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On Trigonometric Moments of the Stereographic Semicircular Gamma Distribution

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Abstract. Phani (2013) constructed a good number of circular and semicircular models induced by inverse stereographic projection. Minh and Farnum (2003) and Toshihiro Abe et al (2010) proposed a new method to derive circular distributions from the existing linear models. In this paper, a new semicircular model, which is coined as Stereographic Semicircular Gamma distribution is derived by inducing modified inverse stereographic projection on Gamma distribution. This distribution generalizes Stereographic Semicircular Exponential model (Phani et al (2013)) and the density and distribution functions of proposed model admit closed form. Explicit expressions for trigonometric moments are derived by applying Meijer's G- function and the new semicircular model is extended to construct Stereographic l- axial Gamma distribution.

2010 Mathematics Subject Classifications: 60E05, 62H11

Key Words and Phrases: Circular Model, Directional Data, Meijer's G-function, Stereographic Projection, Trigonometric Moments.

1. Introduction

Directions in two dimensions can be represented as points on the circumference of a unit circle and models for representing such data are called circular distributions. Quite a lot of work was done on circular models defined on the unit circle (Fisher, 1993; Jammalamadaka and Sen Gupta (2001); Mardia and Jupp, (2000)) and recent publications (Dattatreya Rao et al (2007), Girija (2010), Phani et al (2012)). Most of these models are applicable for dealing circular data. To fit/model certain practical data sets, it is not required to go for full circular models but semicircular/arc models are adequate. Some recent

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papers (Guardiola (2004), Byoung et al (2008) and Phani et al (2013)) address this issue and provided some methodology for constructing distributions suitable for modeling these types of data. For example, when sea turtles emerge from the ocean in search of a nesting site on dry land, a random variable having values on a semicircle is very much sufficient for modeling such data. Given the angles of initial heading and departure, to trace the debris of aircraft lost problem, semicircular models need to be used. A few more examples of semicircular data is available in Ugai et al (1977). A little attention is paid on the study of semicircular distributions. The aim of the present article is to contribute towards filling this gap. In this paper, the modified inverse stereographic projection is used to define a new semicircular model, coined as the Stereographic Semicircular Gamma distribution which generalizes the Stereographic Semicircular Exponential Model (Phani et al (2013)). Explicit expressions for trigonometric moments are derived in terms of Meijer's G- function and it is extended to the Stereographic l - axial Gamma distribution for modeling axial data.

In section 2, methodology of modified inverse stereographic projection is presented. Section 3 is devoted to introduce the proposed distribution and to present graphs of probability density function for various values of parameters. In section 4, the first four trigonometric moments in terms of Meijer's G- function are derived and the proposed model is extended to l- axial distributions for modeling axial data in section 5.

2. Methodology of Modified Inverse Stereographic Projection (Phani et al (2012))

Modified inverse stereographic projection is defined by a one to one mapping given by $T(\theta) = x = vtan\left(\frac{\theta}{2}\right)$, where $x \in (-\infty, \infty)$, $\theta \in [-\pi, \pi)$, $v > 0$. Suppose x is randomly chosen on the interval $(-\infty, \infty)$. Let $F(x)$ and $f(x)$ denote the cumulative distribution and probability density functions of the random variable X respectively. Then $T^{-1}(x) = \theta = 2tan^{-1}\left(\frac{x}{v}\right)$ by Toshihiro Abe et al (2010) is a random point on the unit circle. Let $G(\theta)$ and $g(\theta)$ denote the cumulative distribution and probability density functions of this random point θ respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$using the following theorem.

Theorem 2.1. *For $v > 0$,*

i) $G(\theta) = F\left(vtan\left(\frac{\theta}{2}\right)\right)$

ii) $g(\theta) = v\left(\frac{sec^2\left(\frac{\theta}{2}\right)}{2}\right) f\left(vtan\left(\frac{\theta}{2}\right)\right)$

By applying this modified inverse stereographic projection on linear models with support on $R^+(R)$, new distributions mapped onto $[0, \pi)$ $([-\pi, \pi))$ are derived to study semicircular (*circular*) data.

