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Weakly G-Supplemented modules

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Abstract. In this work, we define weakly g-supplemented modules and cofinitely weak g-supplemented modules. We investigate some properties of these modules. We show that the finite sum of weakly g-supplemented modules is weakly g-supplemented, an arbitrary sum of cofinitely weak g-supplemented modules is cofinitely weak g-supplemented. We also define g-semilocal modules and give some equivalencies for weakly g-supplemented, cofinitely weak g-supplemented and g-semilocal modules.

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1. Introduction

Throughout this paper all rings have an identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. If M/N is finitely generated for $N \leq M$, then N is called a cofinite submodule of M. Let M be an R-module and $T \leq M$. If $K = 0$ for every $K \leq M$ with $T \cap K = 0$, then T is called an essential submodule of M and it is denoted by $T \trianglelefteq M$. K is called a generalized small (briefly, g-small) submodule of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [5], it is called an e-small submodule of M and denoted by $K \ll_e M$). If T is both essential and maximal submodule of M, then T is called a generalized maximal submodule of M. The intersection of all generalized maximal submodules of M is called the generalized radical of M and it is denoted by $Rad_g M$ (in [5], it is denoted by $Rad_e M$). If M have no generalized maximal submodules, then the generalized radical of M is defined by $Rad_g M = M$. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a supplement of U in M. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a g-supplement of U in M. If every submodule of M has a supplement in

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M, then M is called a supplemented module. M is called a g-supplemented module, if every submodule of M has a g-supplement in M. Let $U, V \leq M$. If $M = U + V$ and $U \cap V \ll M$, then V is called a weak supplement of U in M. If every submodule of M has a weak supplement in M, then M is called a weakly supplemented module. If every cofinite submodule of M has a weak supplement in M, then M is called a cofinitely weak supplemented module.

There are some important properties of g-small submodules in [1, 3, 4] and [5].

Lemma 1 ([4, 5]). *Let M be an R-module and K, $N \leq M$. The following conditions are hold.*

(i) If $K \leq N$ and $N \ll_g M$, then $K \ll_g M$.

(ii) If $K \ll_g N$, then K is an g-small submodule in submodules of M which contain N.

(iii) If $f : M \to N$ is an R-module homomorphism and $K \ll_g M$, then $f(K) \ll_g N$.

(iv) If $K \ll_g L$ and $N \ll_g T$ for $L, T \leq M$, then $K + N \ll_g L + T$.

Corollary 1. *Let M be an R-module and $K \leq N \leq M$. If $N \ll_g M$, then $N/K \ll_g M/K$.*

Corollary 2. *Let M be an R-module, $K \ll_g M$ and $L \leq M$. Then $(K + L)/L \ll_g M/L$.*

Lemma 2 ([1]). *Let M be an R-module. Then $Rad_g M = \sum_{L \ll_g M} L$.*

Lemma 3 ([1]). *Let M be an R-module. If M has at least one proper essential submodule and every proper essential submodule of M is contained in a generalized maximal submodule, then $Rad_g M \ll_g M$.*

Lemma 4 ([1]). *If M is a finitely generated R-module and M has at least one proper essential submodule, then every proper essential submodule of M is contained in a generalized maximal submodule of M.*

2. Weakly G-Supplemented Modules

Definition 1. *Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \ll_g M$, then V is called a weak g-supplement of U in M. If every submodule of M has a weak g-supplement in M, then M is called a weakly g-supplemented module.*

Clearly, we see that g-supplemented modules are weakly g-supplemented. We also see that every weakly supplemented module is weakly g-supplemented.

Lemma 5. *Let M be an R-module, $M_1 \leq M$, $U \leq M$ and M_1 be a weakly g-supplemented module. If $M_1 + U$ has a weak g-supplement in M, then U has also a weak g-supplement in M.*

Proof. Let X be a weak g-supplement of M_1+U in M. Then $M_1+U+X=M$ and $(M_1+U)\cap X \ll_g M$. Since M_1 is weakly g-supplemented, $(U+X)\cap M_1$ has a weak g-supplement Y in M_1, i.e. $M_1\cap(U+X)+Y=M_1$ and $M_1\cap(U+X)\cap Y \ll_g M_1$. Following this, we have $M=M_1\cap(U+X)+Y+U+X=U+X+Y$ and

$$U\cap(X+Y)\leq X\cap(U+Y)+Y\cap(U+X)\leq X\cap(M_1+U)+Y\cap M_1\cap(U+X)\ll_g M.$$

