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Tensor Product of Hypervector Spaces

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Abstract. We introduce and study tensor product of hypervector spaces (or hyperspaces) based on Tallini hypervector spaces. Here we introduce the (resp. multivalued) middle linear maps of hyperspaces and construct the categories of linear maps and multivalued linear maps of hyperspaces. It is shown the tensor product of two hypespaces, as an initial object in this category, exists. Also, notion of a quasi-free object in category of hyperspaces is introduced and it is proved that in this category a quasi-free object up to maximum is unique.

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1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory. Hyperstructure theory was first proposed in 1934 by Marty, who defined hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions [19]. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semihypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed ([14], [15], [16],[17] and [23]) (for more see [1, 2, 3, 6, 5, 4, 7, 8, 9]).

M.S. Tallini introduced the notion of hyperspaces (or hypervector spaces) ([20], [21] and [22]) and studied basic properties of them. R. Ameri and O. R. Dehghan introduced and studied dimension of hyperspaces [2]. R. Ameri in [1] introduced and studied categories of hypermodules. Let V and W be two hyperspaces over the fixed filed K (of real or complex numbers). The purpose of this paper is the study of tensor product of hypervector spaces on the sense of Tallini. We introduce the category of multivalued linear maps of hyperspaces and then construct the tensor product of V and W as initial object in this category.

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2. Preliminaries

Which we need to develop our paper.

Definition 1. *Let H be a nonempty set. A map $\cdot : H \times H \longrightarrow P^*(H)$ is called hyperoperation or join operation, where $P^*(H)$ is the set of all nonempty subsets of H. The join operation is extended to nonempty subsets of H in natural way, so that $A \cdot B$ is given by*

$$A \cdot B = \bigcup \{a \cdot b \mid a \in A \text{ and } b \in B\}.$$

the notations $a \cdot A$ and $A \cdot a$ are used for $\{a\} \cdot A$ and $A \cdot \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2. *[14] A hypergroup is a nonempty set H equipped with an associative hyperoperation $\cdot : H \times H \longrightarrow P^*(H)$ which satisfies the property $x \cdot H = H \cdot x = H$, for all $x \in H$. If the hyperoperation $\cdot$ is associative then H is called a semihypergroup.*

A quasicanonical hypergroup is a special kind of a hypergroup, that first time introduced and studied by Bonansinga and Corsini in [10, 11]. After that this kind of hypergroups studied by Comer [13, 12] as the name of polygroups.

Definition 3. *[14, 16] A polygroup is a system $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unary operation on P, $\cdot$ maps $P \times P$ into nonempty subsets of P, and the following axioms hold for all $x, y, z \in P$:*
(P_1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(P_2) $x \cdot e = e \cdot x = x$;
(P_3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, where $A^{-1} = \{a^{-1} \mid a \in A\}$.
A polygroup in which every element has order 2 (i.e., $x^{-1} = x$ for all x) is called symmetric. As in group theory it can be shown that a symmetric polygroup is commutative.

Definition 4. *[15] A semihypergroup $(H, +)$ is called a canonical hypergroup if the following conditions are satisfied:*

(i) $x + y = y + x$ *for all* $x, y \in R$;

(ii) *There exists* $0 \in R$ *(unique) such that for every* $x \in R$, $x \in 0 + x = x$;

(iii) *For every* $x \in R$, *there exists a unique element, say* x' *such that* $0 \in x + x'$ *(we denote* $x' = -x$*)*;

(iv) *For every* $x, y, z \in R$, $z \in x + y \Longleftrightarrow x \in z - y \Longleftrightarrow y \in z - x$; *from the definition it can be easily verified that* $-(-x) = x$ *and* $-(x + y) = -x - y$.

The concept of hyperspace, which is a generalization of the concept of ordinary vector space.