3. Stereographic Semicircular Gamma Model

Gamma distribution plays a prominent role in actuarial science. Here an attempt is made to construct stereographic version of semicircular Gamma distribution by inducing inverse stereographic projection.
A random variable X on the real line is said to have gamma distribution with index parameter$c > 0$, scale parameter $\lambda > 0$ and location parameter α if the probability density, cumulative distribution and characteristic functions of X are respectively given by
(1) $f(x) = \frac{(x-\alpha)^{c-1}}{\lambda^c \Gamma(c)} \exp\left(\frac{-(x-\alpha)}{\lambda}\right)$ for $\lambda,\ c > 0,\ \ x > 0$ and $\alpha\ > 0$
(2)$F(x) = \frac{\Gamma\left(\frac{x}{\lambda}\right)^{(c)}}{\Gamma(c)}$ for $\lambda,\ c > 0,\ \ x > 0$
(3) $\phi_X(t) = \frac{1}{(1-i\lambda t)^c}$ where $t \in R$
By applying inverse stereographic projection defined by a one to one mapping $x = v\ tan\left(\frac{\theta}{2}\right), v > 0, 0 \le \theta < \pi$ a Stereographic Semicircular Gamma Distribution is obtained
A semicircular random variable θ is said to follow Stereographic Semicircular Gamma Distribution with index parameter $c > 0$, location parameter μ and scale parameter $\sigma > 0$ denoted by SSCG (μ, σ, c) if the probability density and cumulative distribution functions are respectively given by

$$g(\theta) = \frac{1}{2\sigma^c \Gamma(c)} \sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right) - \mu\right)^{c-1} \exp\left(-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right) - \mu\right)\right)$$

$$\text{where} \quad 0 \le \theta < \pi, c > 0, \sigma = \frac{\lambda}{v} > 0 \ \text{ and } \quad \mu = \frac{\alpha}{v} \tag{3.1}$$

$$G(\theta) = \frac{\Gamma_t(c)}{\Gamma(c)} \text{ where } t = \frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right), \theta \in [0, \pi) \tag{3.2}$$

Special case: If $c = 1$ in (3.1) reduces to the density function of Stereographic Semicircular Exponential distribution (Phani et al (2013)).

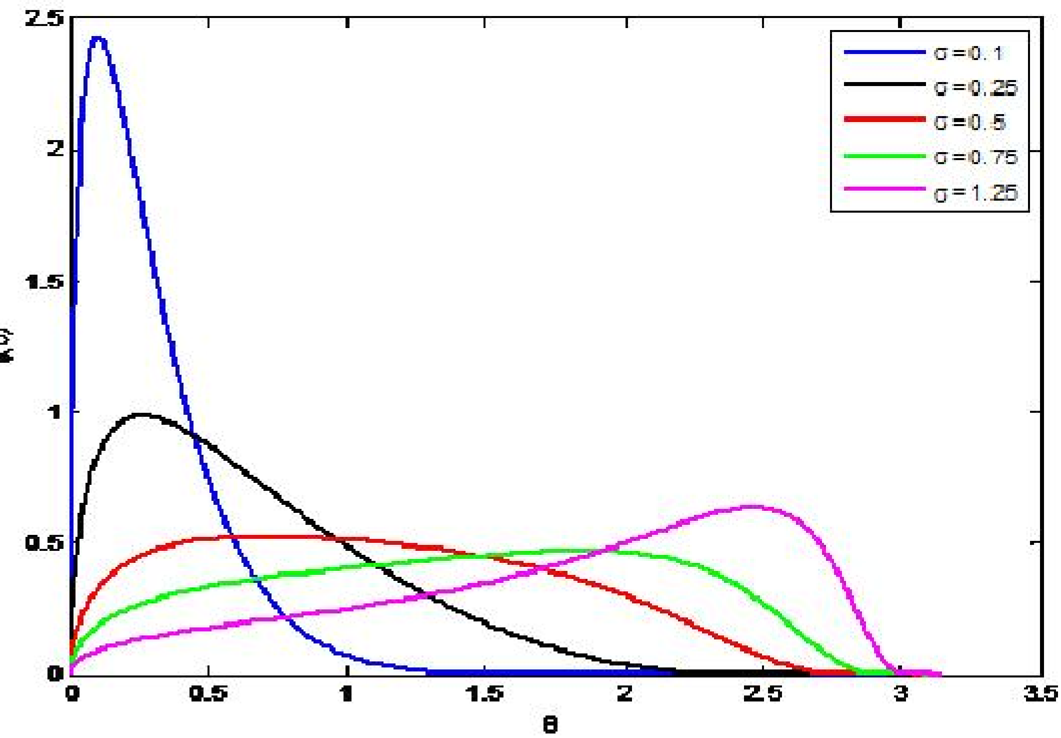
4. Trigonometric moments of Stereographic Semicircular Gamma distribution

