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An unified theorem for mappings in orbitally complete partial metric spaces

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Abstract. In this paper an unified theorem for mappings in orbitally complete partial metric spaces is proved. This theorem generalizes and proves Theorems 8 and 9 [8], Theorem 3.2 [10] and Theorem 2.6 [7].

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1. Introduction

In 1974, Ćirić [4] has first introduced orbitally complete metric spaces and orbitally continuous functions.

Let f be a self mapping of a metric space (X, d). If $x_0 \in X$, every Cauchy sequence of the orbit $O_{x_0}(f) = \{x_0, fx_0, f^2x_0, ...\}$ is convergent to a point $y \in X$, then X is said to be f - orbitally complete in x_0. If f is orbitally complete at each $x \in X$, then X is said to be f - orbitally complete. Every complete metric space is f - orbitally complete for every function f. An orbitally complete metric space may not be a complete metric space [[17], Example 2].

Let f be a self mapping of a metric space (X, d). Then, the mapping f is said to be orbitally continuous at the point $x \in X$ if fy_n converges to fz for any subsequence $y_n \in O_x(f)$ which converges to the point $z \in X$. The function f is said to be orbitally continuous if it is orbitally continuous at each $x \in X$. Any continuous self mappings of a metric space is orbitally continuous. An orbitally continuous mapping may not be continuous [[17], Examples 4, 5].

Some fixed point results for mappings in orbitally complete metric spaces are obtained in [2], [5], [11], [12] and in other papers.

In 1994, Matthews [9] introduced the concept of partial metric spaces as a part of the study of denotional semantics of data flow net work and proved the Banach contraction

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principle in such spaces. Many authors studied the fixed points for mappings satisfying contractive conditions in complete partial metric spaces in [1], [3], [6] and in other papers.

Recently, in [8] the authors initiated the study of fixed points in orbitally complete partial metric spaces.

In [7] and [10] new results are obtained.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [13], [14] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces and G - metric spaces. With this method the proof of some fixed point theorems is more simple. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for mappings satisfying an implicit relation in orbitally metric spaces is initiated in [15], [16] and in other papers. The study of fixed points for mappings satisfying an implicit relation in partial metric spaces is initiated in [18].

The purpose of this paper is to prove a general fixed point theorem for self mappings in orbitally complete partial metric spaces which generalizes and improves Theorem 2.6 [7], Theorem 8 and 9 [8] and Theorem 3.2 [10].

2. Preliminaries

Definition 1 ([9]). *Let X be a nonempty set. A function $p : X \times X \to R_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

$(P_1) : p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$,

$(P_2) : p(x,x) \leq p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space. If $p(x,y) = 0$, then (P_1) and (P_2) implies $x = y$, but the converse does not always hold.

Each partial metric space on X generates a T_0 topology τ_p which has as base the family of open p - balls $\{B_p(x,\epsilon) : x \in X \text{ and } \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) \leq p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

If p is a partial metric on X, then the function $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X. Further, a sequence (x_n) converges in (X, d_p) to a point $x \in X$ if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x,x).$$

Lemma 1 ([2], [9]). *Let (X, p) be a partial metric space and x_n a sequence in X convergent to z, where $p(z, z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Definition 2 ([9]). *Let (X, p) be a partial metric space.*

a) A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

b) A partial metric space is said to be complete if every Cauchy sequence converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Remark 1. *The following examples [9] show that a convergent sequence in partial metric space may not be Cauchy. In particular, it is shown that the limit of a convergent sequence is not unique.*

Example 1 ([9]). *Let $p : R_+ \times R_+ \to R_+$ be a partial metric defined as $p(x, y) = \max\{x, y\}$. Define a sequence $\{x_n\}$ in X as*

$$x_n = \begin{cases} 0, & n = 2k \\ 1, & n = 2k + 1, k \in N. \end{cases}$$

Then $\{x_n\}$ is a convergent sequence but $\lim_{n,m\to\infty} p(x_n, x_m)$ does not exists.

