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On Energy Waves Via Airy Functions in Time-Domain

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Abstract. The main idea is to solve the system of Maxwell's equations in accordance with the causality principle to get the energy quantities via Airy functions in a hollow rectangular waveguide. Evolutionary Approach to Electromagnetics which is an analytical time-domain method is used. The boundary-value problem for the system of Maxwell's equations is reformulated in transverse and longitudinal coordinates. A self-adjoint operator is obtained and the complete set of eigenvectors of the operator initiates an orthonormal basis of the solution space. Hence, the sought electromagnetic field can be presented in terms of this basis. Within the presentation, the scalar coefficients are governed by Klein-Gordon equation. Ultimately, in this study, time-domain waveguide problem is solved analytically in accordance with the causality principle. Moreover, the graphical results are shown for the case when the energy and surplus of the energy for the time-domain waveguide modes are represented via Airy functions.

2010 Mathematics Subject Classifications: 35Q60, 35Q61, 83C50

Key Words and Phrases: Maxwell's equations, wave boundary operators, Airy functions, surplus of energy

1. Introduction

Time-domain waveguide problem deals with the propagation of a signal in a waveguide and this problem can be solved numerically or analytically. Significant publications on approaching time-domain solutions of electromagnetic fields are based on different techniques (see, e.g. [2, 5, 7, 9, 10, 16]). In [11] finite difference time domain method which is a powerful numerical method is studied for time-harmonic fields by Taflove and Hagness.

One of the analytical methods depends on integral transforms such as Fourier and Laplace. In this study, the analytical method 'Evolutionary Approach to Electromagnetics' (EAE) is considered (see, e.g. [1, 3, 4, 8, 14, 15]). As the name suggests, this method deals with solving evolution equations, which contain time derivative. The main idea is to obtain some self-adjoint operators from the system of Maxwell's equations via decomposition. These are called 'Wave Boundary Operators' (WBO) and act on transverse coordinates (see, e.g. [1, 12, 13]).

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Their eigenvector set initiates an orthonormal basis set in the solution space, i.e., Hilbert space $L_2(S)$. Due to the elements of the basis two kinds of solutions, in other words transverse electric (TE) and transverse magnetic (TM) time domain modes occur. Each field component of the modes is a product of the WBO eigenvector function which is the solution of Nuemann or Dirichlet boundary value problem depending on the mode and the modal amplitude depending on axial coordinate z and time t. An evolutionary formulation for the modal amplitudes is obtained by writing the system of Maxwell's equations in terms of basis (see, e.g. [12, 13]). This evolution equation can be solved in compliance with the causality principle.

2. Material and Method

A perfect electric conducting and hollow waveguide with its cross-section domain S, bounded by a closed singly connected contour L is considered. A right-handed triplet of the mutually orthogonal unit vectors $(\mathbf{z}, \mathbf{l}, \mathbf{n})$ is used where $\mathbf{z} \times \mathbf{l} = \mathbf{n}$ and so on. $\mathbf{z}$ is oriented along the Oz axis, $\mathbf{l}$ is tangential to L and $\mathbf{n}$ is the outer normal to S. The three-component vector $\mathbf{R} = (x, y, z)$ is written in the form of $\mathbf{R} = \mathbf{r} + \mathbf{z}z$ where $\mathbf{r} = (x, y)$. The electric and magnetic fields are determined by solving the system of Maxwell's equations

$$\nabla \times \vec{E}(\mathbf{R}, t) = -\mu_0 \partial_t \vec{H}(\mathbf{R}, t), \quad \nabla \cdot \vec{E}(\mathbf{R}, t) = 0, \tag{1}$$

$$\nabla \times \vec{H}(\mathbf{R}, t) = \epsilon_0 \partial_t \vec{E}(\mathbf{R}, t), \quad \nabla \cdot \vec{H}(\mathbf{R}, t) = 0. \tag{2}$$

Due to the perfect electric conductor surface of the waveguide, the field components are subjected to the following boundary conditions

$$\mathbf{n} \cdot \vec{H}(\mathbf{R}, t)\Big|_L = 0, \quad \mathbf{l} \cdot \vec{E}(\mathbf{R}, t)\Big|_L = 0, \quad \mathbf{z} \cdot \vec{E}(\mathbf{R}, t)\Big|_L = 0. \tag{3}$$

