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n-tupled common fixed point theorems via α-series in ordered metric spaces

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Abstract. The aim of this paper is to establish some results about the existence and uniqueness of n-tupled fixed point that extend the previous results, using the concept of an α-series for sequence of mappings having mixed monotone property in the framework of ordered complete metric spaces. The main result is supported with the aid of an illustrative example.

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1. Introduction

For standard terminology and notations in fixed point theory, not specifically mentioned or defined we refer the reader to the standard textbook [21]. Throughout the paper, for a nonempty set X, $\prod_{\lambda=1}^{r} X^{\lambda}$ denote the product space $\prod_{\lambda=1}^{r} X^{\lambda} = X \times X \times X \times \cdots \times X$.

The existence of a fixed point for contraction type mappings in metric spaces along with applications have been taken a considerable attention. The Banach contraction principle is one of the earliest and the most important results in the area of fixed point theory. Several authors have improved, generalized, and extended this classical result in nonlinear analysis. The notion of coupled fixed point is introduced by Bhaskar and Lakshmikantham [7]. Afterwards Lakshmikantham and Ciric [18] extended this notion by defining the g-monotone property in partially ordered spaces. For a detailed study on coupled coincidence and coupled common fixed point results, we refer the reader to [5, 11, 12, 13, 18]. Berinde and Borcut [6] introduced the concept of tripled fixed point. An enough considerable work have been done in this area by several authors (see, for instance, [1, 2, 3, 4, 16, 17, 23]).

In 2010, Samet and Vetro [20] extended the idea of coupled fixed point to higher dimensions by introducing the notion of fixed point of n-order (or n-tupled fixed point,

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where $n \in \mathbb{N}, n \geq 2$) and presented some n-tupled fixed point results in complete metric spaces. In 2011, Gordji and Ramezani [14] introduced and investigated the concept of an n-tupled fixed point. In 2013, Imdad et al. [15] generalized the idea of n-tupled fixed point by considering even-tupled coincidence point by initiating the idea of mixed g-monotone property on X^n and proved an even-tupled coincidence point theorem for nonlinear f-contraction mappings satisfying mixed g-monotone property. However, the concept of n-tupled fixed point given by Imdad et al. [15], which is quite different from the concept of Gordji and Ramezani [14].

In this paper, we shall also point out some useful remarks on mappings whenever found prominent or pertinent as we proceed with this article. Our focus however will be on results that gives the guarantee about the existence and uniqueness of n-tupled fixed point that extend the previous results in the framework of ordered complete metric spaces, using the concept of an α-series for sequence of mappings having mixed monotone property in ordered complete metric spaces. In order to do so, we propose a notion of compatible mapping for mapping $F : \prod_{i=1}^{r} X^i \to X$ and self mapping g akin to compatible mapping as introduced by Choudhary and Kundu [13] for bivariate mapping F and self mapping g. The methodology is analogous to those used in [15]. Finally, the main result of the manuscript is supported with the aid of an illustrative example.

2. Preliminaries

In this section, we collect some definitions, properties and results which will be frequently used in this paper.

As in [19] we define a metric on X, a mapping $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$: (i) $d(x, y) = 0$ if and only if, $x = y$; (ii) $d(x, y) = d(z, x) + d(z, y)$. Thus, in light of the above properties one can easily deduce that $d(x, y) \geq 0$ and $d(y, x) = d(x, y)$ for all $x, y \in X$. The last requirement is called the triangle inequality. If d is a metric on X, then we say that (X, d) is a metric space.

Definition 1. [10] A triple $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.

Imdad et al. [15] introduced the concept of mixed monotone property and g-mixed monotone property for n-tupled mapping $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ in the following way:

Definition 2. Let $(X, \preceq)$ be a partially ordered set and $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is,

$$\begin{cases} \forall\ x_1^1, x_2^1 \in X, x_1^1 \preceq x_2^1 \Rightarrow F(x_1^1, x^2, \ldots, x^r) \preceq F(x_2^1, x^2, \ldots, x^r), \\ \forall\ x_1^2, x_2^2 \in X, x_1^2 \preceq x_2^2 \Rightarrow F(x^1, x_1^2, \ldots, x^r) \succeq F(x^1, x_2^2, \ldots, x^r), \\ \forall\ x_1^3, x_2^3 \in X, x_1^3 \preceq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, \ldots, x^r) \preceq F(x^1, x^2, x_2^3, \ldots, x^r), \\ \vdots \\ \forall\ x_1^r, x_2^r \in X, x_1^r \preceq x_2^r \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^r) \succeq F(x^1, x^2, x^3, \ldots, x_2^r). \end{cases}$$

Definition 3. [15] Let $(X, \preceq)$ be a partially ordered set. Let $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ and $g : X \to X$ be two mappings. Then the mapping F is said to have the mixed g-monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even position arguments, that is

$$\begin{cases} \forall\, x_1^1, x_2^1 \in X, g(x_1^1) \preceq g(x_2^1) \Rightarrow F(x_1^1, x^2, \ldots, x^r) \preceq F(x_2^1, x^2, \ldots, x^r), \\ \forall\, x_1^2, x_2^2 \in X, g(x_1^2) \preceq g(x_2^2) \Rightarrow F(x^1, x_1^2, \ldots, x^r) \succeq F(x^1, x_2^2, \ldots, x^r), \\ \forall\, x_1^3, x_2^3 \in X, g(x_1^3) \preceq g(x_2^3) \Rightarrow F(x^1, x^2, x_1^3, \ldots, x^r) \preceq F(x^1, x^2, x_2^3, \ldots, x^r), \\ \vdots \\ \forall\, x_1^r, x_2^r \in X, g(x_1^r) \preceq g(x_2^r) \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^r) \succeq F(x^1, x^2, x^3, \ldots, x_2^r). \end{cases}$$

Now, we introduce the concept of compatible mapping for mapping $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ and self mapping g akin to compatible mapping as introduced by Choudhary and Kundu [13] for mapping F and self mapping g.