Hence $X+Y$ is a weak g-supplement of U in M. □

Corollary 3. *Let M be an R-module $U\leq M$ and $M_i\leq M$ for $i=1,2,\ldots,n$. If $U+M_1+M_2+\ldots+M_n$ has a weak g-supplement in M and M_i is a weakly g-supplemented module for every $i=1,2,\ldots,n$, then U has a weak g-supplement in M.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let $M=M_1+M_2$. If M_1 and M_2 are weakly g-supplemented modules, then M is a weakly g-supplemented module.*

Proof. Clear from Lemma 5. □

Corollary 4. *A finite sum of weakly g-supplemented modules is weakly g-supplemented.*

Proof. Clear from Lemma 6. □

Lemma 7. *Let M be an R-module, $X\leq U\leq M$ and V be a weak g-supplement of U in M. Then $(V+X)/X$ is a weak g-supplement of U/X in M/X.*

Proof. Since V is a weak g-supplement of U in M, we have $M=U+V$ and $U\cap V\ll_g M$. Thus $(U\cap V+X)/X\ll_g M/X$ by Lemma 1. Since $M=U+V$, it is easy to see that $\frac{M}{X}=\frac{U+V}{X}=\frac{U}{X}+\frac{V+X}{X}$ and $\frac{U}{X}\cap\frac{V+X}{X}=\frac{U\cap V+X}{X}\ll_g\frac{M}{X}$. Therefore $(V+X)/X$ is a weak g-supplement of U/X in M/X. □

Theorem 8. *If M is a weakly g-supplemented module, then every factor module of M is weakly g-supplemented.*

Proof. Clear from Lemma 7. □

Corollary 5. *If M is a weakly g-supplemented module, then the homomorphic image of M is weakly g-supplemented.*

Definition 2. *Let M be an R-module. If M/Rad_gM is semisimple, then M is called a g-semilocal module.*

Clearly, we see that every semilocal module is g-semilocal.

Lemma 9. *For an R-module M, the following statements are equivalent.*

(i) M *is g-semilocal.*

(ii) *For every* $U \leq M$ *there exists a submodule* $V \leq M$ *such that* $U + V = M$ *and* $U \cap V \leq Rad_g M$*.*

(iii) *There exists a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple,* $Rad_g M \trianglelefteq M_2$ *and* $M_2/Rad_g M$ *is semisimple.*

Proof. Clear from [2, Proposition 2.1]. □

Lemma 10. *Any homomorphic image of a g-semilocal module is g-semilocal.*

Proof. Let M and N be R-modules, $f : M \longrightarrow N$ be an R-module epimorphism and M be g-semilocal. Since M is g-semilocal, $M/Rad_g M$ is semisimple. Let

$$\varphi : M/Rad_g M \longrightarrow N/Rad_g N, x + Rad_g M \longrightarrow \varphi\left(x + Rad_g M\right) = f\left(x\right) + Rad_g N$$

be a map. It easy to check that φ is an R-module epimorphism, since $f\left(Rad_g M\right) \leq Rad_g N$. Since every homomorphic image of a semisimple module is semisimple, $N/Rad_g N$ is semisimple. Hence N is g-semilocal. □

Lemma 11. *Let* $M = M_1 + M_2$*. If* M_1 *and* M_2 *are g-semilocal, then* M *is g-semilocal.*

Proof. Since M_1 and M_2 are g-semilocal, $M_1/Rad_g M_1$ and $M_2/Rad_g M_2$ are semisimple. Then $\frac{M_1}{Rad_g M_1} \oplus \frac{M_2}{Rad_g M_2}$ is semisimple. Let

$$f : \frac{M_1}{Rad_g M_1} \oplus \frac{M_2}{Rad_g M_2} \longrightarrow \frac{M}{Rad_g M},$$
$$\left(x_1 + Rad_g M_1, x_2 + Rad_g M_2\right) \longrightarrow f\left(x_1 + Rad_g M_1, x_2 + Rad_g M_2\right) = x_1 + x_2 + Rad_g M$$

be a map. It is easy to check that f is an R-module epimorphism. Since every homomorphic image of a semisimple module is semisimple, $M/Rad_g M$ is semisimple. Hence M is g-semilocal. □

Corollary 6. *Let* $M = M_1 + M_2 + \ldots + M_n$*. If* M_i *is g-semilocal for every* $i = 1, 2, \ldots, n$*, then* M *is g-semilocal.*