In addition, as being a hyperbolic type of PDE, the solution to (1)-(2) should satisfy some given initial conditions

$$\vec{E}(\mathbf{R}, 0) = 0, \quad \vec{H}(\mathbf{R}, 0) = 0. \tag{4}$$

Decomposition of the field vectors and the nabla operator, respectively, into transverse and longitudinal parts as

$$\vec{E}(\mathbf{R}, t) = \vec{\mathbf{E}}(\mathbf{r}, z, t) + \mathbf{z}E_z(\mathbf{r}, z, t), \quad \vec{H}(\mathbf{R}, t) = \vec{\mathbf{H}}(\mathbf{r}, z, t) + \mathbf{z}H_z(\mathbf{r}, z, t), \tag{5}$$

$$\nabla = \nabla_\perp + \mathbf{z}\partial_z, \tag{6}$$

yields two subsystems of equations

$$\nabla_\perp E_z = \mu_0 \partial_t \left[\vec{\mathbf{H}} \times \mathbf{z}\right] + \partial_z \vec{\mathbf{E}}, \quad \epsilon_0 \partial_t E_z = \nabla_\perp \cdot \left[\vec{\mathbf{H}} \times \mathbf{z}\right], \quad \partial_z E_z = -\nabla_\perp \cdot \vec{\mathbf{E}}, \tag{7}$$

$$\nabla_\perp H_z = \epsilon_0 \partial_t \left[\mathbf{z} \times \vec{\mathbf{E}}\right] + \partial_z \vec{\mathbf{H}}, \quad \mu_0 \partial_t H_z = \nabla_\perp \cdot \left[\mathbf{z} \times \vec{\mathbf{E}}\right], \quad \partial_z H_z = -\nabla_\perp \cdot \vec{\mathbf{H}}. \tag{8}$$

The subsystems (7) and (8) can be rewritten in a 4×4 matrix form respectively,

$$W_H \vec{\mathbf{X}} = \begin{pmatrix} 0 & \epsilon_0^{-1}\left[\mathbf{z} \times \nabla_\perp\right]\nabla_\perp \\ \mu_0^{-1}\nabla_\perp\left[\mathbf{z} \times \nabla_\perp\right] & 0 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{E}} \\ \vec{\mathbf{H}} \end{pmatrix}, \tag{9}$$

$$W_E\vec{\mathbf{X}} = \begin{pmatrix} 0 & \epsilon_0^{-1}\nabla_\perp\left[\nabla_\perp \times \mathbf{z}\right] \\ \mu_0^{-1}\left[\nabla_\perp \times \mathbf{z}\right]\nabla_\perp & 0 \end{pmatrix}\begin{pmatrix} \vec{\mathbf{E}} \\ \vec{\mathbf{H}} \end{pmatrix}. \tag{10}$$

where $\mathbf{0}$ is 2×2 zero matrix and $\vec{\mathbf{X}} = (\vec{\mathbf{E}}, \vec{\mathbf{H}})^T = (E_x, E_y, H_x, H_y)^T$.

The operators W_H and W_E are called wave boundary operators (WBO). Together with the boundary conditions (3), $\mathbf{n}\cdot\vec{\mathbf{H}} = 0$, $\mathbf{l}\cdot\vec{\mathbf{E}} = 0$ holds for $\mathbf{r} \in L$ [1, 12, 13]. Because of the physical principle that the electromagnetic field energy is always finite, the initial-boundary value problem (1)-(4) should be solved in a class of integrable vector functions of coordinates and time. This suggests an inner product which is used in [1, 4] for the vector $\vec{\mathbf{X}} = (\vec{\mathbf{E}}, \vec{\mathbf{H}})^T = (E_x, E_y, H_x, H_y)^T$ as

$$\left\langle \vec{\mathbf{X}}_1, \vec{\mathbf{X}}_2 \right\rangle = \frac{1}{2}\int_S \left(\epsilon_0 \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 + \mu_0 \vec{\mathbf{H}}_1 \cdot \vec{\mathbf{H}}_2\right) ds. \tag{11}$$