Definition 4. Let $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ and $g : X \to X$ be two mappings. Then F and g are said to be compatible if

$$\begin{cases} \lim_{n\to+\infty} d(g(F(x_n^1, x_n^2, \ldots, x_n^r)), F(g(x_n^1), g(x_n^2), \ldots, g(x_n^r))) = 0, \\ \lim_{n\to+\infty} d(g(F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1)), F(g(x_n^2), g(x_n^3), \ldots, g(x_n^r), g(x_n^1))) = 0, \\ \lim_{n\to+\infty} d(g(F(x_n^3, x_n^4, \ldots, x_n^1, x_n^2)), F(g(x_n^3), g(x_n^4), \ldots, g(x_n^1), g(x_n^2))) = 0, \\ \vdots \\ \lim_{n\to+\infty} d(g(F(x_n^r, x_n^1, \ldots, x_n^{r-1})), F(g(x_n^r), g(x_n^1), \ldots, g(x_n^{r-1}))) = 0, \end{cases}$$

whenever $\{x_n^1\}$, $\{x_n^2\}, \ldots, \{x_n^r\}$ are sequences in X, such that

$$\begin{cases} \lim_{n\to+\infty} F(x_n^1, x_n^2, \ldots, x_n^r) = \lim_{n\to+\infty} g(x_n^1) = x^1, \\ \lim_{n\to+\infty} F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1) = \lim_{n\to+\infty} g(x_n^2) = x^2 \\ \vdots \\ \lim_{n\to+\infty} F(x_n^r, x_n^1, \ldots, x_n^{r-1}) = \lim_{n\to+\infty} g(x_n^r) = x^r, \end{cases} \tag{1}$$

for all $x^1, x^2, \ldots, x^r \in X$.

The following is the definition of reciprocally continuity and weakly reciprocally continuity for mapping $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ and self mapping g:

Definition 5. Let $F : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ and $g : X \to X$ be two mappings. Then F and g are said to be

(i) Reciprocally continuous if

$$\begin{cases} \lim_{n\to+\infty} g(F(x_n^1, x_n^2, \ldots, x_n^r) = g(x^1) \text{ and} \\ \qquad \lim_{n\to+\infty} F(g(x_n^1), g(x_n^2), \ldots, g(x_n^r)) = F(x^1, x^2, \ldots, x^r); \\ \lim_{n\to+\infty} g(F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1) = g(x^2) \text{ and} \\ \lim_{n\to+\infty} F(g(x_n^2), g(x_n^3), \ldots, g(x_n^r), g(x_n^1)) = F(x^2, x^3, \ldots, x^r, x^1); \\ \vdots \\ \qquad \lim_{n\to+\infty} g(F(x_n^r, x_n^1, \ldots, x_n^{r-1}) = g(x^r) \text{ and} \\ \lim_{n\to+\infty} F(g(x_n^r), g(x_n^1), \ldots, g(x_n^{r-1})) = F(x^r, x^1, \ldots, x^{r-1}). \end{cases}$$

whenever $\{x_n^1\}$, $\{x_n^2\}, \ldots, \{x_n^r\}$ are sequences in X, such that

$$\begin{cases} \lim_{n\to+\infty} F(x_n^1, x_n^2, \ldots, x_n^r) = \lim_{n\to+\infty} g(x_n^1) = x^1, \\ \lim_{n\to+\infty} F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1) = \lim_{n\to+\infty} g(x_n^2) = x^2 \\ \vdots \\ \lim_{n\to+\infty} F(x_n^r, x_n^1, \ldots, x_n^{r-1}) = \lim_{n\to+\infty} g(x_n^r) = x^r, \end{cases}$$

for some $x^1, x^2, \ldots, x^r \in X$.

(ii) Weakly reciprocally continuous if

$$\begin{cases} \lim_{n\to+\infty} g(F(x_n^1, x_n^2, \ldots, x_n^r) = g(x^1) \text{ or} \\ \qquad \lim_{n\to+\infty} F(g(x_n^1), g(x_n^2), \ldots, g(x_n^r)) = F(x^1, x^2, \ldots, x^r); \\ \lim_{n\to+\infty} g(F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1) = g(x^2) \text{ or} \\ \lim_{n\to+\infty} F(g(x_n^2), g(x_n^3), \ldots, g(x_n^r), g(x_n^1)) = F(x^2, x^3, \ldots, x^r, x^1); \\ \vdots \\ \qquad \lim_{n\to+\infty} g(F(x_n^r, x_n^1, \ldots, x_n^{r-1}) = g(x^r) \text{ or} \\ \lim_{n\to+\infty} F(g(x_n^r), g(x_n^1), \ldots, g(x_n^{r-1})) = F(x^r, x^1, \ldots, x^{r-1}). \end{cases} \tag{2}$$

Whenever $\{x_n^1\}$, $\{x_n^2\}, \ldots, \{x_n^r\}$ are sequences in X, such that

$$\begin{cases} \lim_{n\to+\infty} F(x_n^1, x_n^2, \ldots, x_n^r) = \lim_{n\to+\infty} g(x_n^1) = x^1, \\ \lim_{n\to+\infty} F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1) = \lim_{n\to+\infty} g(x_n^2) = x^2 \\ \vdots \\ \lim_{n\to+\infty} F(x_n^r, x_n^1, \ldots, x_n^{r-1}) = \lim_{n\to+\infty} g(x_n^r) = x^r, \end{cases}$$

for some $x^1, x^2, \ldots, x^r \in X$.