The characteristic function of the stereographic semicircular gamma model is

$$\begin{aligned} \phi_\theta(p) &= \int_0^\pi e^{ip\theta} g(\theta) d\theta \\ &= \int_0^\pi e^{ip\theta} \tfrac{1}{2\sigma^c\Gamma(c)} \sec^2\left(\tfrac{\theta}{2}\right)\left(\tan\left(\tfrac{\theta}{2}\right) - \mu\right)^{c-1} \exp\left(-\tfrac{1}{\sigma}\left(\tan\left(\tfrac{\theta}{2}\right) - \mu\right)\right) d\theta \end{aligned}$$

The integration is not tractable. But trigonometric moments can be derived by applying the Meijer G function [Gradshteyn and Ryzhik (2007)]. The trigonometric moments of the distribution are given by $\{\varphi_p : \pm 1, \pm 2, \pm 3, \dots\}$ where $\varphi_p = \alpha_p + i\ \beta_p$ with

Figure 1: **Graphs of probability density function of Stereographic Semicircular gamma distribution for various values of σ and $c = 1.5$**



$\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$ being the p^{th} order cosine and sine moments of the random angle θ, respectively and are required to study the population characteristics.

Theorem 4.1: The trigonometric moments $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$, for $p = 1, 2, 3, 4$ of the stereographic semicircular gamma distribution with $\mu = 0$, are given as follows

$$\alpha_1 = 1 - \frac{1}{\sigma^c \sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\, \begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

$$\beta_1 = \frac{1}{\sigma^c \sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\, \begin{matrix} \frac{1-c}{2} \\ \frac{1-c}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

$$\alpha_2 = 1 + \frac{4}{\sigma^c \sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\, \begin{matrix} -\frac{c}{2} - 1 \\ -\frac{c}{2}, 0, \frac{1}{2} \end{matrix}\right) - \frac{4}{\sigma^c \sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\, \begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

$$\beta_2 = \frac{2}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right) - \frac{4}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{(c+1)}{2} \\ -\frac{(1-c)}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$\alpha_3 = 1 - \tfrac{16}{3\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2}-2 \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right) - \tfrac{24}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2}-1 \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$- \tfrac{9}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$\beta_3 = \frac{3}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\,\begin{matrix} \frac{1-c}{2} \\ \frac{1-c}{2}, 0, \frac{1}{2}\end{matrix}\right) - \frac{8}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{(c+1)}{2} \\ -\frac{(1-c)}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$\alpha_4 = 1 + \tfrac{32}{3\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2}-3 \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right) + \tfrac{80}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2}-1 \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$- \tfrac{64}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2}-2 \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right) - \tfrac{16}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$\beta_4 = \tfrac{4}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} \frac{1-c}{2} \\ \frac{1-c}{2}, 0, \frac{1}{2}\end{matrix}\right) - \tfrac{8}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c+1}{2} \\ \frac{1-c}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

$$- \tfrac{16}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c+1}{2} \\ \frac{3-c}{2}, 0, \frac{1}{2}\end{matrix}\right) - \tfrac{32}{3\sigma^c\sqrt{\pi}\,\Gamma(c)} G^{31}_{13}\left(\tfrac{1}{4\sigma^2}\,\middle|\,\begin{matrix} -\frac{c+3}{2} \\ \frac{3-c}{2}, 0, \frac{1}{2}\end{matrix}\right)$$

where

$$\int_0^\infty x^{2v-1}\left(u^2+x^2\right)^{Q-1} e^{-\mu x}dx = \frac{u^{2v+2Q-2}}{2\sqrt{\pi}\Gamma(1-Q)} G^{31}_{13}\left(\frac{\mu^2u^2}{4}\,\middle|\,\begin{matrix} 1-v \\ 1-Q-v, 0, \frac{1}{2}\end{matrix}\right) \tag{4.1}$$

for $|\arg u\pi| < \frac{\pi}{2}, Re\mu > 0$ and $Rev > 0$ and $G^{31}_{13}\left(\frac{\mu^2u^2}{4}\,\middle|\,\begin{matrix} 1-v \\ 1-Q-v, 0, \frac{1}{2}\end{matrix}\right)$ is called the Meijer's G-function (Gradshteyn and Ryzhik, 2007, formula no. 3.389.2).