Definition 5. *[20] Let K be a field and $(V, +)$ be an abelian group. We define a hyperspace over K (K-hyperspace) to be the quadruplet $(V, +, \circ, K)$, where $\circ$ is a mapping*

$$\circ : K \times V \longrightarrow P^*(V),$$

such that the following conditions hold (for all $x, y \in V$, and $a, b \in K$):

(H_1) *$a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law;*

(H_2) *$(a + b) \circ x \subseteq a \circ x + b \circ x$, left distributive law;*

(H_3) *$a \circ (b \circ x) = (ab) \circ x$, associative law;*

(H_4) *$a \circ (-x) = (-a) \circ x = -(a \circ x)$;*

(H_5) *$x \in 1 \circ x$.*

Remark 1. *(i) In the right hand side of (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have in (H_2).*

(ii) We say that $(V, +, \circ, K)$ is anti-left distributive, if

$$(a + b) \circ x \supseteq a \circ x + b \circ x \text{ for all } a, b \in K, x \in V,$$

and strongly left distributive, if

$$(a + b) \circ x = a \circ x + b \circ x \text{ for all } a, b \in K, x \in V,$$

In a similar way we define the anti-right distributive and strongly right distributive hyperspaces, respectvely. V is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left hand side of (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e. for all $a, b \in K$, and $x \in V$:

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ 0_V$, where 0_V is the zero of $(V, +)$, In [20] it is shown if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$.

Definition 6. *[2] Let V be a hyperspace over a field K. A nonempty subset W of V is called a subhyperspace if W is itself a hyperspace with the hyperoperation on V, i.e.*

$$W \neq \emptyset, \;\; W - W \subseteq W, \;\; a \circ W \subseteq W \text{ for all } a \in K.$$

In this case we write $W \leq V$.

Definition 7. *[2] Let V be a hyperspace over a field K. If W is a nonempty subset of V, then the linear span of W is defined by*

$$\begin{aligned} L(W) &= \{t \in V \mid t \in \sum_{i=1}^{n} a_i \circ w_i, a_i \in K, w_i \in W, n \in \mathbb{N}\} \\ &= \{t_1 + t_2 + ... + t_n \mid t_i \in a_i \circ w_i, a_i \in K, w_i \in W, n \in \mathbb{N}\}. \end{aligned}$$

Lemma 1. *[2] $L(W)$ is the smallest subhyperspace of V containing W.*

Definition 8. *[2] Let V be a hyperspace over a field K. A subset W of V is called linearly independent if for every vectors $v_1, v_2, ..., v_n$ in W, $c_1, c_2, ..., c_n \in K$, and $0_V \in c_1 \circ v_1 + ... + c_n \circ v_n$, implies that $c_1 = c_2 = ... = c_n = 0$. A subset W of V is called linearly dependent if it is not linearly independent.*

Definition 9. *[2] Let V be a hyperspace over a field K. A basis for V is a linearly independent subset of V such that span V. We say that V has finite dimensional if it has a finite basis.*

Example 1. *[2] Consider abelian group $(\mathbb{R}^2, +)$. Define hyper-compositions*

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P^*(\mathbb{R}^2) \\ \quad a \circ (x, y) = ax \times \mathbb{R} \end{cases}$$

and

$$\begin{cases} \diamond : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P^*(\mathbb{R}^2) \\ \quad a \diamond (x, y) = \mathbb{R} \times ay. \end{cases}$$

Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ and $(\mathbb{R}^2, +, \diamond, \mathbb{R})$ are a strongly distributive hyperspaces.

Example 2. *[20] In $(\mathbb{R}^2, +)$ define the hyper-composition $\circ$ as follows (for all $a \in \mathbb{R}$, and $x \in \mathbb{R}^2$):*

$$a \circ x = \begin{cases} \textit{line } ox & if\ x \neq 0_V \\ \{0_V\} & if\ x = 0_V, \end{cases}$$

where $0_V = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left, but not right distributive hyperspace.