Proof. Clear from Lemma 11. □

Lemma 12. *If* M *is a weakly g-supplemented module, then* M *is g-semilocal.*

Proof. Let $U/Rad_g M$ be any submodule of $M/Rad_g M$. Since M is weakly g-supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V \ll_g M$. Since $U \cap V \ll_g M$, then by Lemma 2, $U \cap V \leq Rad_g M$. Then by $\frac{M}{Rad_g M} = \frac{U+V}{Rad_g M} = \frac{U}{Rad_g M} + \frac{V + Rad_g M}{Rad_g M}$ and

$$\frac{U}{Rad_g M} \cap \frac{V + Rad_g M}{Rad_g M} = \frac{U \cap V + Rad_g M}{Rad_g M} = \frac{Rad_g M}{Rad_g M} = 0, \frac{M}{Rad_g M} = \frac{U}{Rad_g M} \oplus \frac{V + Rad_g M}{Rad_g M}.$$

Hence M is g-semilocal. □

Lemma 13. *Assume M be an R-module and $Rad_gM \ll_g M$. If M is g-semilocal, then M is weakly g-supplemented.*

Proof. Let U be any submodule of M. Since M is g-semilocal, $(U + Rad_gM)/Rad_gM$ is a direct summand of M/Rad_gM. Then there exists a submodule V of M such that $Rad_gM \leq V$ and $\frac{M}{Rad_gM} = \frac{U+Rad_gM}{Rad_gM} \oplus \frac{V}{Rad_gM}$. By $\frac{M}{Rad_gM} = \frac{U+Rad_gM}{Rad_gM} \oplus \frac{V}{Rad_gM} = \frac{U+V}{Rad_gM}$, $M = U + V$. Since

$$\frac{U \cap V + Rad_gM}{Rad_gM} = \frac{U + Rad_gM}{Rad_gM} \cap \frac{V}{Rad_gM} = 0, U \cap V \leq Rad_gM \ll_g M.$$

Hence V is a weak g-supplement of U in M. □

Corollary 7. *Assume M be an R-module with $Rad_gM \ll_g M$. Then M is weakly g-supplemented if and only if M is g-semilocal.*

Proof. Clear from Lemma 12 and Lemma 13. □

Lemma 14. *Let M be a finitely generated R-module. Then $Rad_gM \ll_g M$.*

Proof. If M has at least one proper essential submodule, since M is finitely generated, by Lemma 4, every proper essential submodule of M is contained in a generalized maximal submodule of M. Then by Lemma 3, $Rad_gM \ll_g M$. If M have no proper essential submodules, then $Rad_gM = M \ll_g M$ also holds. □

Lemma 15. *Let M be a finitely generated R-module. Then M is weakly g-supplemented if and only if M is g-semilocal.*

Proof. By Lemma 14 and Corollary 7, this is clear. □

Corollary 8. *${}_RR$ is weakly g-supplemented if and only if ${}_RR$ is g-semilocal.*

Proof. By Lemma 15, this is clear. □

Proposition 1. *Let M be a weakly g-supplemented R-module. Then for every $U, V \leq M$ with $M = U + V$, there exists a weak g-supplement K of U in M with $K \leq V$.*

Proof. Assume $U, V \leq M$ with $M = U + V$. Since M is weakly g-supplemented, $U \cap V$ has a weak g-supplement T in M. In this case, $M = U \cap V + T$ and $U \cap V \cap T \ll_g M$. Since $M = U + V = U \cap V + T$, $M = U + V \cap T$. Let $K = V \cap T$. Then $M = U + V \cap T = U + K$ and $U \cap K = U \cap V \cap T \ll_g M$. Hence K is a weak g-supplement of U in M with $K \leq V$. □

Example 1. *Let p and q be prime numbers and consider the ring*

$$R = \mathbb{Z}_{p,q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, p \not| b \text{ and } q \not| b\}.$$

By [2, Remark 3.3], ${}_RR$ is weakly supplemented but not supplemented. Since every nonzero submodule of ${}_RR$ is essential in ${}_RR$, ${}_RR$ is weakly g-supplemented but not g-supplemented.

3. Cofinitely Weak G-Supplemented Modules

Definition 3. *Let M be an R-module. If every cofinite submodule of M has a weak g-supplement in M, then M is called a cofinitely weak g-supplemented module.*

Clearly we see that every weakly g-supplemented module is cofinitely weak g- supplemented.