Proof.

$$\varphi_p = \int_0^\pi \cos(p\theta)d\theta + i\int_0^\infty \sin(p\theta)g(\theta)d\theta = \alpha_p + i\beta_p$$

where

$$\alpha_p = \frac{1}{2\sigma^c\Gamma(c)}\int_0^\pi \cos(p\theta)\sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)}d\theta$$

$$\beta_p = \frac{1}{2\sigma^c\Gamma(c)} \int_0^\pi \sin(p\theta) \sec^2\left(\frac{\theta}{2}\right) \left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

To derive the first order cosine and sine moments

$$\alpha_1 = \frac{1}{2\sigma^c\Gamma(c)} \int_0^\pi \cos(\theta) \sec^2\left(\frac{\theta}{2}\right) \left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

Consider the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\cos\theta = 1 - \frac{2x^2}{1+x^2}$ and formula (4.1)

$$\begin{aligned} \alpha_1 &= \tfrac{1}{\sigma^c\Gamma(c)} \int_0^\pi \left[1 - \tfrac{2x^2}{1+x^2}\right] x^{c-1} e^{\frac{1}{\sigma}x} dx \\ &= 1 - \tfrac{2}{\sigma^c\Gamma(c)} \int_0^\pi x^{c+1}\left(1+x^2\right)^{-1} e^{\frac{1}{\sigma}x} dx \end{aligned}$$

$$\alpha_1 = 1 - \frac{1}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2}\,\middle|\, \begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

$$\beta_1 = \frac{1}{2\sigma^c\Gamma(c)} \int_0^\pi \sin(\theta) \sec^2\left(\frac{\theta}{2}\right) \left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

Consider the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\sin\theta = \frac{2x}{1+x^2}$ and formula (4.1)

$$\begin{aligned} \beta_1 &= \tfrac{1}{\sigma^c\Gamma(c)} \int_0^\pi \left[\tfrac{2x}{1+x^2}\right] x^{c-1} e^{\frac{1}{\sigma}x} dx \\ &= \tfrac{2}{\sigma^c\Gamma(c)} \int_0^\pi x^{c}\left(1+x^2\right)^{-1} e^{\frac{1}{\sigma}x} dx \end{aligned}$$

$$\beta_1 = \frac{1}{\sigma^c\sqrt{\pi}\,\Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2}\,\middle|\, \begin{matrix} \frac{1-c}{2} \\ \frac{1-c}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

To derive the second order cosine and sine moments

$$\alpha_2 = \frac{1}{2\sigma^c\Gamma(c)} \int_0^\pi \cos(2\theta) \sec^2\left(\frac{\theta}{2}\right) \left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

Consider the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\cos 2\theta = 1 + \frac{8x^4}{(1+x^2)^2} - \frac{8x^2}{(1+x^2)}$ and formula (4.1)

$$\alpha_2 = \frac{1}{\sigma^c\Gamma(c)} \int_0^\infty \left[1 + \frac{8x^4}{(1+x^2)^2} - \frac{8x^2}{(1+x^2)}\right] x^{c-1} e^{-\frac{1}{\sigma}x} dx$$

$$= \frac{1}{\sigma^c\Gamma(c)} \int_0^\infty x^{c-1} e^{-\frac{1}{\sigma}x} dx + \frac{8}{\sigma^c\Gamma(c)} \int_0^\infty \frac{x^{c+3}}{(1+x^2)^2} e^{-\frac{1}{\sigma}x} dx - \frac{8}{\sigma^c\Gamma(c)} \int_0^\infty \frac{x^{c+1}}{(1+x^2)} e^{-\frac{1}{\sigma}x} dx$$

$$\alpha_2 = 1 + \frac{4}{\sigma^c\sqrt{\pi}\Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2} \,\middle|\, \begin{matrix} -\frac{c}{2} - 1 \\ -\frac{c}{2},\ 0,\ \frac{1}{2} \end{matrix}\right) - \frac{4}{\sigma^c\sqrt{\pi}\Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2} \,\middle|\, \begin{matrix} -\frac{c}{2} \\ -\frac{c}{2},\ 0,\ \frac{1}{2} \end{matrix}\right)$$

$$\beta_2 = \frac{1}{2\sigma^c\Gamma(c)} \int_0^\pi \sin 2\theta \sec^2\left(\frac{\theta}{2}\right) \left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

