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Generalized Monotone Iterative Method for Caputo Fractional Integro-differential Equations

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Abstract. Using coupled lower and upper solutions we develop the generalized monotone iterative technique to solve Caputo fractional integro-differential equation of order q with periodic boundary condition, via initial value problem (IVP) where $0 < q < 1$. We construct monotone iterates which are solutions of initial value problems associated with linear integro-differential equations, that are easy to obtain. We show that these iterates converge uniformly and monotonically to coupled minimal and maximal solutions of the problem considered. We have obtained explicit solution of the linear IVP of Caputo fractional integro-differential equation.

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1. Introduction

The theory of fractional calculus [3, 8] is more than three centuries old and but its study has been restricted mainly to mathematicians till a few decades ago. The book of Oldham and Spanier [6] attracted the attention of many researchers and study of various areas using fractional derivatives quickly gained impetus. With the monograph published by Prof. V. Lakshmikantham *et al.* [4], there has been extensive work in this area of research.

As the monotone iterative technique MIT combined with method of lower and upper solutions offers a flexible mechanism to obtain a solution of the considered mathematical model, this technique was developed in various setups [5] over the years. The interest in fractional differential equations led to developing MIT for IVPs and BVPs. There have been several papers [1] dealing with iterative techniques for systems involving fractional derivatives.

It is quite obvious that the study of IVPs is relatively simpler than the study of BVPs. Developing iterative techniques for BVPs is quite cumbersome. At this stage Pandit *et al.* [7], obtained the solution of a BVP using the monotone iterates of the corresponding IVP introduced

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the development of MIT of a PBVP through the MIT of a IVP. This work has been extended by J. D. Ramirez and A. S. Vatsala for Caputo fractional differential equations in [9].

Wen-Li Wang and Jing Feng Tian proposed a different approach to obtain a unique solution for the BVP in [10].

In this paper, we consider the PBVP of Caputo fractional integro differential equation and obtain its solution through a sequence of iterates developed for the corresponding IVP.

2. Preliminaries

In this section, we state a few definitions, some properties of fractional derivatives and recall required results pertaining to Caputo fractional integro differential equations which are useful in proving the main result.

Consider the Caputo fractional integro-differential equation of the type

$$^cD^q u = f(t, u, I^q(u)), \tag{1}$$

with

$$u(0) = u_0, \tag{2}$$

where $f, G \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$, $u \in C^1[J, \mathbb{R}]$, $J = [0, T]$,

$$^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s) ds, \tag{3}$$

and

$$I^q(u(t)) = \frac{1}{\Gamma q} \int_0^t (t-s)^{q-1} u(s) ds. \tag{4}$$

We start by stating a couple of lemmas from [3] related to IVPs of Riemann Liouville and Caputo fractional derivative of order q.

Lemma 1. *Let $m(t) \in C^1([0, T]\mathbb{R})$. If there exists $t_1 \in [0, T]$ such that $m(t_1) = 0$ and $m(t) \leq 0$ on $[0, T]$ then $D^q m(t_1) \geq 0$.*

Lemma 2. *Let $m(t) \in C^1([0, T], \mathbb{R})$. If there exists $t_1 \in [0, T]$ such that $m(t_1) = 0$ and $m(t) \leq 0$ on $[0, T]$ then $^cD^q m(t_1) \geq 0$.*

The following theorem, which is a new result, gives the explicit solution of the linear IVP of Caputo fractional integro-differential equation, this is the generalization of the result in [8].

Theorem 1. *If $\lambda \in C^1([0, T], \mathbb{R})$. The solution of $^cD^q \lambda(t) = L_1 \lambda(t) + M I^q(\lambda(t))$ is given by*

$$\lambda(t) = \Sigma_{n=0}^{\infty} \Sigma_{k=0}^{\infty} \frac{2^{n+k} M^n L_1^{kn+k} C_k t^{(2n+1)q}}{\Gamma[(2n+1)q + 1]}$$

where $L_1, M > 0$.

Proof. By hypothesis we have ${}^cD^q\lambda(t) = L_1\lambda(t) + MI^q(\lambda(t))$. Now by applying the Laplace transform on both sides we get

$$\begin{aligned}
s^q\bar{\lambda(s)} - s^{q-1}\lambda(0) =& L_1\bar{\lambda(s)} + Ms^{-q}\\
s^q\bar{\lambda(s)}[s^q - Ms^{-q} - L_1] =& \lambda(0)s^{q-1}\\
\bar{\lambda(s)} =& \frac{s^{q-1}}{[s^q - Ms^{-q} - L_1]}\lambda(0)\\
\bar{\lambda(s)} =& \frac{s^{q-1}}{[s^{2q} - M - L_1s^q]}\lambda(0)\\
L^{-1}(\bar{\lambda(s)}) =& L^{-1}(\frac{s^{q-1}}{[s^{2q} - M - L_1s^q]})\lambda(0)\\
\lambda(t) =& \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{\infty}\frac{2^{n+k}M^nL_1^{kn+k}C_kt^{(2n+1)q}}{\Gamma[(2n+1)q+1]}\lambda(0).
\end{aligned}$$

□

Next we shall establish the following comparison theorem.

