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New Subclasses of Analytic Function Associated with q-Difference Operator

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Abstract. The aim of this paper is to establish the coefficient bounds for certain classes of analytic functions associated with q-difference operator. Certain applications of these results for the functions defined through convolution are also obtained.

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1. Introduction

Recently, the area of q-analysis has attracted the serious attention of researchers. The q-difference calculus or quantum calculus was initiated at the beginning of 19th century, that was initiated by Jackson [6, 7]. He was the first to develop q-integral and q-derivative in a systematic way. The fractional q-difference calculus had its origin in the works by Al.Salam [2] and Agarwal [1]. This great interest is due to its application in various branches of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference and q-integral equations and in q-transform analysis. The generalization q-Taylor's formula in fractional q-calculus was introduced by Purohit and Raina [18]. Mohammed and Darus [12] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk. Purohit and Raina recently in [18, 16] have used the fractional q-calculus operators in investigating certain classes of functions which are analytic in the open disk and Purohit [17] also studied these q-operators are defined by using convolution of normalized analytic

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functions and q-hypergeometric functions. A comprehensive study on applications of q-calculus in operator theory may be found in[4]. Ramachandran et. al. [19] have used the fractional q-calculus operators in investigating certain bound for q-starlike and q-convex functions with respect to symmetric points.

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

defined on the open unit disk $\mathbb{U} = \{z \ : \ z \in \mathbb{C} \ : \ |z| < 1\}$.

If the functions $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$, written as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalence holds (see [11] and [5])

$$f(z) \prec g(z) \quad \Longleftrightarrow \quad f(0) = g(0), \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For function $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, the q-derivative of a function f is defined by (see [6, 7])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0), \tag{2}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (2), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \ a_k z^{k-1}, \tag{3}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{4}$$

As $q \to 1^-$, $[k]_q \to k$. For a function $h(z) = z^k$, we observe that

$$\begin{aligned} D_q(h(z)) &= D_q\left(z^k\right) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q \ z^{k-1}, \\ \lim_{q \to 1^-} \left(D_q(h(z))\right) &= \lim_{q \to 1^-} \left([k]_q z^{k-1}\right) = k z^{k-1} = h'(z), \end{aligned}$$

where h' is the ordinary derivative.

As a right inverse, Jackson [7] introduced the q-integral

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f\left(z q^k\right),$$

provided that the series converges. For a function $h(z) = z^k$, we observe that

$$\begin{aligned}\int_0^z h(t)d_qt &= \int_0^z t^k d_qt = \frac{z^{k+1}}{[k+1]_q} \quad (k \neq -1)\\ \lim_{q\to 1^-}\int_0^z h(t)d_qt &= \lim_{q\to 1^-}\frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t)dt,\end{aligned}$$

where $\int_0^z h(t)dt$ is the ordinary integral.

Making use of the q-derivative $D_qf(z)$, the subclasses $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$ of the class $\mathcal{A}$ for $0 \leq \alpha \leq 1$ are introduced by

$$\mathcal{S}_q^*(\alpha) = \left\{f \in \mathcal{A} : Re\left(\frac{zD_qf(z)}{f(z)}\right) \geq \alpha,\ z \in \mathbb{U}\right\} \tag{5}$$

$$\mathcal{C}_q(\alpha) = \left\{f \in \mathcal{A} : Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) \geq \alpha,\ z \in \mathbb{U}\right\}. \tag{6}$$