Remark 1. Every pair of reciprocally continuous mapping (F, g) is weakly reciprocally continuous but not conversely.

Definition 6. Let $(X, d, \preceq)$ be an ordered metric space. We say that X is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y_n \succeq y$ for all $n \geq 0$.

Definition 7. [22] Let $\{s_n\}$ be a sequence of non-negative real numbers. We say that a series $\sum_{n=1}^{+\infty} s_n$ is an α-series, if there exist $0 < \alpha < 1$ and $n_\alpha \in \mathbb{N}$ such that $\sum_{i=1}^{k} s_i \leq \alpha k$ for each $k \geq n_\alpha$.

Example 1. The series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is an α-series.

Remark 2. [22] It is bring here to notice that each convergent series of non-negative real terms is an α-series. However, there are also divergent series that are α-series. As for instance; the series $\sum_{n=1}^{+\infty} \frac{1}{n}$, is an α-series.

Definition 8. Let X be a nonempty set. An element $(x^1, x^2, \ldots, x^r) \in \prod_{\lambda=1}^{r} X^\lambda$ is called r-tupled fixed point of the mapping $F : \prod_{\lambda=1}^{r} X^\lambda \to X$ if

$$\begin{cases} F(x^1, x^2, \ldots, x^r) = x^1, \\ F(x^2, x^3, \ldots, x^1) = x^2, \\ F(x^3, x^4, \ldots, x^2) = x^3, \\ \vdots \\ F(x^r, x^1, \ldots, x^{r-1}) = x^r. \end{cases} \tag{3}$$

Example 2. Let $(X, d, \preceq)$ be an ordered metric space with $\preceq$ as natural ordering and let $F : \prod_{\lambda=1}^{r} X^\lambda \to X$ be a mapping defined by $F(x^1, x^2, \ldots, x^r) = (x^1 \cdot x^2 \cdots x^r)^2$, for any $x^1, x^2, \ldots, x^r \in X$. Then $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ are both r-tupled fixed points of F.

Definition 9. Let X be a nonempty set. An element $(x^1, x^2, \ldots, x^r) \in \prod_{\lambda=1}^{r} X^\lambda$ is called r-tupled coincidence point of the mappings $F : \prod_{\lambda=1}^{r} X^\lambda \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, \ldots, x^r) = g(x^1), \\ F(x^2, x^3, \ldots, x^1) = g(x^2), \\ F(x^3, x^4, \ldots, x^2) = g(x^3), \\ \vdots \\ F(x^r, x^1, \ldots, x^{r-1}) = g(x^r). \end{cases} \tag{4}$$

Example 3. Let $(X, d, \preceq)$ be an ordered metric space with $\preceq$ as natural ordering and let $F : \prod_{\lambda=1}^{r} X^\lambda \to X$ be a mapping defined by $F(x^1, x^2, \ldots, x^r) = \sin(x^1 \cdot x^2 \cdots x^r)$, for any $x^1, x^2, \ldots, x^r \in X$ and $g : X \to X$ be mapping defined by $g(x) = x^2$. Then $(0, 0, \ldots, 0)$ is r-tupled coincidence point of F and g.

Proposition 1. Let $(X, d, \preceq)$ be an ordered metric space. Let g be a self-mapping on X and $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of mappings from $\prod_{\lambda=1}^{r} X^{\lambda} \to X$. Then the pair of mappings (T_i, g) is said to have **property (A)** if for $x^1, x^2, \ldots, x^r, y^1, y^2, \ldots, y^r \in X$

$$\begin{aligned} d(T_i(x^1, x^2, \ldots, x^r), T_j(y^1, y^2, \ldots, y^r)) \quad & \leq \beta_{i,j}[d(g(x^1), T_i(x^1, x^2, \ldots, x^r)) \\ & \quad + d(g(y^1), T_j(y^1, y^2, \ldots, y^r))] \\ & \quad\quad + \gamma_{i,j} d(g(y^1), g(x^1)) \end{aligned} \tag{5}$$

with

$$\begin{aligned} g(x^1) &\preceq g(y^1), \\ g(x^2) &\succeq g(y^2), \\ g(x^3) &\preceq g(y^3), \\ &\vdots \\ g(x^r) &\succeq g(y^r), \end{aligned}$$

where $0 \leq \beta_{i,j}, \gamma_{i,j} < 1$ for $i, j \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} \sup \beta_{i,n} < 1$.

Proposition 2. Let $(X, \preceq)$ be a partially ordered set and $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of mappings from $\prod_{\lambda=1}^{r} X^{\lambda} \to X$. Then $\{T_i\}_{i\in\mathbb{N}}$ is said to have **property (B)** if for $x^1, x^2, \ldots, x^r, y^1, y^2, \ldots, y^r \in X$

$$\begin{cases} T_i(x^1, x^2, \ldots, x^r) \preceq T_{i+1}(y^1, y^2, \ldots, y^r), \\ T_{i+1}(y^2, y^3, \ldots, y^r, y^1) \preceq T_i(x^2, x^3, \ldots, x^r, x^1), \\ \vdots \\ T_{i+1}(x^r, x^1, x^2 \cdots, x^{r-1}) \preceq T_i(y^r, y^1, y^2, \ldots, y^{r-1}). \end{cases} \tag{6}$$