Theorem 2. *Let $J = [0, T]$, $f \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$, $v, w \in C^1[J, \mathbb{R}]$ and suppose that the following inequalities hold, for all $t \in J$.*

$$ {}^cD^qv(t) \le f(t, v(t), I^q(v(t))), v(0) \le u_0, \tag{5}$$

$$ {}^cD^qw(t) \ge f(t, w(t), I^q(w(t))), w(0) \ge u_0. \tag{6}$$

Suppose further that $f(t, u(t), I^q(u(t)))$ satisfies the following Lipschitz-like condition,

$$ f(t, x, I^q(x)) - f(t, y, I^q(y)) \le L(x - y) + M(I^q(x) - I^q(y)), \tag{7}$$

for $x \ge y$, $L, M > 0$. Then, $v(0) \le w(0)$ implies that

$$ v(t) \le w(t), 0 \le t \le T. \tag{8}$$

Proof. Assume without loss of generality that one of the inequalities in (5), (6) is strict, say ${}^cD^qv(t) < f(t, v(t), I^q(v(t)))$ and $v(0) < w(0)$, where $v(0) = v_0$ and $w(0) = w_0$. We claim that $v(t) < w(t)$ for $t \in J$.

Suppose there exists t_1 such that $0 < t_1 \le T$ for which

$$ v(t_1) = w(t_1), v(t) \le w(t), \text{ for } t < t_1. \tag{9}$$

If we set $m(t) = v(t) - w(t)$. Then $m(t_1) = 0$ and $m(t) = v(t) - w(t) \le 0$ for $t < t_1$.

Then by Lemma 2 we have ${}^cD^qm(t_1) \ge 0$. Thus

$$\begin{aligned}
f(t_1, v(t_1), I^q(v(t_1))) >& {}^cD^qv(t_1)\\
\ge& {}^cD^qw(t_1)\\
\ge& f(t_1, w(t_1), I^q(w(t_1))),
\end{aligned}$$

which is a contradiction. So $v(t) < w(t)$ for $t \in J$. By assuming that the inequalities in (5) and (6) are non strict, we now prove that $v(t) \leq w(t)$. Set $w_\epsilon(t) = w(t) + \epsilon\lambda(t)$ where $\epsilon > 0$ and

$$\lambda(t) = \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{\infty}\frac{2^{n+k}M^nL^{kn+k}C_k t^{(2n+1)q}}{\Gamma[(2n+1)q+1]}$$

is a solution of the Caputo fractional integro differential equation

$$^cD^q\lambda(t) = 2L\lambda(t) + 2MI^q\lambda(t) with \lambda(0) = 1.$$

We have $w_\epsilon(0) = w(0) + \epsilon > w_0$ and $w_\epsilon(t) > w(t)$ for $t \in J$. Using (5), (6), and (7), we find that

$$\begin{aligned}
^cD^q w_\epsilon(t) =& ^cD^q w(t) + \epsilon ^cD^q\lambda(t)\\
\geq& f(t, w(t), I^q(w(t))) + 2L\lambda(t) + 2MI^q(\lambda(t))\\
\geq& f(t, w_\epsilon(t), I^q(w_\epsilon(t))) - L\lambda(t) - MI^q(\lambda(t)) + 2L\lambda(t) + 2MI^q(\lambda(t))\\
\geq& f(t, w_\epsilon(t), I^q(w_\epsilon(t))) + L\lambda(t) + MI^q(\lambda(t))\\
>& f(t, w_\epsilon(t), I^q(w_\epsilon)(t)),
\end{aligned}$$

for $0 \leq t \leq T$. Applying the result for strict inequalities to $v(t)$, $w_\epsilon(t)$ we obtain $v(t) < w_\epsilon(t)$ for $t \in J$, for every $\epsilon > 0$ and consequently as $\epsilon \to 0$, we get that $v(t) \leq w(t)$ for $t \in J$. □

Corollary 1. *Let $m \in C^1[J, \mathbb{R}]$ be such that*

$$^cD^q m(t) \leq Lm(t) + MI^q(m(t)), m(0) = m_0 \leq 1,$$

then

$$m(t) \leq \lambda(t),$$

for $0 \leq t \leq T$ and $L, M > 0$, $\lambda(0) = 1, \lambda(t) = \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{\infty}\frac{2^{n+k}M^nL^{kn+k}C_k t^{(2n+1)q}}{\Gamma[(2n+1)q+1]}$.

Proof. We have $^cD^q m(t) \leq Lm(t) + MI^q(m(t))$ and

$$\begin{aligned}
^cD^q\lambda(t) =& 2L\lambda(t) + 2MI^q(\lambda(t))\\
\geq& L\lambda(t) + MI^q(\lambda(t)),
\end{aligned}$$

for $m(0) = m_0 \leq 1 = \lambda(0)$. Hence from Theorem 2 we conclude that $m(t) \leq \lambda(t)$ for $t \in J$. □

The result of Corollary 1 is still true even if $L = M = 0$, which is given below.