We note that

$$f \in \mathcal{C}_q(\alpha) \Leftrightarrow zD_qf \in \mathcal{S}_q^*(\alpha), \tag{7}$$

and

$$\begin{aligned}\lim_{q\to 1^-}\mathcal{S}_q^*(\alpha) &= \left\{f \in \mathcal{A} : \lim_{q\to 1^-} Re\left(\frac{zD_qf(z)}{f(z)}\right) \geq \alpha,\ z \in \mathbb{U}\right\} = \mathcal{S}^*(\alpha),\\ \lim_{q\to 1^-}\mathcal{C}_q(\alpha) &= \left\{f \in \mathcal{A} : \lim_{q\to 1^-} Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) \geq \alpha,\ z \in \mathbb{U}\right\} = \mathcal{C}(\alpha),\end{aligned}$$

where $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are respectively, the classes of starlike of order α and convex of order α in $\mathbb{U}$ (see Robertson [23]). Kanas and Răducanu in [8] used the Ruscheweyh q-differential operator to introduce and study some properties of (q, k) uniformly starlike functions of order α. It is clear that $D_qf(z) \to f'(z)$ as $q \to 1^-$. This difference operator helps us to generalize the class of starlike functions S^* analytically.

By making use of the q-derivative of a function $f \in \mathcal{A}$ and the principle of subordination, we now introduce the following classes

Definition 1. *Let $\phi(z)$ be a univalent starlike function with respect to 1, which maps the open unit disk $\mathbb{U}$ onto a region in the right half-plane and is symmetric with respect to the real axis, with*

$$\phi(0) = 1 \quad \textit{and} \quad \phi'(0) \geq 0.$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)$ if

$$\left(\frac{zD_qf(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\left(\frac{zD_qf(z)}{f(z)}\right) + \lambda\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right]^{\beta} \prec \phi(z) \tag{8}$$

where $0 \leq \beta \leq 1$; $0 \leq \alpha \leq 1$; $0 \leq \lambda \leq 1$.

We note that

(i) $\lim_{q\to 1^-} \mathcal{M}_{q,\alpha,\beta,\lambda}(\phi) = \mathcal{M}_{\alpha,\beta,\lambda}(\phi)$ (C. Ramachandran et al. [20])

(ii) $\lim_{q\to 1^-} \mathcal{M}_{q,0,1,\lambda}(\phi) = \mathcal{M}(\lambda,\phi)$ (Ali et al. [3])

(iii) $\lim_{q\to 1^-} \mathcal{M}_{q,\alpha,\beta,1}(\phi) = \mathcal{M}_{\alpha,\beta}(\phi)$ (V. Ravichandran et al. [21])

(iv) $\lim_{q\to 1^-} \mathcal{M}_{q,0,1,0}(\phi) = \lim_{q\to 1^-} \mathcal{M}_{q,1,0,\lambda}(\phi) = \mathcal{S}^*(\phi)$ and $\lim_{q\to 1^-} \mathcal{M}_{q,0,1,1}(\phi) = \mathcal{C}(\phi)$ (Ma and Minda [9])

2. Preliminary Results

In order to prove the main results we need the following lemmas.

Lemma 1. *[9] If $p(z) = 1 + c_1z + c_2z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0 \\ 2 & \text{if } 0 \leq \nu \leq 1 \\ 4\nu + 2 & \text{if } \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds true if and only if $p(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z}, \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $\nu = 1$, the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when $\nu = 0$.

Although the above upper bound is sharp, in the case when $0 < \nu < 1$, it can be further improved as follows:

$$|c_2 - \nu c_1^2| + \nu|c_1|^2 \leq 2 \qquad \left(0 < \nu \leq \frac{1}{2}\right)$$

and

$$|c_2 - \nu c_1^2| + (1-\nu)|c_1|^2 \leq 2 \qquad \left(\frac{1}{2} < \nu \leq 1\right).$$

We also need the following result in our investigation.

Lemma 2. *[22] If $p_1(z) = 1 + c_1z + c_2z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$, then*

$$|c_2 - \nu c_1^2| \leq 2\max\{1, |2\nu - 1|\}.$$

The result is sharp for the function $p_1(z) = \dfrac{1+z^2}{1-z^2}$ and $p_1(z) = \dfrac{1+z}{1-z}$.

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that the function $0 < q < 1$, $\phi \in \mathcal{P}$, $[k]_q$ is given by (4) and $z \in \mathbb{U}$.