3. Main results

Theorem 1. *Let $(X, d, \preceq)$ be an ordered metric space. Let g be a continious self-mapping on X and $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of mappings from $\prod_{\lambda=1}^{r} X^{\lambda} \to X$ such that*

i) $T_i(\prod_{\lambda=1}^{r} X^{\lambda}) \subseteq g(X)$, *$g(X)$ is regular and complete subset of X;*

ii) $\{T_i\}_{i\in\mathbb{N}}$ *have g-mixed monotone property, the pair $\{T_i\}_{i\in\mathbb{N}}$ and g are compatible, weakly reciprocally continuous and satisfy property (A) and (B).*

iii) There exists $x_0^1, x_0^2, \ldots, x_0^r \in X$ such that

$$\begin{cases} g(x_0^1) \preceq T_0(x_0^1, x_0^2, \ldots, x_0^r); \\ g(x_0^2) \succeq T_0(x_0^2, x_0^3, \ldots, x_0^r, x_0^1); \\ g(x_0^3) \preceq T_0(x_0^3, x_0^4, \ldots, x_0^1, x_0^2); \\ \vdots \\ g(x_0^r) \succeq T_0(x_0^r, x_0^1, \ldots, x_0^{r-1}). \end{cases} \tag{7}$$

.

If $\sum_{i=1}^{+\infty}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)$ is an α-series, then $\{T_i\}_{i\in\mathbb{N}}$ and g have a r-tupled coincidence point.

Proof. Let $(X,\preceq)$ be a partially ordered set, g be a self-mapping on X and $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of mappings from $\prod_{\lambda=1}^{r}X^{\lambda}\to X$. Since $T_i(\prod_{\lambda=1}^{r}X^{\lambda})\subseteq g(X)$, one can always have $x_1^1,x_1^2,\ldots,x_1^r\in X$ such that

$$\begin{cases} g(x_1^1)=T_1(x_0^1,x_0^2,\ldots,x_0^r),\\ g(x_1^2)=T_1(x_0^2,x_0^3,\ldots,x_0^r,x_0^1),\\ \vdots\\ g(x_1^r)=T_1(x_0^r,x_0^1,\ldots,x_0^{r-1}).\end{cases}$$

Again we can choose $x_2^1,x_2^2,\ldots,x_2^r\in X$ such that

$$\begin{cases} g(x_2^1)=T_2(x_1^1,x_1^2,\ldots,x_1^r),\\ g(x_2^2)=T_2(x_1^2,x_1^3,\ldots,x_1^r,x_1^1),\\ \vdots\\ g(x_2^r)=T_2(x_1^r,x_1^1,\cdots,x_1^{r-1}).\end{cases}$$

Continuing in this way, we can construct the $\{x_m^1\}$, $\{x_m^2\},\ldots,\{x_m^r\}$ sequences as follows :

$$\begin{cases} g(x_{m+1}^1)=T_n(x_m^1,x_m^2,\ldots,x_m^r),\\ g(x_{m+1}^2)=T_n(x_m^2,x_m^3,\ldots,x_m^r,x_m^1),\\ \vdots\\ g(x_{m+1}^r)=T_n(x_m^r,x_m^1,\ldots,x_m^{r-1}).\end{cases}\tag{8}$$

Now our claim is that for all $m\geq 0$.

$$g(x_m^1)\preceq g(x_{m+1}^1),\ g(x_m^2)\preceq g(x_{m+1}^2),\ldots,g(x_m^r)\preceq g(x_{m+1}^r).\tag{9}$$

We shall our claim by the principle of Mathematical induction.
Since

$$\begin{cases} g(x_0^1)\preceq T_0(x_0^1,x_0^2,\ldots,x_0^r)=x_1^1,\\ g(x_0^2)\succeq T_0(x_0^2,x_0^3,\ldots,x_0^r,x_0^1)=x_1^2,\\ g(x_0^3)\preceq T_0(x_0^3,x_0^4,\ldots,x_0^1,x_0^2)=x_1^3,\\ \vdots\\ g(x_0^r)\succeq T_0(x_0^r,x_0^1,\ldots,x_0^{r-1})=x_1^r.\end{cases}$$

and

$$\begin{cases} g(x_1^1)=T_0(x_0^1,x_0^2,\ldots,x_0^r),\\ g(x_1^2)=T_0(x_0^2,x_0^3,\ldots,x_0^r,x_0^1),\\ \vdots\\ g(x_1^r)=T_0(x_0^r,x_0^1,\ldots,x_0^{r-1}).\end{cases}$$

Thus we get

$$\begin{cases} g(x_0^1) \preceq g(x_1^1), \\ g(x_0^2) \succeq g(x_1^2), \\ g(x_0^3) \preceq g(x_1^3), \\ \vdots \\ g(x_0^r) \succeq g(x_1^r). \end{cases}$$

Which shows that (9) holds for $m = 0$. Now assume that (9) holds for some $m > 0$. From (8) and (9), one can deduce that

$$\begin{aligned} g(x_{m+1}^1) \quad & = T_m(x_m^1, x_m^2, \ldots, x_m^r) \\ & \preceq T_{m+1}(x_{m+1}^1, x_m^2, \ldots, x_m^r) \\ & \preceq T_{m+1}(x_{m+1}^1, x_{m+1}^2, \ldots, x_m^r) \\ & \vdots \\ & \preceq T_{m+1}(x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^r) \\ & = g(x_{m+2}^1). \end{aligned}$$

$$\begin{aligned} g(x_{m+1}^2) \quad & = T_{m+1}(x_m^2, x_m^3, \ldots, x_m^r, x_m^1) \\ & \succeq T_{m+1}(x_{m+1}^2, x_m^3, \ldots, x_m^r), x_m^1 \\ & \succeq T_{m+1}(x_{m+1}^2, x_{m+1}^3, \ldots, x_m^r, x_m^1) \\ & \vdots \\ & \succeq T_m(x_{m+1}^2, x_{m+1}^3, \ldots, x_{m+1}^r, x_{m+1}^1) \\ & = g(x_{m+2}^2). \end{aligned}$$

$$\begin{aligned} g(x_{m+1}^3) \quad & = T_m(x_m^3, x_m^4, \ldots, x_m^r, x_m^1, x_m^2) \\ & \preceq T_{m+1}(x_{m+1}^3, x_m^4, \ldots, x_m^r, x_m^1, x_m^2) \\ & \vdots \\ & \preceq T_{m+1}(x_{m+1}^3, x_{m+1}^4, \ldots, x_{m+1}^r, x_{m+1}^1, x_{m+1}^2) \\ & = g(x_{m+2}^3). \end{aligned}$$