Corollary 2. *Let $^cD^q m(t) \leq 0$ on $[0, T]$. If $m(0) \leq 0$ then $m(t) \leq 0$, $t \leq J$.*

Proof. By definition of $^cD^q m(t)$ and by hypothesis,

$$^cD^q m(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q} m'(s)ds \leq 0,$$

which implies that $m'(t) \leq 0$ on $[0, T]$. Therefore $m(t) \leq m(0) \leq 0$ on $[0, T]$. The proof is complete. □

3. The Technique

In this section, we develop generalized monotone iterative technique to obtain a coupled minimal and maximal solutions for the Caputo fractional integro-differential equation of the form

$$^cD^q u = F(t,u,I^q(u)) + G(t,u,I^q(u)), \tag{10}$$

with the boundary condition

$$g(u(0),u(T)) = 0, \tag{11}$$

where $F,G \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$, $u \in C^1[J,\mathbb{R}]$. We begin with various definitions of coupled lower and upper solutions of (10) and (11).

Definition 1. *Let $v_0, w_0 \in C^1[J,\mathbb{R}]$. Then v_0 and w_0 are said to be*

(i) natural lower and upper solutions of (10)*,* (11) *if,*

$$^cD^q v_0(t) \le F(t,v_0(t),I^q(v_0(t))) + G(t,v_0(t),I^q(v_0(t))), g(v_0(0),v_0(T)) \le 0, \tag{12}$$
$$^cD^q w_0(t) \ge F(t,w_0(t),I^q(w_0(t))) + G(t,w_0(t),I^q(w_0(t))), g(w_0(0),w_0(T)) \ge 0, \tag{13}$$

(ii) coupled lower and upper solutions of Type I of (10)*,* (11) *if*

$$^cD^q v_0(t) \le F(t,v_0(t),I^q(v_0(t))) + G(t,w_0(t),I^q(w_0(t))), g(v_0(0),v_0(T)) \le 0, \tag{14}$$
$$^cD^q w_0(t) \ge F(t,w_0(t),I^q(w_0(t))) + G(t,v_0(t),I^q(v_0(t))), g(w_0(0),w_0(T)) \ge 0, \tag{15}$$

(iii) coupled lower and upper solutions of Type II of (10)*,* (11) *if*

$$^cD^q v_0(t) \le F(t,w_0(t),I^q(w_0(t))) + G(t,v_0(t),I^q(v_0(t))), g(v_0(0),v_0(T)) \le 0, \tag{16}$$
$$^cD^q w_0(t) \ge F(t,v_0(t),I^q v_0(t)) + G(t,w_0(t),I^q w_0(t)), g(w_0(0),w_0(T)) \ge 0, \tag{17}$$

(iv) coupled lower and upper solutions of Type III of (10)*,* (11) *if*

$$^cD^q v_0(t) \le F(t,w_0(t),I^q(w_0(t))) + G(t,w_0(t),I^q(w_0(t))), g(v_0(0),v_0(T)) \le 0, \tag{18}$$
$$^cD^q w_0(t) \ge F(t,v_0(t),I^q(v_0(t))) + G(t,v_0(t),I^q(v_0(t))), g(w_0(0),w_0(T)) \ge 0. \tag{19}$$

We note that whenever $v(t) \le w(t)$, $t \in J$, if $F(t,x_1,x_2)$ is nondecreasing in x_1 for each $(t,x_2) \in J \times \mathbb{R}_+$ and is nondecreasing in x_2 for each $(t,x_1) \in J \times \mathbb{R}$, further, if $G(t,y_1,y_2)$ is non increasing in y_1 for each $(t,y_2) \in J \times \mathbb{R}_+$ and is non increasing in y_2 for each $(t,y_1) \in J \times \mathbb{R}$, then the lower and upper solutions defined by (12), (13) and those defined by (18) and (19) reduce to (14), (15), and (16), (17) respectively. Hence it is sufficient to investigate the cases (14), (15) and (16), (17).

Based on the concepts defined above, we now develop the monotone iterative technique for the considered problem. To do so we use sequences of iterates which are solutions sequences of IVP of linear Caputo fractional integro-differential equations. Since the solution of the linear Caputo fractional differential equation is unique, the sequence of iterates is a unique sequence converging to a solution of the considered problem defined by (10) and (11). In this approach, we do not need to prove the existence of solution for BVP of nonlinear Caputo fractional integro differential equation, as it follows from the construction of the monotone sequences.

In the following theorem, we use coupled lower and upper solutions of Type I and obtain monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of the problem defined by (10) and (11).

Theorem 3. *Suppose that*

(A_1) v_0, w_0 *are coupled lower and upper solutions of Type I for problem defined by* (10)*,* (11) *with* $v_0(t) \leq w_0(t)$ *on* J*,*

(A_2) the function $g(u,v) \in C[\mathbb{R}^2, \mathbb{R}]$ *is nonincreasing in* v *for each* u *and there exists a constant* $M > 0$ *such that*

$$g(u_1, v) - g(u_2, v) \leq M(u_1 - u_2), \tag{20}$$

for $v_0(0) \leq u_2 \leq u_1 \leq w_0(0)$*,* $v_0(T) \leq v \leq w_0(T)$*,*

(A_3) $F, G \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$ *and* $F(t, x_1, x_2)$ *is non-decreasing in* x_1 *for each* $(t, x_2) \in J \times \mathbb{R}_+$ *and is nondecreasing in* x_2 *for each* $(t, x_1) \in J \times \mathbb{R}$*. Further,* $G(t, y_1, y_2)$ *is nonincreasing in* y_1 *for each* $(t, y_2) \in J \times \mathbb{R}_+$ *and is nonincreasing in* y_2 *for each* $(t, y_1) \in J \times \mathbb{R}$*.*

