \frac{1}{h})^{q\delta}}\right).
\end{aligned}$$

So we obtain

$$\int_{s\leq|\xi|\leq 2s} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \leq C' \frac{s^{-q\eta}}{(\log s)^{q\delta}},$$

where C' is a positive constant. Now, we have

$$\begin{aligned}
\int_{|\xi|\geq s} |\widehat{f}(\xi)|^q w_l(\xi) d\xi &= \sum_{i=0}^{\infty} \int_{2^i s}^{2^{i+1}s} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \\
&\leq C' \left(\frac{s^{-q\eta}}{(\log s)^{q\delta}} + \frac{(2s)^{-q\eta}}{(\log 2s)^{q\delta}} + \frac{(4s)^{-q\eta}}{(\log 4s)^{q\delta}} + \cdots\right) \\
&\leq C' \frac{s^{-q\eta}}{(\log s)^{q\delta}} \left(1 + 2^{-q\eta} + (2^{-q\eta})^2 + (2^{-q\eta})^3 + \cdots\right) \\
&\leq K_\eta \frac{s^{-q\eta}}{(\log s)^{q\delta}},
\end{aligned}$$

where $K_\eta = C'(1 - 2^{-q\eta})^{-1}$ since $2^{-q\eta} < 1$.
Consequently

$$\int_{|\xi|\geq s} |\widehat{f}(\xi)|^q w_l(\xi) d\xi = O\left(\frac{s^{-q\eta}}{(\log s)^{q\delta}}\right), \quad as \quad s \to \infty.$$

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