By making use of Lemma 1, we first prove the Fekete-Szegö type inequalities asserted by Theorem 1 below.

Theorem 1. *Let $0 \le \mu \le 1$, $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \lambda \le 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \ge 0$.*

If $f(z)$ given by (1) belongs to the function class $\mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{1}{2\xi}\left(2B_2 - \left(\dfrac{\rho^2 + 2\mu\xi - \tau}{\rho^2}\right) B_1^2\right) & \text{if } \mu \le \sigma_1, \\ \dfrac{B_1}{\xi} & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{1}{2\xi}\left(-2B_2 + \left(\dfrac{\rho^2 + 2\mu\xi - \tau}{\rho^2}\right) B_1^2\right) & \text{if } \mu \ge \sigma_2, \end{cases} \tag{9}$$

where, for convenience,

$$\sigma_1 := \frac{2\rho^2(B_2 - B_1) - (\rho^2 - \tau)B_1^2}{2\xi B_1^2}, \tag{10}$$

$$\sigma_2 := \frac{2\rho^2(B_2 + B_1) - (\rho^2 - \tau)B_1^2}{2\xi B_1^2}, \tag{11}$$

$$\sigma_3 := \frac{2\rho^2 B_2 - (\rho^2 - \tau)B_1^2}{2\xi B_1^2}. \tag{12}$$

$$\rho = ([2]_q - 1)\,\alpha + ([2]_q - 1 + \lambda)\,\beta, \tag{13}$$

$$\xi = ([3]_q - 1)\,\alpha + ([3]_q - 1 + \lambda\,([3]_q\,([2]_q - 1) + 1))\,\beta, \tag{14}$$

$$\tau = \left([2]_q^2 - 1\right)\alpha + \left([2]_q^2 - 1 + 2[2]_q^2\lambda + \lambda^2\right)\beta. \tag{15}$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{\rho^2}{\xi B_1}\left(1 - \frac{B_2}{B_1} + \left(\frac{\rho^2 + 2\mu\xi - \tau}{2\rho^2}\right) B_1\right)|a_2|^2 \le \frac{B_1}{\xi}. \tag{16}$$

Furthermore, if $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{\rho^2}{\xi B_1}\left(1 + \frac{B_2}{B_1} - \left(\frac{\rho^2 + 2\mu\xi - \tau}{2\rho^2}\right) B_1\right)|a_2|^2 \le \frac{B_1}{\xi}. \tag{17}$$

Each of these results is sharp.

Proof. If $f(z) \in \mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)$, then there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$\left(\frac{zD_qf(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\left(\frac{zD_qf(z)}{f(z)}\right)+\lambda\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right]^{\beta} = \phi\left(w(z)\right). \tag{18}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots . \tag{19}$$

Since $w(z)$ is a Schwarz function, we see that

$$\Re(p_1(z)) > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad p_1(0) = 1.$$

Now, defining the function $p(z)$ by

$$\begin{aligned} p(z) := & \left(\frac{zD_qf(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\left(\frac{zD_qf(z)}{f(z)}\right)+\lambda\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right]^{\beta} \\ = & \ 1 + b_1z + b_2z^2 + \cdots , \end{aligned} \tag{20}$$

we find from (18) and (19) that

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{21}$$

Thus, by using (19) and (21), we obtain

$$b_1 = \frac{1}{2}B_1c_1 \quad \text{and} \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2c_1^2.$$