Then the iterative scheme given by

$$^cD^q v_{n+1} = F(t, v_n, I^q(v_n)) + G(t, w_n, I^q(w_n)), \tag{21}$$

$$v_{n+1}(0) = v_n(0) - \frac{1}{M} g(v_n(0), v_n(T)), \tag{22}$$

$$^cD^q w_{n+1} = F(t, w_n, I^q(w_n)) + G(t, v_n, I^q(v_n)), \tag{23}$$

$$w_{n+1}(0) = w_n(0) - \frac{1}{M} g(w_n(0), w_n(T)), \tag{24}$$

yields two monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *such that*

$$v_0 \leq v_1 \leq \ldots \leq v_n \leq w_n \leq \ldots \leq w_1 \leq w_0.$$

Further, $v_n \to \rho$ *and* $w_n \to r$ *in* $C^1[J, \mathbb{R}]$ *uniformly and monotonically, such that* ρ *and* r *are respectively the coupled minimal and maximal solutions of the problem defined by* (10) *and* (11)*, that is,* ρ *and* r *satisfy the coupled system*

$$\begin{aligned} ^cD^q \rho &= F(t, \rho, I^q(\rho)) + G(t, r, I^q(r)), \\ g(\rho(0), \rho(T)) &= 0, \\ ^cD^q r &= F(t, r, I^q(r)) + G(t, \rho, I^q(\rho)), \\ g(r(0), r(T)) &= 0. \end{aligned}$$

Proof. Putting $n = 0$ in (21), (22), we get

$$\begin{aligned} {}^cD^q v_1(t) =& F(t, v_0(t), I^q(v_0(t))) + G(t, w_0(t), I^q(w_0(t))), \\ v_1(0) =& v_0(0) - \frac{1}{M} g(v_0(0), v_0(T)). \end{aligned}$$

Clearly the above IVP has a unique solution denoted by $v_1(t), t \in J$. We use induction on n to establish the relation $v_0 \leq v_1 \leq \ldots \leq v_n \leq w_n \leq \ldots \leq w_1 \leq w_0$. We start by showing $v_0 \leq v_1 \leq w_1 \leq w_0$. For this set $p(t) = v_0(t) - v_1(t)$, then

$$\begin{aligned} {}^cD^q p(t) =& {}^cD^q v_0(t) - {}^c D^q v_1(t), \\ \leq& F(t, v_0(t), I^q(v_0(t))) + G(t, w_0(t), I^q(w_0(t))) \\ &- [F(t, v_0(t), I^q(v_0(t))) + G(t, w_0(t), I^q)(w_0(t)))] = 0 \end{aligned}$$

and $p(0) = v_0(0) - v_0(0) + \frac{1}{M} g(v_0(0), v_0(T)) \leq 0$. Thus the hypothesis of Corollary 2 is satisfied and we conclude that $p(t) \leq 0$ on J, and obtain. Similarly we can show that $w_1 \leq w_0$ on J. Next we consider $p(t) = v_1(t) - w_1(t)$, then by adding and subtracting suitable terms, and using the fact that F is nondecreasing in second and third variables, G is nonincreasing in second and third variables, and by taking Caputo fractional derivative we arrive at,

$$\begin{aligned} {}^cD^q p(t) =& {}^cD^q v_1(t) - {}^c D^q w_1(t), \\ =& F(t, v_0(t), I^q(v_0(t))) + G(t, w_0(t), I^q(w_0(t))) \\ &- [F(t, w_0(t), I^q(w_0(t))) + G(t, v_0(t), I^q(v_0(t)))] \\ \leq& 0, \end{aligned}$$

and

$$\begin{aligned} p(0) =& v_0(0) - w_0(0) - \frac{1}{M}[g(v_0(0), v_0(T)) - g(w_0(0), w_0(T)) \\ \leq& v_0(0) - w_0(0) - [v_0(0) - w_0(0)] = 0. \end{aligned}$$

Hence by Corollary 2 we have $p(t) \leq 0$ on J that is, $v_1(t) \leq w_1(t)$, on J. Thus the claim $v_0 \leq v_1 \leq w_1 \leq w_0$ on J is proved.

We now show that v_1, w_1 are the coupled lower and upper solutions of Type I for (10), (11), Using the fact that $v_0 \leq v_1$, $w_1 \leq w_0$ and from the assumption *(A_3)*, proceeding as earlier we obtain ${}^cD^q v_1(t) \leq 0$. Also,

$$\begin{aligned} g(v_1(0), v_1(T)) =& g(v_1(0), v_1(T)) - g(v_0(0), v_0(T)) - Mv_1(0) + Mv_0(0) \\ \leq& M(v_1(0) - v_0(0)) - M(v_1(0) - v_0(0)) = 0. \end{aligned}$$

Similarly we can show that w_1 satisfies reverse inequalities. Hence v_1, w_1 are coupled lower and upper solutions of (10), (11).