According to the inner product (11) it can be shown that $\left\langle W_H\vec{\mathbf{X}}_1, \vec{\mathbf{X}}_2\right\rangle = \left\langle \vec{\mathbf{X}}_1, W_H\vec{\mathbf{X}}_2\right\rangle$ and $\left\langle W_E\vec{\mathbf{X}}_1, \vec{\mathbf{X}}_2\right\rangle = \left\langle \vec{\mathbf{X}}_1, W_E\vec{\mathbf{X}}_2\right\rangle$ which means the operators W_H and W_E are both self-adjoint. Therefore, the eigenvalue equations $W_H\vec{Y}_m(\mathbf{r}) = p_m\vec{Y}_m(\mathbf{r})$ and $W_E\vec{Z}_n(\mathbf{r}) = q_n\vec{Z}_n(\mathbf{r})$ hold where p_m and q_n are the real eigenvalues, respectively. All the eigenvalues are situated symmetrically on the real axis and they can be put in order as $p_{+m} = -p_{-m} > 0$, $q_{+n} = -q_{-n} > 0$. The formulation for these eigenvalues are obtained in [12, 13] as $p_{\pm m} = \pm\upsilon_m^2/\sqrt{\epsilon_0\mu_0}$, $q_{\pm n} = \pm\kappa_n^2/\sqrt{\epsilon_0\mu_0}$ by solving Neumann and Dirichlet boundary eigenvalue problems, respectively,

$$\left(\nabla_\perp^2 + \nu_m^2\right)\psi_m(\mathbf{r}) = 0, \quad \left.\frac{\partial}{\partial\mathbf{n}}\psi_m\right|_L = 0, \quad \frac{\nu_m^2}{S}\int_S \left|\psi_m\right|^2 ds = 1, \tag{12}$$

$$\left(\nabla_\perp^2 + \kappa_n^2\right)\phi_n(\mathbf{r}) = 0, \quad \phi_n\big|_L = 0, \quad \frac{\kappa_n^2}{S}\int_S \left|\phi_n\right|^2 ds = 1. \tag{13}$$

The WBO eigenvectors $\vec{Y}_m(\mathbf{r})$ and $\vec{Z}_n(\mathbf{r})$ corresponding to the eigenvalues p_m and q_n, respectively, are presented by the scalar potentials $\psi_m(\mathbf{r})$, $\phi_n(\mathbf{r})$ in [12, 13] which are eigensolutions to problems in (12)-(13), as

$$\begin{aligned} \vec{Y}_{\pm m}(\mathbf{r}) &= \left(\sqrt{\epsilon_0^{-1}}\left[\nabla_\perp\psi_m \times \mathbf{z}\right], \pm\sqrt{\mu_0^{-1}}\nabla_\perp\psi_m\right)^T, \\ \vec{Z}_{\pm n}(\mathbf{r}) &= \left(\sqrt{\epsilon_0^{-1}}\nabla_\perp\phi_n, \pm\sqrt{\mu_0^{-1}}\left[\mathbf{z}\times\nabla_\perp\phi_n\right]\right)^T. \end{aligned} \tag{14}$$

Consequently elements of the orthonormal basis is specified on the cross section S via $\left\{\vec{Y}_{\pm m}(\mathbf{r})\right\}_{m=1}^{\infty}$, $\left\{\vec{Z}_{\pm n}(\mathbf{r})\right\}_{n=1}^{\infty}$ in Hilbert space $L_2(S)$. Then the vector $\vec{\mathbf{X}} = \left(\vec{\mathbf{E}}, \vec{\mathbf{H}}\right)^T$ can be presented in terms of the basis elements. Due to the completeness of $\left\{\vec{\psi}_m(\mathbf{r})\right\}_{m=0}^{\infty}$ and $\left\{\vec{\phi}_n(\mathbf{r})\right\}_{n=0}^{\infty}$ in $L_2(S)$, the field components E_z and H_z can be written in terms of ψ_m and ϕ_n respectively [1].

The basis set implies that (7) and (8) have two kinds of solutions as TE and TM time domain waveguide modes. The solutions of the Neumann and Dirichlet boundary value problems