An easy computation would show that

$$\begin{aligned} &\left(\frac{zD_qf(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\left(\frac{zD_qf(z)}{f(z)}\right)+\lambda\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right]^{\beta} = \\ &\quad 1 + \left[\left([2]_q - 1\right)\alpha + \left([2]_q - 1 + \lambda\right)\beta\right]a_2z + \\ &\quad \left[\left([3]_q - 1\right)\alpha + \left([3]_q - 1 + \lambda\left([3]_q\left([2]_q - 1\right) + 1\right)\right)\beta\right]a_3z^2 + \\ &\quad \left[\frac{\alpha}{2}\left([2]_q - 1\right)\left((\alpha-1)\left([2]_q - 1\right) - 2\right) + \frac{\beta(\beta-1)}{2}\left([2]_q - 1 + \lambda\right)^2 + \right. \\ &\quad \left. \alpha\left([2]_q - 1\right)\beta\left([2]_q - 1 + \lambda\right) - \left(\left([2]_q - 1\right) + \left([2]_q\left([2]_q - 1\right) + 1\right)\lambda\right)\beta\right]a_2^2z^2 + \cdots \end{aligned}$$

which, in view of (20), yields

$$b_1 = \left[\left([2]_q - 1\right)\alpha + \left([2]_q - 1 + \lambda\right)\beta\right]a_2$$

and

$$\begin{aligned} b_2 &= \left[\left([3]_q-1\right)\alpha+\left([3]_q-1+\lambda\left([3]_q\left([2]_q-1\right)+1\right)\right)\beta\right]a_3+\\ &\left[\frac{\alpha}{2}\left([2]_q-1\right)\left((\alpha-1)\left([2]_q-1\right)-2\right)+\frac{\beta(\beta-1)}{2}\left([2]_q-1+\lambda\right)^2+\right.\\ &\left.\alpha\left([2]_q-1\right)\beta\left([2]_q-1+\lambda\right)-\left(\left([2]_q-1\right)+\left([2]_q\left([2]_q-1\right)+1\right)\lambda\right)\beta\right]a_2^2. \end{aligned}$$

Equivalently, we have

$$a_2=\frac{B_1c_1}{2\left[\left([2]_q-1\right)\alpha+\left([2]_q-1+\lambda\right)\beta\right]}$$

and

$$a_3=\frac{B_1}{2\left[\left([3]_q-1\right)\alpha+\left([3]_q-1+\lambda\left([3]_q\left([2]_q-1\right)+1\right)\right)\beta\right]}\left[c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}+\Lambda_0B_1\right)c_1^2\right]$$

where

$$\begin{aligned} \Lambda_0 &= \frac{1}{\left[\left([2]_q-1\right)\alpha+\left([2]_q-1+\lambda\right)\beta\right]^2}\left[\frac{\alpha}{2}\left([2]_q-1\right)\left((\alpha-1)\left([2]_q-1\right)-2\right)+\right.\\ &\frac{\beta(\beta-1)}{2}\left([2]_q-1+\lambda\right)^2+\alpha\left([2]_q-1\right)\beta\left([2]_q-1+\lambda\right)-\\ &\left.\left(\left([2]_q-1\right)+\left([2]_q\left([2]_q-1\right)+1\right)\lambda\right)\beta\right]. \end{aligned}$$

Therefore, we obtain

$$a_3-\mu a_2^2=\frac{B_1}{2\left[\left([3]_q-1\right)\alpha+\left([3]_q-1+\lambda\left([3]_q\left([2]_q-1\right)+1\right)\right)\beta\right]}\left(c_2-\nu c_1^2\right) \tag{22}$$

where

$$\begin{aligned} \nu &= \frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1}{2\left[\left([2]_q-1\right)\alpha+\left([2]_q-1+\lambda\right)\beta\right]^2}\left[\left(\left([2]_q-1\right)\alpha+\left([2]_q-1+\lambda\right)\beta\right)^2\right.\right.\\ &+2\mu\left(\left([3]_q-1\right)\alpha+\left([3]_q-1+\lambda\left([3]_q\left([2]_q-1\right)+1\right)\right)\beta\right)-\\ &\left.\left.\left(\left([2]_q^2-1\right)\alpha+\left([2]_q^2-1+2[2]_q^2\lambda+\lambda^2\right)\beta\right)\right]\right). \end{aligned}$$

The assertion of Theorem 1 now follows by an application of Lemma 1.