Proposition 1. *[20] Every strongly right distributive hyperspace is strongly left distributive hyperspace. Let $(V, +)$ be an abelian group, Ω a subgroup of V and K a field such that $W = V/\Omega$ is a classical vector space over a field K. If $p : V \longrightarrow W$ is the canonical projection of $(V, +)$ onto $(W, +)$ and set:*

$$\begin{cases} \circ : K \times V \longrightarrow P^*(V) \\ a \circ x = p^{-1}(a \cdot p(x)). \end{cases}$$

Then $(V, +, \circ, K)$ is a strongly distributive hyperspace over a field K. Moreover every strongly distributive hyperspace can be obtained in such a way.

Proposition 2. *[20] If $(V, +, \circ, K)$ is a left distributive hyperspace, then for all $a \in K$ and $x \in V$:*

(i) $0 \circ x$ *is a subgroup of* $(V, +)$;

(ii) Ω_V *is a subgroup of* $(V, +)$;

(iii) $a \circ 0_V = \Omega_V = a \circ \Omega_V$;

(iv) $\Omega_V \subseteq 0 \circ x$;

(v) $x \in 0 \circ x \Longleftrightarrow 1 \circ x = 0 \circ x \Longleftrightarrow a \circ x = 0 \circ x$.

Remark 2. *Let* $(V, +, \circ, K)$ *be a hyperspace and* W *be a subhyperspace of* V*. Consider the quotient abelian group* $(V/W, +)$*. Define the rule*

$$\begin{cases} * : K \times V/W \longrightarrow P^*(V/W) \\ \quad (a, x + W) \longmapsto a \circ x + W. \end{cases}$$

Then it is easy to verify that $(V/W, +, *, K)$ *is a hyperspace over* K *and it is called the quotient hyperspace of* V *over* W*.*

Definition 10. *[2] Let* V *and* W *be two hyperspaces over a field* K*. A mapping* $T : V \longrightarrow W$ *is called (for all* $x, y \in V$*, and* $a \in K$*):*

(i) weak linear transformation (WLT) iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \cap a \circ T(x) \neq \emptyset;$$

(ii) linear transformation (LT) iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \subseteq a \circ T(x);$$

(iii) strong linear transformation (SLT) iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) = a \circ T(x).$$

A (resp. weak, strong) linear isomorphism is defined as usual. If $T : V \longrightarrow W$ is a (resp. weak, strong) linear isomorphism, then it is denoted by (resp. $V \cong_w W$, $V \cong_s W$) $V \cong W$.

Definition 11. *[2] Let* V *and* W *be two hyperspaces over a field* K *and* $T : V \longrightarrow W$ *be a linear transformation. The kernel and image of* T *are denoted by* $KerT$ *and* ImT*, respectively, are defined by*

$$KerT = \{x \in V \mid T(x) \in \Omega_W\}.$$

and

$$ImT = \{y \in W \mid y = T(x) \text{ for some } x \in V\}.$$

Proposition 3. *[2] Let* $T : V \longrightarrow W$ *be a strong linear transformation.*

(i) If Z *is a subhyperspace of* V*, then the image of* Z*,* $T(Z)$ *is a subhyperspace of* W*. In particular* ImT *is a subhyperspace of* W*.*

(ii) If L is a subhyperspace of W, then the preimage of L, $T^{-1}(L)$ is a subhyperspace of V containing $KerT$.

Definition 12. *Let V and W be two hyperspaces over a field K. A multivalued linear transformation (MLT), $T : V \longrightarrow P^*(W)$ is a mapping such that for all $x, y \in V$, and $a \in K$:*

(i) $T(x + y) \subseteq T(x) + T(y)$;

(ii) $T(a \circ x) \subseteq a \circ T(x)$;

(iii) $T(-x) = -T(x)$;

(iv) $T(0) = \{0\}$.

Remark 3. *(i) In Definition 12(i) and (ii), if the equality holds, then T is called a strong multivalued linear transformation $(SMLT)$.*

(ii) In Definition 12, if we consider T as a mapping $T : V \longrightarrow W$, then it is called a linear transformation. Here we consider only inclusion and equality cases.