(12)-(13), generate the TE and TM time-domain modal fields, respectively, with the following components

$$\begin{aligned} \vec{\mathbf{E}}_m(\mathbf{r},z,t) &= -\sqrt{\epsilon_0^{-1}}\partial_{ct}h_m(z,t)\left[\nabla_\perp\psi_m(\mathbf{r})\times\mathbf{z}\right], \quad E_{zm}(\mathbf{r},z,t)=0, \\ \vec{\mathbf{H}}_m(\mathbf{r},z,t) &= \sqrt{\mu_0^{-1}}\partial_z h_m(z,t)\nabla_\perp\psi_m(\mathbf{r}), \quad H_{zm}(\mathbf{r},z,t)=\sqrt{\mu_0^{-1}}\upsilon_m^2 h_m(z,t)\psi_m(\mathbf{r}), \end{aligned} \tag{15}$$

and

$$\begin{aligned} \vec{\mathbf{E}}_n(\mathbf{r},z,t) &= \sqrt{\epsilon_0^{-1}}\partial_z e_n(z,t)\nabla_\perp\phi_n(\mathbf{r}), \quad E_{zm}(\mathbf{r},z,t)=\sqrt{\epsilon_0^{-1}}\kappa_n^2 e_n(z,t)\phi_n(\mathbf{r}), \\ \vec{\mathbf{H}}_n(\mathbf{r},z,t) &= -\sqrt{\mu_0^{-1}}\partial_{ct}e_n(z,t)\left[\vec{\mathbf{z}}\times\nabla_\perp\phi_n(\mathbf{r})\right], \quad H_{zm}(\mathbf{r},z,t)=0, \end{aligned} \tag{16}$$

where $\partial_{ct}=(1/c)\partial_t$ and $c=1/\sqrt{\epsilon_0\mu_0}$. The potentials $h_m(z,t)$ and $e_n(z,t)$ in eq. (15)-(16) are governed by Klein-Gordon equations

$$\left(\partial^2_{\upsilon_m ct}-\partial^2_{\upsilon_m z}+\upsilon_m^2\right)h_m(z,t)=0, \quad \left(\partial^2_{\kappa_n ct}-\partial^2_{\kappa_n z}+\kappa_n^2\right)e_n(z,t)=0 \tag{17}$$

which are obtained by projecting the system of Maxwell's equations on to the basis [15].

The KGE in (17) can be written in the general form

$$\left(\partial_\tau^2-\partial_\xi^2+1\right)f(\xi,\tau)=0 \tag{18}$$

where $f(\xi,\tau)$ is either $h_m(z,t)$ provided that $\xi=\upsilon_m z$ and $\tau=\upsilon_m ct$ for TE-modes or $e_n(z,t)$ provided that $\xi=\kappa_n z$ and $\tau=\kappa_n ct$ for TM-modes (τ is the scaled time and ξ is the scaled coordinate) [15]. Depending on W. Miller's idea [6] the solution $f(\xi,\tau)$ for KGE is interpreted as a function of new variables as $f(u(\xi,\tau),\nu(\xi,\tau))$. This idea propose 11 suitable functions which enables the factorization of the solution as $f(u,\nu)=U(u)V(\nu)$. In this study, the function pairs $u+\nu=(\xi+\tau)/2$, $u-\nu=\pm\sqrt{\xi+\tau}$ are considered and substitution in equation (18) yields

$$f(\xi,\tau)=\begin{cases} 0 & ,\tau<0 \\ \left[c_1Ai(u)+c_2Bi(u)\right]\left[c_3Ai(\nu)+c_4Bi(\nu)\right] & ,0\le\xi\le\tau\ . \\ 0 & ,\xi>\tau \end{cases} \tag{19}$$

In accordance with the causality principle where $c_{1,2,3,4}$ are arbitrary constants, Ai and Bi are Airy functions. Their arguments are the functions of time, τ and axial coordinate, ξ. All possible combinations of the Airy functions are

$$\begin{aligned} f_1(\xi,\tau) &= Ai(u)Ai(\nu), \quad f_2(\xi,\tau)=Ai(u)Bi(\nu) \\ f_3(\xi,\tau) &= Bi(u)Ai(\nu), \quad f_4(\xi,\tau)=Bi(u)Bi(\nu). \end{aligned} \tag{20}$$