To show that the bounds asserted by Theorem 1 are sharp, we define the following functions:

$$\mathcal{K}_{\phi_n}(z)\quad(n\in\mathbb{N}\backslash\{1\};\mathbb{N}:=\{1,2,3,\cdots\}),$$

with

$$\mathcal{K}_{\phi_n}(0)=0=\mathcal{K}'_{\phi_n}(0)-1,$$

by

$$\left(\frac{z\mathcal{K}'_{\phi_n}(z)}{\mathcal{K}_{\phi_n}(z)}\right)^{\alpha}\left[(1-\lambda)\left(\frac{z\mathcal{K}'_{\phi_n}(z)}{\mathcal{K}_{\phi_n}(z)}\right)+\lambda\left(\frac{\mathcal{K}'_{\phi_n}(z\mathcal{K}'_{\phi_n}(z))}{\mathcal{K}'_{\phi_n}(z)}\right)\right]^{\beta}=\phi(z^{n-1}),$$

and the functions $\mathcal{F}_\eta$ and $\mathcal{G}_\eta$ $(0 \le \eta \le 1)$ with

$$\mathcal{F}_\eta(0) = 0 = \mathcal{F}'_\eta(0) - 1 \qquad \text{and} \qquad \mathcal{G}_\eta(0) = 0 = \mathcal{G}'_\eta(0) - 1$$

by

$$\left(\frac{z\mathcal{F}'_\eta(z)}{\mathcal{F}_\eta(z)}\right)^\alpha \left[(1-\lambda)\left(\frac{z\mathcal{F}'_\eta(z)}{\mathcal{F}_\eta(z)}\right) + \lambda\left(\frac{\mathcal{F}'_\eta(z\mathcal{F}'_\eta(z))}{\mathcal{F}'_\eta(z)}\right)\right]^\beta = \phi\left(\frac{z(z+\eta)}{1+\eta z}\right)$$

and

$$\left(\frac{z\mathcal{G}'_\eta(z)}{\mathcal{G}_\eta(z)}\right)^\alpha \left[(1-\lambda)\left(\frac{z\mathcal{G}'_\eta(z)}{\mathcal{G}_\eta(z)}\right) + \lambda\left(\frac{\mathcal{G}'_\eta(z\mathcal{G}'_\eta(z))}{\mathcal{G}'_\eta(z)}\right)\right]^\beta = \phi\left(-\frac{z(z+\eta)}{1+\eta z}\right)$$

respectively. Then, clearly, the functions $\mathcal{K}_{\phi_n}$, $\mathcal{F}_\eta$, $\mathcal{G}_\eta \in \mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)$. Also we write

$$\mathcal{K}_\phi := \mathcal{K}_{\phi_2}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality in Theorem 1 holds true if and only if f is $\mathcal{K}_\phi$ or one of its rotations. When $\sigma_1 \le \mu \le \sigma_2$, then the equality holds true if and only if f is $\mathcal{K}_{\phi_3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds true if and only if f is $\mathcal{F}_\eta$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if f is $\mathcal{G}_\eta$ or one of its rotations.

By making use of Lemma 2, we immediately obtain the following Fekete-Szegö type inequality.

Theorem 2. *Let $0 \le \mu \le 1$, $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \lambda \le 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \ge 0$.*
If $f(z)$ given by (1) belongs to the function class $\mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \le \frac{B_1}{\xi} \max\left\{1, \left|-\frac{B_2}{B_1} + \left(\frac{\rho^2 + 2\mu\xi - \tau}{2\rho^2}\right) B_1\right|\right\} \qquad (\mu \in \mathbb{C}),$$

where ρ, ξ and τ are defined by (13), (14) and (15). The result is sharp.