Continuing in this way

$$\begin{aligned} g(x_{m+1}^r) \quad & = T_m(x_m^r, x_m^1, x_m^2, \ldots, x_m^{r-1}) \\ & \succeq T_{m+1}(x_{m+1}^r, x_m^1, x_m^2, \ldots, x_m^{r-1}) \\ & \succeq T_{m+1}(x_{m+1}^r, x_{m+1}^1, x_m^2, \ldots, x_m^{r-1}) \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\succeq T_{m+1}(x^r_{m+1}, x^1_{m+1}, x^2_{m+1}, \cdots, x^{r-1}_{m+1})\\
&= g(x^r_{m+2}).
\end{aligned}
$$

Thus by the principle of Mathematical induction, we conclude that (9) holds for all $n \geq 0$. Therefore,

$$
\begin{array}{rcl}
g(x^1_m) & \preceq & g(x^1_{m+1})\\
g(x^2_m) & \succeq & g(x^2_{m+1})\\
g(x^3_m) & \preceq & g(x^3_{m+1})\\
 & \vdots & \\
g(x^r_m) & \succeq & g(x^r_{m+1}).
\end{array}
$$

We consider the sequences $\{x^1_m\}$, $\{x^2_m\}, \ldots, \{x^r_m\}$ in X constructed in (8) and represent them by δ_m such that

$$
\delta_m = d(g(x^1_m), g(x^1_{m+1})) + d(g(x^2_m), g(x^2_{m+1})), + \ldots, +d(g(x^r_m), g(x^r_{m+1})).
$$

Now from the property (A) of $\{T_i\}_{i\in\mathbb{N}}$ and g in (5) we get

$$
\begin{aligned}
d(g(x^1_1), g(x^1_2)) &= d(T_0(x^1_0, x^2_0, \ldots x^r_0)), T_1(x^1_1, x^2_1, \ldots x^r_1)\\
&\leq \beta_{0,1}[d(g(x^1_0), T_0(x^1_0, x^2_0, \ldots x^r_0)) + d(g(x^1_1), T_1(x^1_1, x^2_1, \ldots x^r_1))]\\
&\quad +\gamma_{0,1}[d(g(x^1_0), g(x^1_1))]\\
&= \beta_{0,1}[d(g(x^1_0), g(x^1_1)) + d(g(x^1_1), g(x^1_2))]\\
&\quad +\gamma_{0,1}[d(g(x^1_0), g(x^1_1))].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(1-\beta_{0,1})d(g(x^1_1), g(x^1_2)) &\leq (\beta_{0,1}+\gamma_{0,1})d(g(x^1_0), g(x^1_1)),\\
\text{or equivalently, } d(g(x^1_1), g(x^1_2)) &\leq \left(\tfrac{\beta_{0,1}+\gamma_{0,1}}{1-\beta_{0,1}}\right) d(g(x^1_0), g(x^1_1)).
\end{aligned}
$$

Now

$$
\begin{aligned}
d(g(x^1_2), g(x^1_3)) &= d(T_1(x^1_1, x^2_1, \ldots x^r_1), T_2(x^1_2, x^2_2, \ldots x^r_2))\\
&\leq \left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-\beta_{1,2}}\right) d(g(x^1_1), g(x^1_2))\\
&\leq \left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-\beta_{1,2}}\right)\left(\frac{\beta_{0,1}+\gamma_{0,1}}{1-\beta_{0,1}}\right) d(g(x^1_0), g(x^1_1)).
\end{aligned}
$$

Continuing in this way, we get

$$
d(g(x^1_m), g(x^1_{m+1})) \leq d(T_{m-1}(x^1_{m-1}, x^2_{m-1}, \ldots x^r_{m-1}), T_m(x^1_m, x^2_m, \ldots x^r_m))
$$

$$\leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(g(x_0^1), g(x_1^1)). \tag{10}$$

$$\begin{aligned} d(g(x_m^2), g(x_{m+1}^2)) &\leq d(T_{m-1}(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^r, x_{m-1}^1), T_m(x_m^2, x_m^3, \ldots x_m^r, x_m^1)) \\ &\leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(g(x_0^2), g(x_1^2)). \end{aligned}$$

Similarly, one can inductively write

$$\begin{cases} d(g(x_m^2), g(x_{m+1}^2)) \leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(g(x_0^2), g(x_1^2)). \\ d(g(x_m^3), g(x_{m+1}^3)) \leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(g(x_0^3), g(x_1^3)). \\ \vdots \\ d(g(x_m^r), g(x_{m+1}^r)) \leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) d(g(x_0^r), g(x_1^r)). \end{cases} \tag{11}$$