Definition 3 ([8]). *Let (X, p) be a partial metric space. A mapping $T : X \to X$ is called orbitally continuous if $\lim_{i\to\infty} p(T^{n_i}x, z) = p(z, z)$ implies $\lim_{i\to\infty} p(TT^{n_i}x, z) = p(Tz, z)$ for each $x \in X$.*

Definition 4 ([8]). *A partial metric space is called orbitally complete if every Cauchy sequence $\{T^{n_i}x\}_{i=1}^{\infty}$ converges in (X, p), that is*

$$\lim_{i,j\to\infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i\to\infty} p(T^{n_i}x, z) = p(z, z).$$

Theorem 2 ([8]). *Let $T : X \to X$ be an orbitally continuous function on an orbitally complete partial metric space (X, p). If*

$$\min\{p\,(Tx, Ty)\,, p\,(x, Tx)\,, p\,(y, Ty)\} \le ap\,(x, y)$$

for some $0 \le a < 1$ and all $x, y \in X$, then the sequence $\{T^n x\}$ converges to a fixed point of T in X.

Theorem 3 ([8]). *Let $T : X \to X$ be an orbitally continuous function on an orbitally complete partial metric space (X, p). If*

$$\frac{\min\{p\,(Tx, Ty) \cdot p(x, y), p\,(x, Tx) \cdot p\,(y, Ty)\}}{\min\{p\,(x, Tx)\,, p\,(y, Ty)\}} \le ap\,(x, y)$$

for some $0 \le a < 1$ and all $x, y \in X$ such that $p(x, Tx) \ne 0$ and $p(y, Ty) \ne 0$, then the sequence $\{T^n x\}$ converges to a fixed point of T.

Theorem 4 ([10]). *Let (X,p) be an orbitally complete partial metric space and let $T: X \to X$ be an orbitally continuous function that satisfy*

$$p(Tx,Ty) \leq ap(x,y) + b\frac{p(x,Tx) + p(y,Ty)\}}{1 + p(x,y)}$$

for all $x \neq y$, where $a, b \geq 0$ and $a + b < 1$. Then T has a fixed point z in X. Moreover, $p(z,Tz) = p(Tz,Tz) = p(z,z) = 0$.

Theorem 5 ([7]). *Let $T: X \to X$ be an orbitally continuous function on an orbitally complete partial metric space. Suppose that*

$$\min\{p^2(x,Tx), p^2(y,Ty), p(x,y) \cdot p(Tx,Ty)\} \leq ap(x,Tx) \cdot p(y,Ty)$$

for all all $x, y \in X$ and for some $0 \leq a < 1$. Then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

3. Implicit relations

Definition 5. *Let $\mathcal{F}_{op}$ be the sets of all continuous functions $F(t_1, t_2, ..., t_5) : R_+^5 \to R$ satisfying the following conditions:*

(F_1) : F is not increasing in variable t_5,

(F_2) : There exists $h \in (0,1)$ such that for all $u \geq 0, v > 0$, $F(u,v,v,u,u+v) \leq 0$ implies $u \leq hv$.

In the following examples, the condition (F_1) is obviously.

Example 2. *$F(t_1,...,t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, where $a > 0$, $b,c,d \geq 0$ and $a + b + c + 2d < 1$.*

(F_2) : Let $u \geq 0$, $v > 0$ and $F(u,v,v,u,u+v) = u - av - bv - cu - d(u+v) \leq 0$. Then $u \leq hv$, where $0 < h = \frac{a+b+d}{1-(c+d)} < 1$.

Example 3. *$F(t_1,...,t_5) = t_1 - k\max\{t_2,t_3,t_4,t_5\}$, where $k \in \left(0, \frac{1}{2}\right)$.*

(F_2) : Let $u \geq 0$, $v > 0$ and $F(u,v,v,u,u+v) = u - k(u+v) \leq 0$. Then $u \leq hv$, where $0 < h = \frac{k}{1-k} < 1$.

Example 4. *$F(t_1,...,t_5) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}$, where $k \in (0,1)$.*

(F_2) : Let $u \geq 0$, $v > 0$ and $F(u,v,v,u,u+v) = u - k\max\left\{u,v,\frac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $0 < h = k < 1$.

Example 5. *$F(t_1,...,t_5) = t_1^2 - at_2t_3 - bt_4^2 - ct_5^2$, where $a > 0$, $b,c \geq 0$ and $a + b + 4c < 1$.*

(F_2) : Let $u \geq 0$, $v > 0$ and $F(u,v,v,u,u+v) = u^2 - av^2 - bu^2 - c(u+v)^2 \leq 0$. If $u > v$, then $u^2[1 - (a+b+4c)] \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $0 < h = \sqrt{a+b+4c} < 1$.