Remark 1. *The coefficient bounds for $|a_2|$ and $|a_3|$ are special cases of those asserted by Theorem 1.*

Remark 2. *In its special case when $\lim\limits_{q\to 1^-}$, Theorem 1 reduces to the result obtained in [20]. Note that there were few typographical errors in the assertion of [20, Theorem 1] and the following result is the corrected one:*

Corollary 1. *[20, Theorem 1] Let $0 \le \mu \le 1$, $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \lambda \le 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \ge 0$.*

If $f(z)$ given by (1) belongs to the function class $\mathcal{M}_{\alpha,\beta,\lambda}(\phi)$, then

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{1}{4\xi}\left(2B_2-\left(\dfrac{\rho^2+4\mu\xi-\tau}{\rho^2}\right)B_1^2\right) & \text{if } \mu\leq\sigma_1, \\ \dfrac{B_1}{2\xi} & \text{if } \sigma_1\leq\mu\leq\sigma_2, \\ \dfrac{1}{4\xi}\left(-B_2+\left(\dfrac{\rho^2+4\mu\xi-\tau}{\rho^2}\right)B_1^2\right) & \text{if } \mu\geq\sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{2\rho^2(B_2-B_1)-(\rho^2-\tau)B_1^2}{4\xi B_1^2},$$

$$\sigma_2 := \frac{2\rho^2(B_2+B_1)-(\rho^2-\tau)B_1^2}{4\xi B_1^2},$$

$$\sigma_3 := \frac{2\rho^2 B_2-(\rho^2-\tau)B_1^2}{4\xi B_1^2}.$$

$$\begin{aligned} \rho &= \alpha+(1+\lambda)\,\beta, \\ \xi &= \alpha+(1+2\lambda)\,\beta, \\ \tau &= (3)\,\alpha+\left(3+8\lambda+\lambda^2\right)\beta. \end{aligned}$$

If $\sigma_1\leq\mu\leq\sigma_3$, then

$$|a_3-\mu a_2^2|+\frac{\rho^2}{2\xi B_1}\left(1-\frac{B_2}{B_1}+\left(\frac{\rho^2+4\mu\xi-\tau}{2\rho^2}\right)B_1\right)|a_2|^2\leq\frac{B_1}{2\xi}.$$

Furthermore, if $\sigma_3\leq\mu\leq\sigma_2$, then

$$|a_3-\mu a_2^2|+\frac{\rho^2}{2\xi B_1}\left(1+\frac{B_2}{B_1}-\left(\frac{\rho^2+4\mu\xi-\tau}{2\rho^2}\right)B_1\right)|a_2|^2\leq\frac{B_1}{2\xi}.$$

Each of these results is sharp.

Remark 3. *When $\lim\limits_{q\to1^-}\mathcal{M}_{q,\alpha,\beta,1}(\phi)=\mathcal{M}_{\alpha,\beta}(\phi)$, Theorem 1 reduces to the result obtained by V. Ravichandran et al. [21].*

Remark 4. *Special case if $\mathcal{M}_{q,0,1,0}(\phi)=\mathcal{M}_{q,1,0,\lambda}(\phi)=\mathcal{S}_q^*(\phi)$ Theorem 1 reduces to starlike function with q-difference operator and $\mathcal{M}_{q,0,1,1}(\phi)=\mathcal{C}_q(\phi)$, Theorem 1 reduces to convex function with q-difference operator which was obtained by Seoudy et al. [24].*

Remark 5. *Special case if* $\lim_{q\to 1^-} \mathcal{M}_{q,0,1,0}(\phi) = \lim_{q\to 1^-} \mathcal{M}_{q,1,0,\lambda}(\phi) = \mathcal{S}^*(\phi)$ *Theorem 1 reduces to starlike function and* $\lim_{q\to 1^-} \mathcal{M}_{q,0,1,1}(\phi) = \mathcal{C}(\phi)$, *Theorem 1 reduces to convex function which was obtained by Ma and Minda [9].*

4. Applications to analytic functions defined by using fractional calculus operators and convolution

During the past three decades, the subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance. There are two most recent works on this subject of widespread investigations, namely rather comprehensive treatises on the theory, applications of fractional differential equations by Podlubny [15] and Kilbas et al. [10].