Adding (10) and (11), we have

$$\begin{aligned} \delta_m &= d(g(x_m^1), g(x_{m+1}^1)) + d(g(x_m^2), g(x_{m+1}^2) + \cdots + d(g(x_m^r), g(x_{m+1}^r) \\ &\leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) [d(g(x_0^1), g(x_1^1)) + d(g(x_0^2), g(x_1^2)) + \cdots + d(g(x_0^r), g(x_1^r))] \\ &= \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) \delta_0. \end{aligned}$$

Moreover, for $p > 0$ and by repeated use of the triangle inequality, ones obtain

$$\begin{aligned} & d(g(x_m^1), g(x_{m+p}^1)) + d(g(x_m^2), g(x_{m+p}^2)) + \cdots + d(g(x_m^r), g(x_{m+p}^r)) \\ & \leq [(d(g(x_m^1), g(x_{m+1}^1)) + d(g(x_m^2), g(x_{m+1}^2)) + \cdots + d(g(x_m^r), g(x_{m+1}^r))) \\ & +(d(g(x_m^1), g(x_{m+2}^1)) + d(g(x_m^2), g(x_{m+2}^2)) + \cdots + d(g(x_m^r), g(x_{m+2}^r))) \\ & \cdots + (d(g(x_{m+p-1}^1), g(x_{m+p}^1)) + d(g(x_{m+p-1}^2), g(x_{m+p}^2)) + \\ & \cdots + d(g(x_{m+p-1}^r), g(x_{m+p}^r)))] \\ & \leq \prod_{i=0}^{m-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) \delta_0 + \prod_{i=0}^{m}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) \delta_0 + \cdots \\ & + \prod_{i=0}^{m+p-2}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) \delta_0 \\ & = \sum_{k=0}^{p-1} \prod_{i=0}^{m+k-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) \delta_0 = \sum_{k=m}^{m+p-1} \prod_{i=0}^{k-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right) \delta_0. \end{aligned}$$

Now using the fact that the Geometric mean of non-negative numbers is always less or equal to the Arithmetic mean, one have

$$\begin{aligned}
&d(g(x_m^1), g(x_{m+p}^1)) + d(g(x_m^2), g(x_{m+p}^2)) + \cdots + d(g(x_m^r), g(x_{m+p}^r)) \\
&\leq \sum_{k=m}^{m+p-1} [\frac{1}{k} \prod_{i=0}^{k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right)]^k \delta_0. \\
&\leq (\sum_{k=m}^{m+p-1} \alpha^k) \delta_0. \\
&\leq \frac{\alpha^m}{1-\alpha} \delta_0
\end{aligned}$$

Now, proceeding the limit as $m \to +\infty$, ones deduce that

$$\begin{aligned}
&\lim_{m \to +\infty} \delta_m = \lim_{m \to +\infty} \frac{\alpha^m}{1-\alpha} \delta_0 = 0 \text{ as } \alpha < 1. \\
&\text{i.e., } \lim_{m \to +\infty} [d(g(x_m^1), g(x_{m+p}^1)) + d(g(x_m^2), g(x_{m+p}^2)) + \\
&\cdots + d(g(x_m^r), g(x_{m+p}^r))] = 0.
\end{aligned}$$

Which further implies that

$$\begin{aligned}
\lim_{m \to +\infty} d(g(x_m^1), g(x_{m+p}^1)) &= \lim_{n \to +\infty} d(g(x_m^2), g(x_{m+p}^2)) = \cdots \\
&= \lim_{n \to +\infty} d(g(x_m^r), g(x_{m+p}^r)) = 0.
\end{aligned}$$

$\{g(x_m^1)\}$, $\{g(x_m^2)\}, \ldots, \{g(x_m^1)\}$ are all Cauchy sequences in $(X, d, \preceq)$.

Since $g(X)$ is complete subspace of X, and hence there exists $(x_0^1, x_0^2, \ldots, x_0^r) \in \prod_{\lambda=1}^r X^\lambda$ such that

$$\begin{cases} g(x_0^1) = x^1, \\ g(x_0^2) = x^2, \\ \vdots \\ g(x_0^r) = x^r. \end{cases} \tag{12}$$

By the continuity of g and (12), we have

$$\begin{cases} g(g(x_0^1)) = g(x^1), \\ g(g(x_0^2)) = g(x^2), \\ \vdots \\ g(g(x_0^r)) = g(x^r). \end{cases} \tag{13}$$

Now using the above equations we have

$$\begin{cases} \lim_{m \to +\infty} g(x_{m+1}^1) = \lim_{m \to +\infty} T_m(x_m^1, x_m^2, \ldots, x_m^r) = x^1, \\ \lim_{m \to +\infty} g(x_{m+1}^2) = \lim_{m \to +\infty} T_m(x_m^2, x_m^3, \ldots, x_m^r, x_m^1) = x^2, \\ \vdots \\ \text{and } \lim_{m \to +\infty} g(x_{m+1}^r) = \lim_{m \to +\infty} T_m(x_m^r, x_m^1, \ldots, x_m^{r-1}) = x^r. \end{cases}$$

Since $\{T_i\}_{i\in\mathbb{N}}$ and g are weakly reciprocally continuous, thus

$$\begin{cases} \lim\limits_{m\to+\infty} g(T_m(x_m^1, x_m^2, \ldots, x_m^r)) = g(x^1), \\ \lim\limits_{m\to+\infty} g(T_m(x_m^2, x_m^3, \ldots, x_m^r, x_m^1) = g(x^2), \\ \vdots \\ \text{and} \lim\limits_{m\to+\infty} g(T_m(x_m^r, x_m^1, \ldots, x_m^{r-1})) = g(x^r). \end{cases}$$

On the other hand, the compatibility of $\{T_i\}_{i\in\mathbb{N}}$ and g yields.