Definition 13. *The category of hyperspaces over a field K denoted by $\mathcal{HV}_K$ is defined as follows:*

(i) The objects of $\mathcal{HV}_K$ are all hyperspaces over K;

(ii) For the objects V and W of $\mathcal{HV}_K$, the set of all morphisms from V to W denoted by $Hom_K(V, W)$, is the set of all MLT from V to W.

(iii) The composition $ST : V \longrightarrow P^*(W)$ of morphisms $T : V \longrightarrow P^*(L)$ and $S : L \longrightarrow P^*(W)$ is defined as follows:

$$ST(x) = \bigcup_{t \in T(x)} S(t).$$

(iv) For any object V, the morphism $1_V : V \longrightarrow P^*(V)$, $x \longrightarrow \{x\}$ is the identity.

(v) The category of hyperspaces over a field K with (resp. SLT) LT is denoted by (resp. $\mathcal{H}^s_K$) $\mathcal{H}_K$.

Remark 4. *If in Definition 13 part (ii) we replace $Hom_K(V, W)$ by $Hom^s_K(V, W)$, the set of all $SMLT$, then we will obtain a new category, which it denotes by $\mathcal{HV}^s_K$. In fact, $\mathcal{HV}^s_K \preceq \mathcal{HV}_K$ (by $A \preceq B$ we mean A is a subcategory of B). Also, denote the category of all vector spaces over a field K (K-vector spaces) by $\mathcal{V}_K$. Clearly, $\mathcal{V}_K \preceq \mathcal{H}_K \preceq \mathcal{H}^s_K \preceq \mathcal{HV}^s_K \preceq \mathcal{HV}_K$ (for more details see [1]).*

Definition 14. *[1] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow P^*(W)$ be a $SMLT$. Then multivalued kernel and multivalued image of T, denoted by $\overline{Ker}T$ and $\overline{Im}T$, respectively, are defined as follows:*

$$\overline{Ker}T = \{x \in V \mid 0_W \in T(x)\};$$

and

$$\overline{Im}T = \{y \in W \mid y \in T(x) \text{ for some } x \in V\}.$$

Remark 5. *(i) Note that $\overline{Ker}T \neq \emptyset$, by Definition 12(iv).*

(ii) For hyperspaces V and W over a field K, by $Hom_K(V,W)$ and $Hom^s_K(V,W)$, we mean the set of all MLT and $SMLT$, respectively and sometimes we use morphism instead multivalued linear transformation, respectively.

Definition 15. *Let $T : V \longrightarrow P^*(W)$ be a $SMLT$ of hyperspaces. We say that T is weakly injective if for all $x, y \in V$:*

$$T(x) \cap T(y) \neq \emptyset \Longrightarrow x = y.$$

We say that T is strongly injective if for all $x, y \in V$:

$$T(x) = T(y) \Longrightarrow x = y.$$

Remark 6. *Clearly, every weakly injective morphism is also strongly injective. Note that T is strongly injective, means that T is injective as a function with values in $P^*(W)$. In the following example we show that a strongly injective morphism need not to be weakly injective.*

Similarly, we introduce the notions of weakly and strongly surjective. A morphism $T : V \longrightarrow P^*(W)$ of hyperspaces is said to be weakly surjective if for every $y \in W$ there exists $x \in V$ such that $y \in T(x)$ and is strongly surjective, if for every nonempty subset Z of W, there exists $x \in V$ such that $Z = T(x)$.