For the applications of the results given in the preceding sections, we first introduce the class $\mathcal{M}^{\delta}_{q,\alpha,\beta,\lambda}(\phi)$, which is defined by means of the Hadamard product (or convolution) and a certain operator of fractional calculus, known as the Owa-Srivastava operator (see, for details, [25] and [27]; see also [13], [14], and [26]).

Definition 2. *The fractional integral of order* δ *is defined, for a function* $f(z)$, *by*

$$\mathcal{D}_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \qquad (\delta > 0), \tag{23}$$

where the function $f(z)$ *be analytic in a simply connected domain of the complex* z*-plane containing the origin and the multiplicity of* $(z-\zeta)^{\delta-1}$ *is removed by requiring that* $\log(z-\zeta)$ *to be real when* $z-\zeta > 0$.

Definition 3. *The fractional integral of order* δ *is defined, for a function* $f(z)$, *by*

$$\mathcal{D}_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \qquad (0 \le \delta < 1), \tag{24}$$

where $f(z)$ *is constrained, and the multiplicity of* $(z-\zeta)^{-\delta}$ *is removed, as in Definition 2.*

Definition 4. *Under the hypotheses of Definition 3, the fractional derivative of order* $n+\delta$ *is defined, for a function* $f(z)$, *by*

$$\mathcal{D}_z^{n+\delta} f(z) = \frac{d^n}{dz^n}\left(\mathcal{D}_z^{\delta} f(z)\right) \qquad (0 \le \delta < 1;\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{25}$$

Using Definitions 2, 3 and 4 of fractional derivatives and fractional integrals, Owa and Srivatsava [14] introduced what is popularly referred to in the current literature as the Owa-Srivastava operator $\Omega^{\delta} : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^{\delta} f)(z) := \Gamma(2-\delta) z^{\delta} \mathcal{D}_z^{\delta} f(z), \qquad (\delta \ne 2, 3, 4 \cdots). \tag{26}$$

In terms of the Owa-Srivastava operator Ω^{δ} defined by (26), we now introduce the function class $\mathcal{M}^{\delta}_{q,\alpha,\beta,\lambda}(\phi)$ in the following way:

$$\mathcal{M}^{\delta}_{q,\alpha,\beta,\lambda}(\phi) := \{f \;:\; f \in \mathcal{A} \;\text{ and }\; \Omega^{\delta} f \in \mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)\}. \tag{27}$$

It is easily seen that the function class $\mathcal{M}^{\delta}_{q,\alpha,\beta,\lambda}(\phi)$ is a special case of the function class $\mathcal{M}^{g}_{q,\alpha,\beta,\lambda}(\phi)$when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n. \tag{28}$$

Suppose now that

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \qquad (g_n > 0).$$

Then, since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{M}^{g}_{q,\alpha,\beta,\lambda}(\phi) \Longleftrightarrow (f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathcal{M}_{q,\alpha,\beta,\lambda}(\phi) \tag{29}$$

we can obtain the coefficient estimates for functions in the class $\mathcal{M}^{g}_{q,\alpha,\beta,\lambda}(\phi)$ from the corresponding estimates for functions in the class $\mathcal{M}_{q,\alpha,\beta,\lambda}(\phi)$. By applying Theorem 1 to the following Hadamard product (or convolution):

$$(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots,$$

we get Theorem 3 below after an obvious change of the parameter μ.