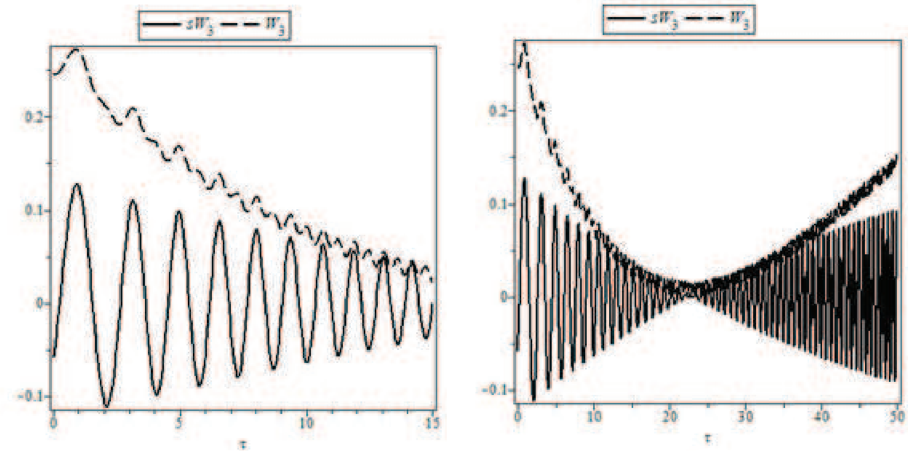
On the other hand, energy waves are propagating along the waveguide together with the electromagnetic waves. So the energy density stored in electric and magnetic fields and the difference of energy densities are investigated in terms of the source functions, i.e. Airy functions. Surplus of the energy is the difference between stored energies.

The energy and surplus of energy is given in [4] with the following formulas, respectively,

$$W(\xi,\tau)=\left[A^2(\xi,\tau)+B^2(\xi,\tau)+f^2(\xi,\tau)\right]/2, \quad sW(\xi,\tau)=\left[A^2(\xi,\tau)-B^2(\xi,\tau)\right]/2$$

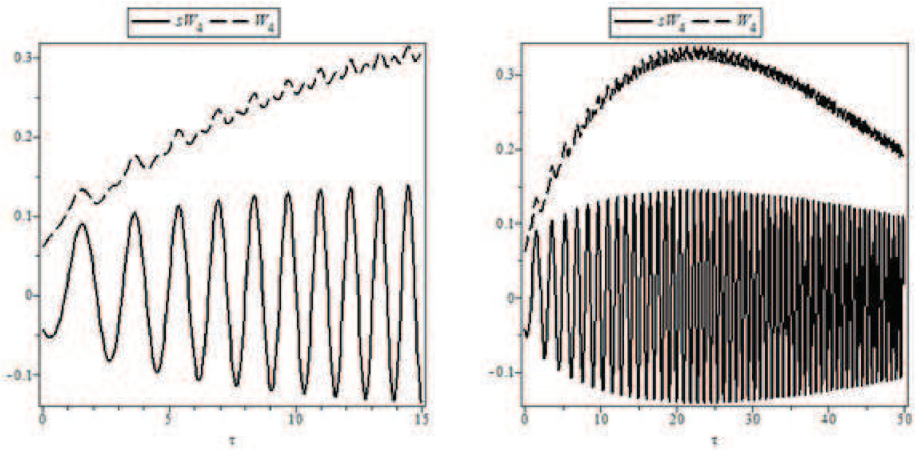
where $A(\xi,\tau)=-\frac{\partial}{\partial\tau}f(\xi,\tau)$, $B(\xi,\tau)=\frac{\partial}{\partial\xi}f(\xi,\tau)$ and $f(\xi,\tau)$ is the Airy function.

In this work, energetic quantities are specially discussed for the Airy functions. In Figure 1 and Figure 2 dependence on time, τ of energy density, $W_{3,4}(\xi,\tau)$ and surplus of the energy, $sW_{3,4}(\xi,\tau)$ are exhibited for on fixed position, $\xi=\tau-0.05$ of the cross-section.



(a) $0 \le \tau \le 15$ (b) $0 \le \tau \le 50$

Figure 1: Time dependence of W_3 and sW_3.



(a) $0 \le \tau \le 15$ (b) $0 \le \tau \le 50$

Figure 2: Time dependence of W_4 and sW_4.

3. Conclusion

In this study, the time-domain waveguide modes are expressed analytically by a method of Evolutionary Approach to Electromagnetics (EAE). A hollow waveguide is considered with perfect electric conductor surface. Energy quantities for the time-domain fields are analysed in details and the energy waves which are also propagating accompanying the electromagnetic field waves are obtained. Especially the energy and surplus of the energy are presented via Airy functions. Thus, the energetic wave process of exchange by energy stored in the longitudinal and transverse field components is introduced in the time-domain, directly. In further studies, the other possible solutions proposed from the Miller's eleven cases will be considered for the solution of different problems such as partially filled lossless and lossy waveguides.

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