Remark 7. *Clearly, every strongly surjective morphism is weakly surjective. But the converse is not true. For example the identity function on every hyperspace is weakly surjective, but is not strongly surjective.*

Theorem 2. *[18] Let K be a field. The following conditions on a K-vector space F are equivalent:*

(i) F has a nonempty basis;

(ii) F is the internal direct sum of a family of cyclic K-vector spaces, each of which is isomorphic as a K-vector space to K;

(iii) F is K-vector space isomorphic to a direct sum of copies of the K-vector space K;

(iv) There exists a nonempty set X and a function $\iota : X \longrightarrow F$ with the following property: given any K-vector space V and function $f : X \longrightarrow V$, there exists a unique K-vector space homomorphism $\overline{f} : F \longrightarrow V$ such that $\overline{f}\iota = f$. In other words, F is a free object in the category of K-vector spaces.

Remark 8. *A vector space F over a field K, which satisfies the equivalent conditions of Theorem 2, is called a free K-vector space on the set X. By Theorem 2 (iv), F is a free object in the category of all K-vector spaces.*

Definition 16. *[18] Let V and W be two vector space over a field K, and Z is an (additive) abelian group. Then a middle linear map from $V \times W$ to Z is a function $f : V \times W \longrightarrow Z$ such that (for all $v, v_i \in V,\ w, w_i \in W,\ a \in K$, and $i = 1, 2$):*

(i) $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$;

(ii) $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$;

(iii) $f(av, w) = f(v, aw)$.

For fixed V and W consider the category $\mathcal{ML}(V, W)$ whose objects are all middle linear maps on $V \times W$. By definition a morphism in $\mathcal{ML}(V, W)$ from the middle linear map $f : V \times W \longrightarrow Z$ to the middle linear map $g : V \times W \longrightarrow Z'$ is a group homomorphism $h : Z \longrightarrow Z'$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dd, "h"] \\ V \times W \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & Z' \end{tikzcd}$$

is commutative. Verify that $\mathcal{ML}(V, W)$ is a category, that 1_H is the identity morphism from f to f, and that h is an equivalence in $\mathcal{ML}(V, W)$ if and only if h is an isomorphism of groups.

3. Quasi-free object

Definition 17. *Let $(F, \cdot)$ is an object in the category $\mathcal{HV}^s_K$ and $i : X \hookrightarrow F$ is an inclusion map of sets. We say that F is quasi-free on the subset X provided that:*

(i) $F = \langle X \rangle$;

(ii) For any object V in $\mathcal{HV}^s_K$ and any multivalued map $\lambda : X \longrightarrow P^*(V)$, there is a maximum SMLT, $\overline{\lambda} : F \longrightarrow P^*(V)$ such that for all $x \in X$, we have $\overline{\lambda} i(x) = \lambda(x)$.

Theorem 3. *Let F be a strongly distributive hyperspace over a field K and X be a basis for F. Then*

(i) If $j : X \hookrightarrow F$ is a inclusion map, then for all K-hyperspace V and map $f : X \longrightarrow P^*(V)$, there is a maximum $SMLT$, $\varphi : F \longrightarrow P^*(V)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"] & F \arrow[dl, "\varphi"] \\ P^*(V) & \end{tikzcd}$$

is commutative.

(ii) For all K-hyperspace V and $f : X \longrightarrow P^*(V)$ induced maximum $SMLT$, $\varphi : F \longrightarrow P^*(V)$, means there is a maximum $SMLT$, $\varphi : F \longrightarrow P^*(V)$ such that $\varphi|_X = f$.

Proof.

(i) Since for every $u \in F$, there exists scalars $c_1, ..., c_n \in K$ such that

$$(*) \qquad u \in \sum_{i=1}^{n} c_i \circ x_i,$$

then we define a map $\varphi : F \longrightarrow P^*(V)$ as follows:

$$\varphi(u) = \varphi(\sum_{i=1}^{n} c_i \circ x_i) = \sum_{i=1}^{n} c_i \circ f(x_i).$$