Lemma 16. *Assume M be a finitely generated R-module. If M is cofinitely weak g-supplemented, then M is weakly g-supplemented.*

Proof. Clear, since every submodule of M is cofinite. □

Lemma 17. *Let M be a cofinitely weak g-supplemented module. Then every factor module of M is cofinitely weak g-supplemented.*

Proof. Let M/X be any factor module of M and U/X be a cofinite submodule of M/X. Since $\frac{M}{U} \cong \frac{M/X}{U/X}$, U is a cofinite submodule of M. Since M is cofinitely weak g-supplemented, U has a weak g-supplement V in M. Then by Lemma 7, $(V+X)/X$ is a weak g-supplement of U/X in M/X. Hence M/X is cofinitely weak g-supplemented. □

Corollary 9. *Any homomorphic image of a cofinitely weak g-supplemented module is cofinitely weak g-supplemented.*

Proof. Clear from Lemma 17. □

Lemma 18. *Let M be an R-module, $M_1 \leq M$, U be a cofinite submodule of M and M_1 be a cofinitely weak g-supplemented module. If $M_1 + U$ has a weak g-supplement in M, then so does U.*

Proof. Let X be a weak g-supplement of $M_1 + U$ in M. Then $M_1 + U + X = M$ and $(M_1 + U) \cap X \ll_g M$. Since U is a cofinite submodule of M, $U + X$ is also a cofinite submodule of M. Then by $\frac{M_1}{M_1 \cap (U+X)} \cong \frac{M_1+U+X}{U+X} = \frac{M}{U+X}$, $M_1 \cap (U + X)$ is a cofinite submodule of M_1. Since M_1 is cofinitely weak g-supplemented, $M_1 \cap (U + X)$ has a weak g-supplement Y in M_1, i.e. $M_1 \cap (U + X) + Y = M_1$ and $M_1 \cap (U + X) \cap Y \ll_g M_1$. Following this, we have $M = M_1 \cap (U + X) + Y + U + X = U + X + Y$ and

$$U \cap (X + Y) \leq X \cap (U + Y) + Y \cap (U + X) \leq X \cap (M_1 + U) + Y \cap M_1 \cap (U + X) \ll_g M.$$

Hence $X + Y$ is a weak g-supplement of U in M. □

Corollary 10. *Let M be an R-module, U be a cofinite submodule of M and $M_i \leq M$ for $i = 1, 2, \ldots, n$. If $U + M_1 + M_2 + \ldots + M_n$ has a weak g-supplement in M and M_i is a cofinitely weak g-supplemented module for every $i = 1, 2, \ldots, n$, then U has a weak g-supplement in M.*

Proof. Clear from Lemma 18. □

Lemma 19. *Any sum of cofinitely weak g-supplemented modules is cofinitely weak g-supplemented.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of cofinitely weak g-supplemented submodules of an R-module M and $M = \sum_{i\in I} M_i$. Let U be any cofinite submodule of M. Since U is cofinite submodule of M, there exists a finite subset $\{i_1, i_2, \ldots, i_n\}$ of I such that
$M = U + M_{i_1} + M_{i_2} + \ldots + M_{i_n}$. Since $U + M_{i_1} + M_{i_2} + \ldots + M_{i_n}$ has a weak g-supplement 0 in M and M_{i_k} is cofinitely weak g-supplemented for $k = 1, 2, \ldots, n$, then by Corollary 10, U has a weak g-supplement in M. □

Proposition 2. *Let R be a ring. The following statements are equivalent.*

(1) ${}_RR$ is g-semilocal.

(2) ${}_RR$ is weakly g-supplemented.

(3) Every finitely generated R-module is g-semilocal.

(4) Every finitely generated R-module is weakly g-supplemented.

(5) $R^{(I)}$ is cofinitely weak g-supplemented for every index set I.

(6) Every R-module is cofinitely weak g-supplemented.

Proof. (1) ⇔ (2) Clear from Corollary 8.

(1) ⇒ (3) Assume M be a finitely generated R-module and let $M = \langle m_2, m_2, \ldots, m_n\rangle$. Then $M = Rm_1 + Rm_2 + \ldots + Rm_n$. Since ${}_RR$ is g-semilocal and Rm_i $(i = 1, 2, \ldots, n)$ is an homomorphic image of ${}_RR$, by Lemma 10, Rm_i is g-semilocal. Then by Corollary 6, M is g-semilocal.

(3) ⇔ (4) Obtained from Lemma 15.

(4) ⇒ (5) By hypothesis, ${}_RR$ is weakly g-supplemented. Hence ${}_RR$ is cofinitely weak g-supplemented. Because of this, by Lemma 19, $R^{(I)}$ is cofinitely weak g-supplemented for every index set I.

(5) ⇒ (6) Clear from Corollary 9, since every R-module is ${}_RR$-generated.

(6) ⇒ (2) By hypothesis, ${}_RR$ is cofinitely weak g-supplemented. Since ${}_RR$ is finitely generated, by Lemma 16, ${}_RR$ is weakly g-supplemented. □

Proposition 3. *Let M be weakly g-supplemented R-module and U be a cofinite submodule of M. Then for every $V \leq M$ with $M = U + V$ and $U \cap V$ is a cofinite submodule of M, there exists a weak g-supplement K of U with $K \leq V$.*

Proof. Similar to proof of Proposition 1. □

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