Consider the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\sin 2\theta = \frac{4x}{(1+x^2)} - \frac{8x^3}{(1+x^2)^2}$ and by formula (4.1)

$$\beta_2 = \frac{1}{\sigma^c\Gamma(c)} \int_0^\infty \left[\frac{4x}{(1+x^2)} - \frac{8x^3}{(1+x^2)^2}\right] x^{c-1} e^{-\frac{1}{\sigma}x} dx$$

$$= \frac{4}{\sigma^c\Gamma(c)} \int_0^\infty \frac{x^c}{(1+x^2)} e^{-\frac{1}{\sigma}x} dx - \frac{8}{\sigma^c\Gamma(c)} \int_0^\infty \frac{x^{c+2}}{(1+x^2)^2} e^{-\frac{1}{\sigma}x} dx$$

$$\beta_2 = \frac{2}{\sigma^c\sqrt{\pi}\Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2} \,\middle|\, \begin{matrix} \frac{1-c}{2} \\ \frac{1-c}{2},\ 0,\ \frac{1}{2} \end{matrix}\right) - \frac{4}{\sigma^c\sqrt{\pi}\Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2} \,\middle|\, \begin{matrix} -\frac{(c+1)}{2} \\ \frac{(1-c)}{2},\ 0,\ \frac{1}{2} \end{matrix}\right)$$

To derive the third cosine and sine moments

$$\alpha_3 = \frac{1}{\sigma^c 2\Gamma(c)} \int_0^\pi \cos 3\theta \sec^2\left(\frac{\theta}{2}\right) \left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

Consider the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\cos 3\theta = 1 - \frac{32x^6}{(1+x^2)^3} + \frac{48x^4}{(1+x^2)^2} - \frac{18x^2}{(1+x^2)}$

$$\alpha_3 = \frac{1}{\sigma^c\Gamma(c)} \int_0^\infty \left[1 - \frac{32x^6}{(1+x^2)^3} + \frac{48x^4}{(1+x^2)^2} - \frac{18x^2}{(1+x^2)}\right] x^{c-1} e^{-\frac{1}{\sigma}x} dx$$

$$= 1 - \frac{32}{\sigma^c\Gamma(c)} \int_0^\infty x^{c+5}(1+x^2)^{-3} e^{-\frac{1}{\sigma}x} dx + \frac{48}{\sigma^c\Gamma(c)} \int_0^\infty x^{c+3}(1+x^2)^{-2} e^{-\frac{1}{\sigma}x} dx$$

$$- \frac{18}{\sigma^c\Gamma(c)} \int_0^\infty x^{c+1}(1+x^2)^{-1} e^{-\frac{1}{\sigma}x} dx$$

$$\alpha_3 = 1-\frac{16}{3\sigma^c\sqrt{\pi}\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\ \middle|\ \begin{array}{c}-\frac{c}{2}-2\\ -\frac{c}{2},\ 0,\ \frac{1}{2}\end{array}\right)+\frac{24}{\sigma^c\sqrt{\pi}\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\ \middle|\ \begin{array}{c}-\frac{c}{2}-1\\ -\frac{c}{2},\ 0,\ \frac{1}{2}\end{array}\right)$$

$$-\frac{9}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\ \middle|\ \begin{array}{c}-\frac{c}{2}\\ -\frac{c}{2},\ 0,\ \frac{1}{2}\end{array}\right)$$

$$\beta_3 = \frac{1}{2\sigma^c\Gamma(c)}\int_0^{\pi}\sin 3\theta\sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1}e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)}d\theta$$