Since $(*)$ is unique, then φ is well-defined. Now, we check that φ is a $SMLT$. Let $u, v \in F$ and scalars $d_1, ..., d_n \in K$. Then $u \in \sum_{i=1}^{n} c_i \circ x_i$ and $v \in \sum_{i=1}^{n} d_i \circ x_i$, thus we have $\varphi(u) = \sum_{i=1}^{n} c_i \circ f(x_i)$ and $\varphi(v) = \sum_{i=1}^{n} d_i \circ f(x_i)$. Now since $u + v \in \sum_{i=1}^{n}(c_i + d_i) \circ x_i$, then we obtain:

$$\begin{aligned} \varphi(u+v) &= \varphi(\sum_{i=1}^{n}(c_i + d_i) \circ x_i) \\ &= \sum_{i=1}^{n}(c_i + d_i) \circ f(x_i) \\ &= \sum_{i=1}^{n} c_i \circ f(x_i) + \sum_{i=1}^{n} d_i \circ f(x_i) \end{aligned}$$

$$= \varphi(u) + \varphi(v).$$

Also, it is clear that $(c \circ \varphi)(x_i) = c \circ \varphi(x_i)$. Hence, φ is a multivalued linear transformation.
Also, for all $x \in X$, $\varphi j(x) = \varphi(x) = f(x)$, thus $\varphi j = f$, means that φ is a $SMLT$, where the diagram is commutative. Now, If there is a $SMLT$, $\psi : F \longrightarrow P^*(V)$ such that the diagram is commutative, then for all $u \in F$:

$$\begin{aligned}
\varphi(u) &= \varphi(\sum_{i=1}^{n} c_i \circ x_i) \\
&= \sum_{i=1}^{n} c_i \circ f(x_i) \\
&= \sum_{i=1}^{n} c_i \circ \psi j(x_i) \\
&= \sum_{i=1}^{n} c_i \circ \psi(x_i) \\
&= \psi(\sum_{i=1}^{n} c_i \circ x_i) \supseteq \psi(u),
\end{aligned}$$

Therfore $\varphi \supseteq \psi$.

(ii) Let V be a K-hyperspace and $f : X \longrightarrow P^*(V)$ be a map. By part (i), there exists maximum $SMLT$, $\varphi : F \longrightarrow P^*(V)$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, "f"'] & F \arrow[dl, "\varphi"] \\
P^*(V) &
\end{tikzcd}$$

is commutative, means that $\varphi j = f$. Therefore for all $x \in X$,

$$\varphi(x) = \varphi j(x) = f(x).$$

So $\varphi|_X = f$.

Remark 9. *By Theorem 3 every strongly distributive hyperspace is a quasi-free on every of its basis.*

4. Tensor product

Definition 18. *Let V, W be two hyperspaces over a field K, and Z be an (additive) abelian group. Then a multivalued middle linear map from $V \times W$ to $P^*(Z)$ is a multivalued function $f : V \times W \longrightarrow P^*(Z)$ such that (for all $v, v_i \in V$, $w, w_i \in W$, $a \in K$, and $i = 1, 2$):*

(i) $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$*;*

(ii) $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$*;*

(iii) $f(a \circ v, w) = f(v, a \circ w)$*, where* $f(a \circ v, w) = \bigcup_{t \in a \circ v} f(t, w)$.

For fixed V and W consider the category $\mathcal{MML}(V, W)$ whose objects are all multivalued middle linear maps on $V \times W$. By definition a morphism in $\mathcal{MML}(V, W)$ from the multivalued middle linear map $f : V \times W \longrightarrow P^*(Z)$ to the multivalued middle linear map $g : V \times W \longrightarrow P^*(Z')$ is a map $\overline{h} : P^*(Z) \longrightarrow P^*(Z')$ such that the diagram

$$\begin{tikzcd} & P^*(Z) \arrow[dd, "\overline{h}"] \\ V \times W \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & P^*(Z') \end{tikzcd}$$

is commutative. Verify that $\mathcal{MML}(V, W)$ is a category, that 1_H is the identity morphism from f to f. In Theorem 4 we shall construct a universal object in the category $\mathcal{MML}(V, W)$. First, however, we need

Definition 19. *Let V and W be two hyperspaces over a field K. Let F be the free abelian group on the set $V \times W$. Let H be the subgroup of F generated by all elements of the following forms (for all $v, v' \in V$, $w, w' \in W$, and $a \in K$):*

(i) $(v + v', w) - (v, w) - (v', w)$*;*

(ii) $(v, w + w') - (v, w) - (v, w')$*;*

(iii) $(a \circ v, w) - (v, a \circ w)$*, where* $(a \circ v, w) = \bigcup_{t \in a \circ v} (t, w)$.