Assume that $v_{k-1} \leq v_k \leq w_k \leq w_{k-1}$ on J for $k > 1$, where v_{k-1}, v_k are the solutions of the IVP (21), (22) and w_{k-1}, w_k are the solutions of the IVP (23), (24) for $n = k-1$, $n = k$ respectively. We claim that the following relation holds.

$$v_k \leq v_{k+1} \leq w_{k+1} \leq w_k$$

on J. To prove this we take $p(t) = v_k(t) - v_{k+1}(t)$, then by adding and subtracting suitable terms and working as earlier we arrive at,

$$\begin{aligned}{}^cD^q p(t) =& {}^cD^q v_k(t) - {}^c D^q v_{k+1}(t)\\ \le& [F(t, v_{k-1}(t), I^q(v_{k-1}(t))) + G(t, w_{k-1}(t), I^q(w_{k-1)}(t))]\\ &- [F(t, v_k(t), I^q(v_k(t))) + G(t, w_k(t), I^q(w_k(t)))] \le 0.\end{aligned}$$

Further, $p(0) = v_k(0) - v_{k+1}(0) = v_k(0) - v_k(0) + \frac{1}{M} g(v_k(0), v_k(T)) \le 0$. An application of Corollary 2 yields that $p(t) \le 0$ and consequently, $v_k(t) \le v_{k+1}(t)$, on J. In a similar manner we can prove that $w_{k+1}(t) \le w_k(t)$.

Next to prove $v_{k+1}(t) \le w_{k+1}(t)$, on J, consider $p(t) = v_{k+1}(t) - w_{k+1}(t)$ and again following the earlier approach we deduce that

$${}^cD^q p(t) = {}^c D^q v_{k+1}(t) - {}^c D^q w_{k+1}(t) \le 0$$

and

$$\begin{aligned}p(0) =& v_{k+1}(0) - w_{k+1}(0)\\ =& v_k(0) - \frac{1}{M} g(v_k(0), v_k(T)) - w_k(0) + \frac{1}{M} g(w_k(0), w_k(T))\\ \le& v_k(0) - w_k(0) + \frac{1}{M}[g(w_k(0), w_k(T)) - g(v_k(0), v_k(T)) \le 0.\end{aligned}$$

which yields $p(t) \le 0$ on using Corollary 2. Thus we obtain two monotone sequences $\{v_n\}$ and $\{w_n\}$ satisfying

$$v_0 \le v_1 \le \ldots \le v_n \le w_n \le \ldots \le w_1 \le w_0.$$

Now we claim that these sequences are equicontinuous and uniformly bounded. By hypothesis both $v_0(t)$, $w_0(t)$ are bounded on $[0, T]$ and the sequences $\{v_n\}$ and $\{w_n\}$ are such that

$$v_0 \le v_1 \le \ldots \le v_n \le w_n \le \ldots \le w_1 \le w_0.$$

Therefore $\{v_n\}$ and $\{w_n\}$ are uniformly bounded. Next we prove that $\{v_n\}$ is equicontinous. To do so for given $\epsilon > 0$ choose $\delta = (\frac{(\epsilon\Gamma(q+1))}{2M_2})^{\frac{1}{q}}$. Next for $t_1, t_2 \in J$ such that $t_2 > t_1$ consider

$$\begin{aligned}&|v_n(t_1) - v_n(t_2)|\\ &\quad= |v_n(0) + \frac{1}{\Gamma q}\int_0^{t_1}(t_1 - s)^{q-1}[F(s, v_{n-1}(s), I^q(v_{n-1}(s)) + G(s, w_{n-1}(s), I^q(w_{n-1}(s)))]ds\\ &\quad\quad - v_n(0) + \frac{1}{\Gamma q}\int_0^{t_2}(t_2 - s)^{q-1}[F(s, v_{n-1}(s), I^q(v_{n-1}(s))) + G(s, w_{n-1}(s), I^q(w_{n-1}(s)))]ds|\\ &\quad\le \frac{1}{\Gamma q}\int_0^{t_1}[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}]|F(s, v_{n-1}(s), I^q(v_{n-1}(s))) + G(s, w_{n-1}(s), I^q(w_{n-1}(s)))|ds\\ &\quad\quad + \int_{t_1}^{t_2}(t_2 - s)^{q-1}|F(s, v_{n-1}(s), I^q(v_{n-1}(s))) + G(s, w_{n-1}(s), I^q(w_{n-1}(s)))|ds\end{aligned}$$

$$\leq \frac{M_2}{\Gamma q}\{\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]ds+\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\}$$
$$\leq \frac{M_2}{\Gamma q+1}[(t_1)^q+(t_2-t_1)^q-(t_2)^q+(t_2-t_1)^q]$$
$$\leq \frac{M_2}{\Gamma q+1}[t_1^q-t_2^q]+\frac{2M_2}{\Gamma q+1}(t_2-t_1)^q$$
$$\leq \frac{2M_2}{\Gamma q+1}(t_2-t_1)^q,$$

here we have used the fact that $\{v_n\}$, $\{I^q v_n\}$, $\{w_n\}$, $\{I^q v_n\}$ are uniformly bounded and $F(t,x_1,x_2)$, $G(t,y_1,y_2)$ are continuous on $[0,T]$.