Theorem 3. *Let $0 \leq \mu \leq 1$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \lambda \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where the coefficients B_n are real with $B_1 > 0$, $B_2 \geq 0$ and $B_n > 0$ $(n \in \mathbb{N}\backslash\{1,2\})$.*

If $f(z)$ given by (1) belongs to the function class $\mathcal{M}^{g}_{q,\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{2\xi g_3}\left(2B_2 - \dfrac{B_1^2}{\rho^2}\gamma_2\right) & \text{if } \;\mu \leq \sigma_4, \\[2ex] \dfrac{B_1}{\xi g_3} & \text{if } \;\sigma_4 \leq \mu \leq \sigma_5, \\[2ex] \dfrac{1}{2\xi g_3}\left(-2B_2 + \dfrac{B_1^2}{\rho^2}\gamma_2\right) & \text{if } \;\mu \geq \sigma_5, \end{cases}$$

where, for convenience,

$$\sigma_4 := \quad \frac{g_3}{g_2^2}\left(\frac{2\rho^2(B_2 - B_1) - (\rho^2 - \tau)B_1^2}{2\xi B_1^2}\right),$$

$$\sigma_5 := \quad \frac{g_3}{g_2^2}\left(\frac{2\rho^2(B_2+B_1)-(\rho^2-\tau)B_1^2}{2\xi B_1^2}\right),$$

and

$$\gamma_2 := \left(\rho^2 + \frac{2\mu\xi g_3}{g_2^2} - \tau\right) \tag{30}$$

and ρ*,* ξ *and* τ *are defined as in* (13)*,* (14) *and* (15)*, respectively. These results are sharp.*

Since, by (1) and the definition 4,

$$(\Omega^\delta f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n, \tag{31}$$

we readily obtain

$$g_2 := \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta} \tag{32}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\delta)}{\Gamma(4-\delta)} = \frac{6}{(2-\delta)(3-\delta)}, \tag{33}$$

For g_2 and g_3 given by (32) and (33), respectively, Theorem 3 reduces to the following interesting result.

Theorem 4. *Let* $0 \le \mu \le 1$*,* $0 \le \alpha \le 1$*,* $0 \le \beta \le 1$ *and* $0 \le \lambda \le 1$*. Also let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$*, where the coefficients* B_n *are real with* $B_1 > 0$*,* $B_2 \ge 0$*.*

If $f(z)$ *given by* (1) *belongs to the function class* $\mathcal{M}^g_{q,\alpha,\beta,\lambda}(\phi)$*, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(2-\delta)(3-\delta)}{12\xi}\left(2B_2 - \dfrac{B_1^2}{\rho^2}\gamma_3\right) & \text{if} \quad \mu \le \sigma_4, \\[2ex] \dfrac{(2-\delta)(3-\delta)}{6\xi} B_1 & \text{if} \quad \sigma_4 \le \mu \le \sigma_5, \\[2ex] \dfrac{(2-\delta)(3-\delta)}{12\xi}\left(-2B_2 + \dfrac{B_1^2}{\rho^2}\gamma_3\right) & \text{if} \quad \mu \ge \sigma_5, \end{cases}$$

where, for convenience,

$$\sigma_4 := \quad \frac{2(3-\delta)}{3(2-\delta)}\left(\frac{2\rho^2(B_2-B_1)-(\rho^2-\tau)B_1^2}{2\xi B_1^2}\right),$$

$$\sigma_5 := \quad \frac{2(3-\delta)}{3(2-\delta)}\left(\frac{2\rho^2(B_2+B_1)-(\rho^2-\tau)B_1^2}{2\xi B_1^2}\right),$$

and

$$\gamma_3 := \left(\rho^2 + 2\mu\xi\frac{2(3-\delta)}{3(2-\delta)} - \tau\right) \tag{34}$$

and ρ*,* ξ *and* τ *are defined as in* (13)*,* (14) *and* (15)*, respectively.*

Remark 6. *In its special case when* $\lim_{q\to 1^-} \mathcal{M}_{q,\alpha,\beta,\lambda}(\phi) = \mathcal{M}_{\alpha,\beta,\lambda}(\phi)$ *Theorem 4 coincide with the result obtained earlier by C. Ramachandran et al. [20].*

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