Consider the transformation $x=\tan\left(\frac{\theta}{2}\right), \sin 3\theta = \frac{6x}{(1+x^2)} - \frac{32x^3}{(1+x^2)^3}$

$$\beta_3 = \frac{1}{\sigma^c\Gamma(c)}\int_0^{\infty}\left[\frac{6x}{(1+x^2)} - \frac{32x^3}{(1+x^2)^3}\right]x^{c-1}e^{-\frac{1}{\sigma}x}dx$$

$$= \frac{6}{\sigma^c\Gamma(c)}\int_0^{\infty}x^c(1+x^2)^{-1}e^{-\sigma x}dx - \frac{32}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{c+2}(1+x^2)^{-3}e^{-\frac{1}{\sigma}x}dx$$

$$\beta_3 = \frac{3}{\sigma^c\sqrt{\pi}\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\ \middle|\ \begin{array}{c}\frac{1-c}{2}\\ \frac{1-c}{2},\ 0,\ \frac{1}{2}\end{array}\right)-\frac{8}{\sigma^c\sqrt{\pi}\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\ \middle|\ \begin{array}{c}-\frac{(c+1)}{2}\\ \frac{1-c}{2},\ 0,\ \frac{1}{2}\end{array}\right)$$

To derive the fourth cosine and sine moments

$$\alpha_4 = \frac{1}{2\sigma^c\Gamma(c)}\int_0^{\pi}\cos 4\theta\sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1}e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)}d\theta$$

Consider the transformation

$x=\tan\left(\frac{\theta}{2}\right)$, $\cos 4\theta = 1+\frac{128x^8}{(1+x^2)^4}+\frac{160x^4}{(1+x^2)^2}-\frac{256x^6}{(1+x^2)^3}-\frac{32x^2}{(1+x^2)}$ and formula (4.1)

$$\alpha_4 = 1+\frac{128}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{c+7}(1+x^2)^{-4}e^{-\frac{1}{\sigma}x}dx+\frac{160}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{c+3}(1+x^2)^{-2}e^{-\frac{1}{\sigma}x}dx$$

$$-\frac{256}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{c+5}(1+x^2)^{-3}e^{-\frac{1}{\sigma}x}dx-\frac{32}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{c+1}(1+x^2)^{-1}e^{-\frac{1}{\sigma}x}dx$$

$$= 1+\frac{128}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{2\left(\frac{c}{2}+4\right)-1}(1+x^2)^{-3-1}e^{-\frac{1}{\sigma}x}dx+\frac{160}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{2\left(\frac{c}{2}+2\right)-1}(1+x^2)^{-1-1}e^{-\frac{1}{\sigma}x}dx$$

$$-\frac{256}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{2\left(\frac{c}{2}+3\right)-1}(1+x^2)^{-2-1}e^{-\frac{1}{\sigma}x}dx-\frac{32}{\sigma^c\Gamma(c)}\int_0^{\infty}x^{2\left(\frac{c}{2}+1\right)-1}(1+x^2)^{0-1}e^{-\frac{1}{\sigma}x}dx$$

$$\alpha_4 = 1 + \frac{32}{3\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{c}{2}-3 \\ -\frac{c}{2},\ 0,\ \frac{1}{2} \end{array}\right) + \frac{80}{\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{c}{2}-1 \\ -c,\ 0,\ \frac{1}{2} \end{array}\right)$$

$$-\frac{64}{\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{c}{2}-2 \\ -\frac{c}{2},\ 0,\ \frac{1}{2} \end{array}\right) - \frac{16}{\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{c}{2} \\ -\frac{c}{2},\ 0,\ \frac{1}{2} \end{array}\right)$$

$$\beta_4 = \frac{1}{2\sigma^c\Gamma(c)} \int_0^{\pi} \sin 4\theta \sec^2\left(\frac{\theta}{2}\right)\left(\tan\left(\frac{\theta}{2}\right)\right)^{c-1} e^{-\frac{1}{\sigma}\tan\left(\frac{\theta}{2}\right)} d\theta$$

$\sin 4\theta = \frac{8x}{(1+x^2)} - \frac{16x^3}{(1+x^2)^2} - \frac{64x^3}{(1+x^2)^3} + \frac{128x^5}{(1+x^2)^4}$ and formula (4.1)

$$\beta_4 = \frac{8}{\sigma^c\Gamma(c)} \int_0^{\infty} x^c(1+x^2)^{-1} e^{-\frac{1}{\sigma}x} dx - \frac{16}{\sigma^\lambda\Gamma(c)} \int_0^{\infty} x^{c+2}(1+x^2)^{-2} e^{-\frac{1}{\sigma}x} dx$$

$$-\frac{64}{\sigma^c\Gamma(c)} \int_0^{\infty} x^{c+2}(1+x^2)^{-3} e^{-\frac{1}{\sigma}x} dx + \frac{128}{\sigma^c\Gamma(c)} \int_0^{\infty} x^{c+4}(1+x^2)^{-4} e^{-\frac{1}{\sigma}x} dx$$