Thus for any given $\epsilon > 0$ there exists $\delta > 0$ independent of n such that for each n, $|v_n(t_1)-v_n(t_2)| < \epsilon$ whenever $\delta = (\frac{(\epsilon\Gamma(q+1))}{2M_2})^{\frac{1}{q}}$. Therefore $\{v_n\}$ is equicontinuous. Similarly we can prove that $\{w_n\}$ is equicontinuous. Hence by Arzela-Ascoli's theorem there exist subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$ which converge uniformly to $\rho(t)$ and $r(t)$ respectively. Since the sequences are monotone, the entire sequences converge uniformly to ρ and r respectively on J.

To prove that ρ and r are coupled minimal and maximal solutions of (10) and (11) respectively, we need to show that if u is any solution of (10) and (11), such that $v_0 \leq u \leq w_0$, then $\rho \leq u_1$, $u_2 \leq r$. Assume that there exists a positive integer n such that $v_n \leq u \leq w_n$ on J. Then using the monotone nature of F, G we have

$$\begin{aligned}
{}^cD^q p(t) =& {}^cD^q v_{n+1}(t) - {}^c D^q u(t),\\
\leq& [F(t,v_n(t),I^q(v_n(t)))+G(t,w_n(t),I^q(w_n(t)))]\\
&-[F(t,u(t),I^q(u(t)))+G(t,u(t),I^q(u(t)))]\\
{}^cD^q p(t) \leq& 0
\end{aligned}$$

and

$$\begin{aligned}
p(0) =& v_{n+1}(0)-u(0)\\
=& v_n(0)-\frac{1}{M}g(v_n(0),v_n(T))-u(0)\\
\leq& v_n(0)-u(0)-\frac{1}{M}[g(v_n(0),v_n(T))-g(u(0),u(T))] \leq 0
\end{aligned}$$

so $v_{n+1}(t) \leq u_1(t)$, on J follows from Corollary 2. Similarly we can show that $u(t) \leq w_{n+1}(t)$, on J. By applying induction on n we conclude that $v_{n+1} \leq u \leq w_{n+1}$ on J. Taking limit as $n \to \infty$, we get $\rho \leq u \leq r$, $t \in J$. Hence

$$v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq \ldots \leq \rho \leq u \leq r \leq \ldots \leq w_n \leq \ldots w_1 \leq w_0,$$

on J, where ρ and r are coupled minimal and maximal solutions of (10) and (11). Thus the proof is complete. □

Remark 1.

(i) *In Theorem 3, if $G(t,u,I^q(u)) = 0$, then we get a result when F is nondecreasing in first and second variables,*

(ii) *If $F(t,u,I^q u) = 0$, in Theorem 3 then we obtain the results for G nonincreasing in first and second variables.*

Theorem 4. *Assume that conditions (A_1), (A_2), and (A_3) of Theorem 3 are true. Then for any solution $u(t)$ of* (10), (11) *with $v_0 \le u \le w_0$. on J, we have the iterates $\{v_{2n}, w_{2n+1}\}$ and $\{v_{2n+1}, w_{2n}\}$ satisfying*

$$v_0 \le w_1 \le \ldots \le v_{2n} \le w_{2n+1} \le u \le v_{2n+1} \le w_{2n} \le \ldots \le v_1 \le w_0. \tag{25}$$

for each $n \ge 1$ on J, Further more $\{v_{2n}, w_{2n+1}\} \to \rho$ and $\{v_{2n+1}, w_{2n}\} \to r$ in $C^1[J,\mathbb{R}]$ uniformly and monotonically, such that ρ and r are coupled minimal and maximal solutions of (10) *and* (11)*, respectively, that is, $\rho \le u \le r$, ρ and r satisfy the coupled system*

$$\begin{aligned} {}^cD^q\rho &= F(t,\rho,I^q(\rho)) + G(t,r,I^q(r)),\\ g(\rho(0),\rho(T)) &= 0,\\ {}^cD^q r &= F(t,r,I^q(r)) + G(t,\rho,I^q(\rho))\\ g(r(0),r(T)) &= 0. \end{aligned}$$

Proof. Consider the following IVP

$$\begin{aligned} {}^cD^q v_{n+1} &= F(t,w_n,I^q(w_n)) + G(t,v_n,I^q(v_n)), && (26)\\ v_{n+1}(0) &= w_n(0) - \frac{1}{M} g(w_n(0),w_n(T)), && (27)\\ {}^cD^q w_{n+1} &= F(t,v_n,I^q(v_n)) + G(t,w_n,I^q(w_n)), && (28)\\ w_{n+1}(0) &= v_n(0) - \frac{1}{M} g(v_n(0),v_n(T)), && (29) \end{aligned}$$

where $v_0 \le w_0$. Our aim is to show that the solutions v_{n+1}, w_{n+1} of (26), (27), and (28), (29) satisfy

$$v_0 \le w_1 \le \ldots \le v_{2n} \le w_{2n+1} \le u \le v_{2n+1} \le w_{2n} \le \ldots \le v_1 \le w_0.$$