$$\begin{cases} \lim\limits_{m\to+\infty} d(g(T_m(x_m^1, x_m^2, \ldots, x_m^r)), T_m(g(x_m^1), g(x_m^2), \ldots, g(x_m^r))) = 0, \\ \lim\limits_{m\to+\infty} d(g(T_m(x_m^2, x_m^3, \ldots, x_m^r, x_m^1)), T_m(g(x_m^2), g(x_m^3), \ldots, g(x_m^r), g(x_m^1))) = 0, \\ \vdots \\ \text{and} \lim\limits_{m\to+\infty} d(g(T_m(x_m^r, x_m^1, \ldots, x_m^{r-1})), T_m(g(x_m^r), g(x_m^1), \ldots, g(x_m^{r-1}))) = 0. \end{cases}$$

Thus the above expression turns out to be

$$\begin{cases} \lim\limits_{m\to+\infty} T_m(g(x_m^1), g(x_m^2), \ldots, g(x_m^r)) = g(x^1) \\ \lim\limits_{m\to+\infty} T_m(g(x_m^2), g(x_m^3), \ldots, g(x_m^r, x_m^1)) = g(x^2) \\ \vdots \\ \text{and} \lim\limits_{m\to+\infty} T_m(g(x_m^r), g(x_m^1), \ldots, g(x_m^{r-1})) = g(x^r). \end{cases}$$

Since $\{g(x_m^i)\}$ are non-decreasing or non-increasing according as i is odd or even, respectively. Using the regularity of $g(X)$, we have $g(x_m^i) \preceq x^i$, when i is odd; and $g(x_m^i) \succeq x^i$, when i is even. Therefore

$$\begin{aligned} &g(g(x_m^i)) \preceq g(x^i), \text{when } i \text{ is odd; and} \\ &g(g(x_m^i)) \succeq g(x^i), \text{when } i \text{ is even.} \end{aligned}$$

Then by (5), ones obtain

$$\begin{aligned} &d(T_i(x^1, x^2, \ldots, x^r), T_m(g(x_m^1), g(x_m^2), \ldots, g(x_m^r))) \\ &\leq \beta_{i,m}[d(g(x^1), T_i(x^1, x^2, \ldots, x^r) \\ &+d(g(g(x_m^1)), T_m(g(x_m^1), g(x_m^2), \ldots, g(x_m^r)] \\ &+\gamma_{i,m}[d(g(g(x_m^1)), g(x^1))]. \end{aligned}$$

Proceeding limit as $m \to +\infty$ in the above inequality, using (13) with the fact that $\beta_{i,m} < 1$, we get $T_i(x^1, x^2, \ldots, x^r = g(x^1)$. Similarly, it can also be shown that

$$g(x^2) \quad = \quad T_i(x^2, x^3, \ldots, x^r, x^1)$$

$$\begin{aligned} g(x^3) &= T_i(x^3, x^4, \ldots, x^r, x^1, x^2) \\ &\vdots \\ g(x^r) &= T_i(x^r, x^1, \ldots, x^{r-1}). \end{aligned}$$

This shows that $(x^1, x^2, \ldots, x^r) \in \prod_{\lambda=1}^r X^\lambda$ is a r-tupled coincidence point of $\{T_i\}_{i\in\mathbb{N}}$ and g. ∎

Theorem 2. *Let X be regular and $(X, d, \preceq)$ be a complete ordered metric space. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of mappings from $\prod_{\lambda=1}^r X^\lambda \to X$ such that for all $x^1, x^2, \ldots, x^r, y^1, y^2, \ldots, y^r \in X$ with $x^1 \preceq y^1, x^2 \succeq y^2, x^3 \preceq y^3, \ldots, x^r \succeq y^r$, $\{T_i\}_{i\in\mathbb{N}}$ satisfy the following conditions:*

i) $T_m(x^1, x^2, \ldots, x^r) \preceq T_{m+1}(y^1, y^2, \ldots, y^r)$;

ii) $d(T_i(x^1, x^2, \ldots, x^r), T_j(y^1, y^2, \ldots, y^r))$

$$\leq \beta_{i,j}[d(x, T_i(x^1, x^2, \ldots, x^r)) + d(y^1, T_j(y^1, y^2, \ldots, y^r))] + \gamma_{i,j} d(y^1, x^1).$$

where $0 \leq \beta_{i,j}, \gamma_{i,j} < 1 \ \forall \ i, j \in \mathbb{N}$

iii) There exists $(x_0^1, x_0^2, \ldots, x_0^r) \in \prod_{\lambda=1}^r X^\lambda$ *such that*

$$\begin{cases} x_0^1 \preceq T_0(x_0^1, x_0^2, \ldots, x_0^r), \\ x_0^2 \succeq T_0(x_0^2, x_0^3, \ldots, x_0^r, x_0^1), \\ x_0^3 \preceq T_0(x_0^3, x_0^4, \ldots, x_0^1, x_0^2), \\ \vdots \\ x_0^r \succeq T_0(x_0^r, x_0^1, \ldots, x_0^{r-1}). \end{cases}$$

If $\sum_{i=1}^{+\infty}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)$ *is an α-series, then $\{T_i\}_{i\in\mathbb{N}}$ has r-tupled fixed point.*

Proof. The proof easily follows from the proof of Theorem 1 by taking g to be an identity mapping. ∎

Now, we give useful conditions for existence and uniqueness of a n-tupled common fixed point.