Example 6. $F(t_1, ..., t_5) = t_1^2 + \frac{t_1}{1+t_5} - \left(at_2^2 + bt_3^2 + ct_4^2\right)$, *where* $a > 0$, $b, c \geq 0$ *and* $a + b + c < 1$.

(F_2) : *Let* $u \geq 0$, $v > 0$ *and* $F(u, v, v, u, u + v) = u^2 + \frac{u}{1+u+v} - \left(av^2 + bv^2 + cu^2\right) \leq 0$, *which implies* $u^2 - \left(av^2 + bv^2 + cu^2\right) \leq 0$. *Hence,* $u \leq hv$, *where* $0 < h = \sqrt{\frac{a+b}{1-c}} < 1$.

Example 7. $F(t_1, ..., t_5) = t_1 - at_2 - \frac{b(1+t_3)t_4}{1+t_2} - ct_5$, *where* $a > 0$, $b, c \geq 0$ *and* $a+b+2c < 1$.

(F_2) : *Let* $u \geq 0$, $v > 0$ *and* $F(u, v, v, u, u + v) = u - av - bu - c(u + v) \leq 0$, *which implies* $u \leq hv$, *where* $0 < h = \frac{a+c}{1-(b+c)} < 1$.

Example 8. $F(t_1, ..., t_5) = \min\{t_1, t_3, t_4\} - at_2 - b\min\{t_3, t_5\}$, *where* $a, b \geq 0$ *and* $0 < a + b < 1$.

(F_2) : *Let* $u \geq 0$, $v > 0$ *and* $F(u, v, v, u, u + v) = \min\{u, v\} - av - bv \leq 0$. *If* $u > v$, *then* $v(1 - (a + b)) \leq 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u \leq hv$, *where* $0 < h = a + b < 1$.

Example 9. $F(t_1, ..., t_5) = \min\{t_1t_2, t_3t_4\} - at_2\min\{t_3, t_4\} - b\min\{t_2^2, t_5^2\}$, *where* $a, b \geq 0$ *and* $0 < a + b < 1$.

(F_2) : *Let* $u \geq 0$, $v > 0$ *and* $F(u, v, v, u, u + v) = uv - av\min\{u, v\} - b\min\{v^2, (u + v)^2\} \leq 0$. *If* $u > v$, *then* $v^2(1 - (a + b)) \leq 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u \leq hv$, *where* $0 < h = \sqrt{a + b} < 1$.

Example 10. $F(t_1, ..., t_5) = \min\{t_3^2, t_1t_2, t_4^2\} - at_3t_4 - bt_5^2$, *where* $a, b \geq 0$ *and* $0 < a+4b < 1$.

(F_2) : *Let* $u \geq 0$, $v > 0$ *and* $F(u, v, v, u, u + v) = \min\{v^2, uv, u^2\} - auv - b(u + v)^2 \leq 0$. *If* $u > v$, *then* $v^2(1 - (a + 4b)) \leq 0$, *a contradiction. Hence,* $u \leq v$ *which implies* $u \leq hv$, *where* $0 < h = \sqrt{a + 4b} < 1$.

Example 11. $F(t_1, ..., t_5) = \min\{t_1t_3, t_2t_4\} - at_2\min\{t_3, t_4\} - b\min\{t_3^2, t_5^2\}$, *where* $a, b \geq 0$ *and* $0 < a + b < 1$.

The proof is similar to the proof of Example 9.

4. Main result

Theorem 6. *Let* (X, p) *be an orbitally complete metric space and* $T : X \to X$ *be orbitally continuous such that*

$$F(p(Tx, Ty), p(x, y), p(x, Tx), p(y, Ty), p(x, Ty) + p(y, Tx)) \leq 0 \tag{1}$$

for all $x \neq y \in X$ *and* $F \in \mathcal{F}_{op}$. *Then,* T *has an fixed point* z *such that* $p(z, z) = p(Tz, Tz) = p(z, Tz) = 0$.