Clearly the IVPs (26), (27), and (28), (29) have unique solutions for each $n = 0,1,2,\ldots$ denoted by v_{n+1}, w_{n+1}. First we show that $v_0 \le v_1 \le w_1 \le w_0$. Since v_0 is a coupled lower solution of Type I for (10), (11) we have

$${}^cD^q v_0(t) \le F(t,v_0(t),I^q(v_0(t))) + G(t,w_0(t),I^q(w_0(t))), g(v_0(0),v_0(T)) \le 0.$$

Setting $n = 0$ in (26), (27), we get that v_1 is a solution of the boundary value problem,

$$\begin{aligned} {}^cD^q v_1(t) &= F(t,w_0(t),I^q(w_0(t))) + G(t,v_0(t),I^q(v_0(t))),\\ v_1(0) &= w_0(0) - \frac{1}{M} g(w_0(0),w_0(T)). \end{aligned}$$

Set $p(t) = v_0(t) - v_1(t)$, then by taking the Caputo fractional derivative on both sides and due to the fact that F and G are in monotonic in the second and third variable we get that

$$\begin{aligned} {}^cD^q p(t) =& {}^cD^q v_0(t) - {}^cD^q v_1(t) \\ \leq& [F(t, v_0(t), I^q(v_0(t))) + G(t, w_0(t), I^q(w_0(t)))] \\ & - [F(t, w_0(t), I^q(w_0(t))) + G(t, v_0(t), I^q(v_0(t)))] \end{aligned}$$

which means that

$${}^cD^q p(t) \leq 0.$$

Now $p(0) = v_0(0) - w_0(0) + \frac{1}{M} g(w_0(0), w_0(T)) \leq \frac{1}{M} g(v_0(0), v_0(T)) \leq 0$. On applying Corollary 2 we arrive at $v_0(t) \leq v_1(t)$, on J. In a similar fashion we get $w_1(t) \leq w_0(t)$, on J. Next we proceed to show that

$$v_0 \leq w_1 \leq v_2 \leq w_3 \leq u \leq v_3 \leq w_2 \leq v_1 \leq w_0. \tag{30}$$

Writing $p = u - v_1$, and working as earlier, we get ${}^cD^q p(t) \leq 0$ and $p(t) \leq 0$ Again an application of Corollary 2 gives $u(t) \leq v_1(t)$, on J. A similar argument yields $w_1 \leq u$, $v_2 \leq u$, $u \leq w_2$, $u \leq v_3$ and $w_3 \leq u$. Our next claim is that $v_0 \leq w_1 \leq v_2 \leq w_3$ and $v_3 \leq w_2 \leq v_1 \leq w_0$. For this, let $p(t) = v_0(t) - w_1(t)$, then ${}^cD^q p(t) \leq 0$ due to the fact that $v_0 \leq w_0$, also $p_0 \leq 0$. By applying Corollary 2 we get $p(t) \leq 0$. Thus $v_0 \leq w_1$. Proceeding in the same way we can obtain $w_1 \leq v_2$, $v_1 \leq w_0$, $v_2 \leq w_3$, $v_3 \leq w_2$, $w_2 \leq v_1$ on J. Thus we arrive at relation (30). Suppose there exists an integer $k \geq 2$ such that

$$w_{2k-1} \leq v_{2k} \leq w_{2k+1} \leq u \leq v_{2k+1} \leq w_{2k} \leq v_{2k-1}$$

holds, then we claim that

$$w_{2k+1} \leq v_{2k+2} \leq w_{2k+3} \leq u \leq v_{2k+3} \leq w_{2k+2} \leq v_{2k+1}.$$

Setting $p(t) = w_{2k+1}(t) - v_{2k+2}(t)$.

$$\begin{aligned} {}^cD^q p(t) =& {}^cD^q w_{2k+1}(t) - {}^cD^q v_{2k+2}(t) \\ \leq& [F(t, v_{2k}(t), I^q(v_{2k}(t))) + G(t, w_{2k}(t), I^q(w_{2k}(t)))] \\ & - [F(t, w_{2k+1}(t), I^q(w_{2k+1}(t))) + G(t, v_{2k+1}(t), I^q(v_{2k+1}(t)))] \\ \leq& 0, \end{aligned}$$

which is obtained by adding and subtracting suitable terms and on using the monotone nature of F, G. Next

$$\begin{aligned} p(0) =& w_{2k+1}(0) - v_{2k+2}(0) \\ =& v_{2k}(0) - w_{2k+1}(0) + \frac{1}{M}[g(w_{2k+1}(0), w_{2k+1}(T)) - g(v_{2k}(0), v_{2k}(T)) \\ \leq& 0. \end{aligned}$$

Corollary 2 yields that $p(t) \leq 0$ and consequently, $w_{2k+1} \leq v_{2k+2}$, on J. Similarly, we obtain $w_{2k+2} \leq v_{2k+1}$, $v_{2k+2} \leq w_{2k+3}$, $v_{2k+3} \leq w_{2k+2}$. Finally consider $p(t) = v_{2k+2}(t) - u(t)$, and working in a similar fashion we arrive at

$$\begin{aligned}
{}^cD^q p(t) =& {}^cD^q v_{2k+2}(t) - {}^c D^q u(t) \\
\leq& F(t, w_{2k+1}(t), I^q(w_{2k+1}(t))) + G(t, v_{2k+1}(t), I^q(v_{2k+1}(t))) \\
&- [F(t, u(t), I^q(u(t))) + G(t, u(t), I^q(u(t)))] \\
\leq& 0,
\end{aligned}$$

and $p(0) \leq 0$. So by applying Corollary 2 we get $u(t) \leq v_{2k+1}(t)$. The relations $u \leq v_{2k+2}$, $w_{2k+3} \leq u$, $w_{2k+2} \leq u$, $u \leq v_{2k+3}$ can be proved by working as in the previous case.