Theorem 3. *In addition to the hypotheses of Theorem 1, suppose that the set of coincidence points is comparable with respect to g, then $\{T_i\}_{i\in\mathbb{N}}$ and g have a unique r-tupled common fixed point.*

Proof. It is clear from Theorem 1 that the set of r-tupled coincidence points is nonempty. Let $(x^1, x^2, \ldots, x^r)$ and $(y^1, y^2, \ldots, y^r) \in \prod_{\lambda=1}^r X^\lambda$ be two r-tupled coincidence points of $\{T_i\}_{i\in\mathbb{N}}$ and g. Then

$$\begin{aligned} g(x^1) &= T_i(x^1, x^2, \ldots, x^r) \\ g(x^2) &= T_i(x^2, x^3, \ldots, x^r, x^1) \end{aligned}$$

$$\begin{array}{rcl} & \vdots & \\ g(x^r) & = & T_i(x^r, x^1, \ldots, x^{r-1}). \end{array}$$

and

$$\begin{array}{rcl} g(y^1) & = & T_i(y^1, y^2, \ldots, y^r) \\ g(y^2) & = & T_i(y^2, y^3, \ldots, y^r, x^1) \\ & \vdots & \\ g(y^r) & = & T_i(y^r, y^1, \ldots, y^{r-1}). \end{array}$$

Now our aim is to show that

$$\begin{array}{rcl} g(x^1) & = & g(y^1) \\ g(x^2) & = & g(y^2) \\ & \vdots & \\ g(x^r) & = & g(y^r). \end{array}$$

Since the set of coincidence points is comparable, using property (A) to these points we obtain,

$$\begin{array}{rcl} d(g(x^1), g(y^1)) & = & d(T_i(x^1, x^2, \ldots x^r)), T_j(y^1, y^2, \ldots y^r) \\ & & \leq \beta_{i,j}[d(g(x^1), T_i(x^1, x^2, \ldots x^r)) \\ & & +d(g(y^1), T_j(y^1, y^2, \ldots y^r))] \\ & & +\gamma_{i,j}[d(g(y^1), g(x^1))] \\ \text{i.e., } (1-\gamma_{i,j})d(g(x^1), g(y^1)) & & \leq \beta_{i,j}[d(g(x^1), T_i(x^1, x^2, \ldots x^r)) \\ & & +d(g(y^1), T_j(y^1, y^2, \ldots y^r))] \\ \Rightarrow\ d(g(x^1), g(y^1)) & & \leq \dfrac{\beta_{i,j}}{(1-\gamma_{i,j})}[d(g(x^1), T_i(x^1, x^2, \ldots x^r)) \\ & & +d(g(y^1), T_j(y^1, y^2, \ldots y^r))]. \end{array}$$

Also $\gamma_{i,j} < 1$ and the elements of coincidence point are comparable, thus $d(g(x^1), g(y^1)) = 0 \Rightarrow\ g(x^1) = g(y^1)$. Using the argument analogous to those used above, one can show that

$$\begin{array}{rcl} g(x^2) & = & g(y^2) \\ g(x^3) & = & g(y^3) \\ & \vdots & \\ g(x^r) & = & g(y^r). \end{array}$$

Hence $\{T_i\}_{i\in\mathbb{N}}$ and g have a unique r-tupled coincidence point. It is well-known that two compatible mappings are also weakly compatible, (i.e., they commute at their coincidence points). Thus, $\{T_i\}_{i\in\mathbb{N}}$ and g have a unique r-tupled common fixed point. Hence the result. ■

Example 4. Take $X = [0,1]$ endowed with usual metric $d = |x-y|$ for all $x, y \in X$ and $\preceq$ be defined as "greater equal" the $(X, d, \preceq)$ be ordered metric space. Let $T_i : \prod_{\lambda=1}^{r} X^{\lambda} \to X$ be mapping defined as

$$T_i(x^1, x^2, \ldots, x^r) = \frac{1}{r}[\frac{x^1 + x^2 + \ldots + x^r}{i}]; i \in \mathbb{N}$$

and g is a self mapping defined as $g(x) = x^2$. By choosing the sequences

$$\begin{aligned} \{x_m^1\} &= \frac{1}{m} \\ \{x_m^2\} &= \frac{1}{m+1} \\ &\vdots \\ \{x_m^2\} &= \frac{1}{m+r-1}. \end{aligned}$$

One can easily observe that (1) $\{T_i\}_{i\in\mathbb{N}}$ have g-mixed monotone property; (2) $\{T_i\}_{i\in\mathbb{N}}$ and g are compatible, weakly reciprocally continuous (3) g is continuous. By taking $0 < \beta_{i,j} < 1$ and $0 \leq \gamma_{i,j} < 1$ it is easy to verify property (A) and (B). Thus all the hypotheses of Theorem 1 are satisfied and $(0, 0, 0, \ldots, 0)$ is the only r-tupled coincident point of g and T_i for all $i, i \in \mathbb{N}$.

The following remarks depicts that if we take T_i to be a single mapping rather than the sequence of mappings $\{T_i\}_{i\in\mathbb{N}}$, then the pair T_i and g may have more than one r-tupled coincident points which depends upon the value of i.

Remark 3. One can notice that if we consider X, d, $\preceq$, T_i and g as taken in the above example, then $(1, 1, 1, \ldots, 1)$ is an r-tupled coincident point of g and T_1. Also $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is an r-tupled coincident point of g and T_2. Similarly, $(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$ is r-tupled coincident point of g and T_k for a fixed $k, k \in \mathbb{N}$.

Remark 4. One can also notice that if we consider X, d, $\preceq$, T_i as taken in the above example and $g(x) = x$, then $(0, 0, 0, \ldots, 0)$ is only r-tupled coincident point of g and T_i, $\forall\ i \in \mathbb{N}$. However $(0, 0, 0, \ldots, 0)$ and $(1, 1, 1, \ldots, 1)$ are only r-tupled coincident point of g and T_i only for $i = 1$.

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