$$= \frac{8}{\sigma^c\Gamma(c)} \int_0^{\infty} x^{2\left(\frac{c}{2}+\frac{1}{2}\right)-1}(1+x^2)^{0-1} e^{-\frac{1}{\sigma}x} dx - \frac{16}{\sigma^c\Gamma(c)} \int_0^{\infty} x^{2\left(\frac{c}{2}+\frac{3}{2}\right)-1}(1+x^2)^{-1-1} e^{-\frac{1}{\sigma}x} dx$$

$$-\frac{64}{\sigma^c\Gamma(c)} \int_0^{\infty} x^{2\left(\frac{c}{2}+\frac{3}{2}\right)-1}(1+x^2)^{-2-1} e^{-\frac{1}{\sigma}x} dx + \frac{128}{\sigma^c\Gamma(c)} \int_0^{\infty} x^{2\left(\frac{c}{2}+\frac{5}{2}\right)-1}(1+x^2)^{-3-1} e^{-\frac{1}{\sigma}x} dx$$

$$\beta_4 = \frac{4}{\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} \frac{1-c}{2} \\ \frac{1-c}{2},\ 0,\ \frac{1}{2} \end{array}\right) - \frac{8}{\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{(c+1)}{2} \\ \frac{1-c}{2},\ 0,\ \frac{1}{2} \end{array}\right)$$

$$-\frac{16}{\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{(c+1)}{2} \\ \frac{3-c}{2},\ 0,\ \frac{1}{2} \end{array}\right) - \frac{32}{3\sigma^c\sqrt{\pi}\Gamma(c)} G^{31}_{13}\left(\frac{1}{4\sigma^2} \middle| \begin{array}{c} -\frac{(c+3)}{2} \\ \frac{3-c}{2},\ 0,\ \frac{1}{2} \end{array}\right)$$

On the similar lines the higher-order moments can be obtained .

5. Stereographic -l-axial Semicircular Gamma Distribution

The above proposed model is extended to the l-axial distribution, which is applicable to any arc of arbitrary length say $\frac{2\pi}{l}$ for $l \in$ N, . It is possible and useful to extend the Stereographic Semicircular Gamma distribution to construct the Stereographic-l-axial Gamma distribution.

The density function of stereographic semicircular gamma distribution by using the transformation $\phi = \frac{2\theta}{l}, l \in$ N. on the probability density function of ϕ is given by

$$g(\phi) = \frac{1}{2\sigma^c\Gamma(c)} \sec^2\left(\frac{l\phi}{4}\right)\left(\tan\left(\frac{l\phi}{4}\right)\right)^{c-1} \exp\left(-\frac{1}{\sigma}\left(\tan\left(\frac{l\phi}{4}\right)\right)\right) \tag{5.1}$$

$0 < \phi < \frac{2\pi}{l}, \sigma > 0, c > 0$ and $l = 1, 2, \ldots$

It is coined as Stereographic -l-axial Gamma Distribution

Case (1) When $l = 1$, in the probability density function (5.1), we get the density function

$$g(\phi) = \frac{1}{2\sigma^c \Gamma(c)} \sec^2\left(\frac{\phi-\mu}{4}\right)\left(\tan\left(\frac{\phi-\mu}{4}\right)\right)^{c-1} \exp\left(-\frac{1}{\sigma}\left(\tan\left(\frac{\phi-\mu}{4}\right)\right)\right) \tag{5.2}$$

$0 < \phi < 2\pi, \sigma > 0$ and $c > 0$

It is named by us as Stereographic Circular Gamma Distribution.

Case (2) When $l = 2$, the probability density function (5.1) is the same as that of Stereographic Semicircular Gamma Distribution.

Case (3) When $l = 2$ and $c = 1$, in the probability density function (5.1) we get the density function of Stereographic Semicircular Exponential Distribution [Phani et al (2013)].

6. Conclusion

In this paper, we derived the semicircular distribution induced by modified inverse stereographic projection on Gamma distribution is discussed and named it by us as Stereographic Semicircular Gamma distribution. The density and distribution function of Stereographic semicircular gamma distribution admit explicit forms, as do trigonometric moments. As this distribution is asymmetric, it is suitable for modeling skewed directional data.

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