Now by induction we have

$$v_0 \leq w_1 \leq \ldots \leq v_{2n} \leq w_{2n+1} \leq u \leq v_{2n+1} \leq w_{2n} \leq \ldots \leq v_1 \leq w_0.$$

By arguing as in Theorem 3, we get the sequences $\{v_{2n}, w_{2n+1}\} \to \rho$ and $\{v_{2n+1}, w_{2n}\} \to r$ in $C^1[J, \mathbb{R}]$ uniformly and monotonically, such that ρ and r are coupled minimal and maximal solutions of Type I for (10), (11). Hence the proof of the theorem. □

To avoid repetition, we will state next two theorems without proof since it follows the same of pattern as that for Theorem 3 and Theorem 4.

Theorem 5. *Assume that the hypothesis* (A_1), (A_2), (A_3) *of Theorem 3 hold and* v_0, w_0 *are coupled lower and upper solutions of Type II for* (10), (11) *with* $v_0(t) \leq w_0(t)$ *on* J. *Then the iterative scheme given by*

$$\begin{aligned}
{}^cD^q v_{n+1} =& F(t, v_n, I^q(v_n)) + G(t, w_n, I^q(w_n)), \\
v_{n+1}(0) =& v_n(0) - \frac{1}{M} g(v_n(0), v_n(T)), \\
{}^cD^q w_{n+1} =& F(t, w_n, I^q(w_n)) + G(t, v_n, I^q(v_n)), \\
w_{n+1}(0) =& w_n(0) - \frac{1}{M} g(w_n(0), w_n(T)),
\end{aligned}$$

result in two monotone sequences $\{v_n(t)\}$, $\{w_n(t)\}$ *satisfying*

$$v_0 \leq v_1 \leq \ldots \leq v_n \leq w_n \leq \ldots \leq w_1 \leq w_0.$$

Further more $v_n \to \rho$ *and* $w_n \to r$ *in* $C^1[J, \mathbb{R}]$ *uniformly and monotonically, such that* ρ *and* r *are coupled minimal and maximal solutions of Type II for* (10), (11), *respectively, provided that* $v_0 \leq w_0$. *Thus* ρ *and* r *satisfy the coupled system*

$$\begin{aligned}
{}^cD^q \rho =& F(t, \rho, I^q(\rho)) + G(t, r, I^q(r)), \\
& g(\rho(0), \rho(T)) = 0, \\
{}^cD^q r =& F(t, \rho, I^q(\rho)) + G(t, r, I^q(r)), \\
& g(r(0), r(T)) = 0.
\end{aligned}$$

Theorem 6. *Let (A_2), (A_3) of Theroem 3 hold and v_0, w_0 are coupled lower and upper solutions of Type II for* (10), (11) *with $v_0(t) \leq w_0(t)$ on J. Then the iterative scheme given by*

$$\begin{aligned} {}^cD^q v_{n+1} =& F(t, w_n, I^q(w_n)) + G(t, v_n, I^q(v_n)), \\ v_{n+1}(0) =& w_n(0) - \frac{1}{M} g(w_n(0), w_n(T)), \\ {}^cD^q w_{n+1} =& F(t, v_n, I^q(v_n)) + G(t, w_n, I^q(w_n)), \\ w_{n+1}(0) =& v_n(0) - \frac{1}{M} g(v_n(0), v_n(T)), \end{aligned}$$

yields alternating monotone sequences $\{v_{2n}, w_{2n+1}\}$ and $\{v_{2n+1}, w_{2n}\}$ satisfying

$$v_0 \leq w_1 \leq \ldots \leq v_{2n} \leq w_{2n+1} \leq u \leq v_{2n+1} \leq w_{2n} \leq \ldots \leq v_1 \leq w_0,$$

for each $n \geq 1$ on J, provided that $v_0 \leq u \leq w_0$. Furthermore $\{v_{2n}, w_{2n+1}\} \to \rho$ and $\{v_{2n+1}, w_{2n}\} \to r$ in $C^1[J, \mathbb{R}]$ uniformly and monotonically, such that ρ and r are coupled minimal and maximal solutions of (10) , (11), *respectively, that is, if $v_0 \leq u \leq w_0$ then $\rho \leq u \leq r$, and ρ and r satisfy the coupled system.*

$$\begin{aligned} {}^cD^q \rho =& F(t, \rho, I^q(\rho)) + G(t, r, I^q(r)), \\ & g(\rho(0), \rho(T)) = 0, \\ {}^cD^q r =& F(t, \rho, I^q(\rho)) + G(t, r, I^q(r)), \\ & g(r(0), r(T)) = 0. \end{aligned}$$

4. conclusion

We consider periodic boundary value problem of Caputo fractional integro differential equation and obtained its maximal and minimal solutions. We obtained this by using monotone iterative technique of initial value problems.

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