Proof. Let $x_0 \in X$ and $x_{n+1} = T^n x_0$ and so $x_{n+1} = Tx_n$. If there exists $n \in N$ such that $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point. Suppose that $x_n \neq x_{n+1}$. Hence, $p(x_n, x_{n+1}) > 0$ for all $n \in N$. By (1) we have successively

$$F(p(Tx_n, Tx_{n+1}), p(x_n, x_{n+1}), p(x_n, Tx_n), \\ p(x_{n+1}, Tx_{n+1}), p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)) \leq 0,$$

$$F(p(x_{n+1},x_{n+2}),p(x_n,x_{n+1}),p(x_n,x_{n+1}),\\ p(x_{n+1},x_{n+2}),p(x_n,x_{n+2})+p(x_{n+1},x_{n+1}))\leq 0.$$

By (P_3):

$$p(x_n,x_{n+2})\leq p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})-p(x_{n+1},x_{n+1}).$$

Then by (F_1) we obtain

$$F(p(x_{n+1},x_{n+2}),p(x_n,x_{n+1}),p(x_n,x_{n+1}),\\ p(x_{n+1},x_{n+2}),p(x_n,x_{n+1})+p(x_{n+1},x_{n+2}))\leq 0.$$

By (F_2) we obtain

$$p(x_{n+1},x_{n+2})\leq hp(x_{n+1},x_n).$$

For $m>n$, using (P_4) we obtain

$$\begin{aligned} p(x_n,x_m) &\leq p(x_n,x_{n+1})+...+p(x_{n+m-1},x_{n+m}) \\ &\leq \left(h^n+h^{n+1}+...+h^{m-1}\right)p(x_0,x_1) \\ &\leq \frac{h^n}{1-h}p(x_0,x_1). \end{aligned}$$

Thus, $\lim_{n,m\to\infty}p(x_n,x_m)=0$. Hence, $\{x_n\}$ is a Cauchy sequence in (X,p) and since (X,p) is orbitally complete, then $\{T^nx_0\}$ converges to a limit $z\in X$ such that $\lim_{n\to\infty}p(T^nx_0,T^mx_0)=\lim_{n\to\infty}p(T^nx_0,z)=p(z,z)=0$. Since T is orbitally continuous, $\lim_{n\to\infty}p(T^nx_0,z)=p(z,z)$ implies

$$\lim_{n\to\infty}p\left(T^{n+1}x_0,Tz\right)=p(Tz,Tz).$$

On the other hand

$$\begin{aligned} p(z,Tz) &\leq p\left(z,T^{n+1}x_0\right)+p\left(T^{n+1}x_0,Tz\right) \\ &= p(z,x_{n+2})+p\left(T^{n+1}x_0,Tz\right). \end{aligned}$$

Using Lemma 1 and letting n tends to infinity we obtain

$$p(z,Tz)\leq p(Tz,Tz).$$

By (1) we have

$$F(p\left(Tz,T^2z\right),p(z,Tz),p(z,Tz),\\ p\left(Tz,T^2z\right),p\left(z,T^2z\right)+p(Tz,Tz))\leq 0.$$

Then by (P_4),

$$p\left(z,T^2z\right)\leq p(z,Tz)+p\left(Tz,T^2z\right)-p(Tz,Tz).$$

By (F_1) we obtain

$$F(p\left(Tz,T^2z\right),p\left(z,Tz\right),p\left(z,Tz\right),\\ p\left(Tz,T^2z\right),p\left(z,Tz\right)+p\left(Tz,T^2z\right))\leq 0$$

which implies by (F_2) that

$$p\left(Tz,T^2z\right)\leq hp\left(z,Tz\right).$$

Hence, by (P_2) we obtain

$$p\left(z,Tz\right)\leq p\left(Tz,Tz\right)\leq p\left(Tz,T^2z\right)\leq hp\left(z,Tz\right)$$

which implies

$$p\left(z,Tz\right)\left(1-h\right)\leq 0,$$

i.e.

$$p\left(z,Tz\right)=0.$$

Hence, $z=Tz$ and z is a fixed point of T.

Since

$$p\left(z,Tz\right)\leq p\left(Tz,Tz\right)\leq p\left(z,Tz\right).$$

Hence

$$p\left(z,Tz\right)=p\left(Tz,Tz\right).$$

Therefore

$$p\left(z,z\right)=p\left(z,Tz\right)=p\left(Tz,Tz\right)=0.$$

Remark 2. *a) By Theorem 6 and Example 8 with $b=0$, we obtain a generalization of Theorem 2.*

b) By Theorem 6 and Example 9 we obtain Theorem 3.

c) By Theorem 6 and Example 7 with $c=0$, we obtain Theorem 4.

d) By Theorem 6 and Example 10 we obtain Theorem 5.

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