The quotient group F/H is called the tensor product of V and W; it is denoted $V \otimes_K W$. The coset $(v, w) + K$ of the element (v, w) in F is denoted $v \otimes w$; the coset of $(0, 0)$ is denoted 0.

Since F is generated by the set $V \times W$, the quotient group $F/H = V \otimes_K W$ is generated by all elements (cosets) of the form $v \otimes w$ ($v \in V,\ w \in W$). But it is not true that every element of $V \otimes_K W$ is of the form $v \times w$. For the typical element of F is a sum $\sum_{i=1}^{r} n_i(v_i, w_i)$ ($n_i \in \mathbb{Z}$, $v_i \in V$, and $w_i \in W$) and hence its coset in $V \otimes_K W = F/H$ is of the form $\sum_{i=1}^{r} n_i(v_i \otimes w_i)$. Furthermore, since it is possible to choose different representatives for a coset, one may have $v \otimes w = v' \otimes w'$ in $V \otimes_K W$, but $v \neq v'$ and $w \neq w'$. It is also possible to have $V \otimes_K W = 0$ even though $V \neq 0$ and $W \neq 0$.

Definition 19 implies that the generators $v \otimes w$ of $V \otimes_K W$ satisfy the following relations (for all $v, v_i \in V,\ w, w_i \in W,\ a \in K$, and $i = 1, 2$):

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w; \tag{1}$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2; \tag{2}$$

$$(a \circ v) \otimes w = v \otimes (a \circ w). \tag{3}$$

The proof of these facts is straightforward; for example, since $(v_1 + v_2, w) - (v_1, w) - (v_2, w) \in H$, the zero coset, we have

$$[(v_1 + v_2, w) + H] - [(v_1, w) + H] - [(v_2, w) + H] = H;$$

or in the notation $(v, w) + H = v \otimes w$,

$$(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w = 0.$$

Also, since $(a \circ v, w) - (v, a \circ w) = \bigcup_{t \in a \circ v}(t, w) - \bigcup_{s \in a \circ w}(v, s) \in H$, the zero coset, we have

$$[(a \circ v, w) + H] - [(v, a \circ w) + H] = [\bigcup_{t \in a \circ v}(t, w) + H] - [\bigcup_{s \in a \circ w}(v, s) + H] = H;$$

or in the notation $\bigcup_{t \in a \circ v}(t, w) + H = (a \circ v) \otimes w$,

$$(a \circ v) \otimes w - v \otimes (a \circ w) = 0.$$

Indeed an alternate definition of $V \otimes_K W$ is that it is the abelian group with generators all symbols $v \otimes w$ ($v \in V,\ w \in W$), subject to the relations $(1) - (3)$ above. Furthermore, since 0 is the only element of a group satisfying $x + x = x$, it is easy to see that for all $v \in V,\ w \in W$:

$$v \otimes 0 = 0 \otimes w = 0 \otimes 0 = 0.$$

Given hyperspaces V and W over a field K, it is easy to verify that the map $i : V \times W \longrightarrow V \otimes_K W$ given by $(v, w) \longmapsto v \otimes w$ is a middle linear map. The map i is called canonical middle linear map. Its importance is seen in

Theorem 4. *Let V, W be two hyperspaces over a field K, and Z be an abelian group. If $g : V \times W \longrightarrow Z$ is a middle linear map, then there exists a unique group homomorphism $\overline{g} : V \otimes_K W \longrightarrow Z$ such that $\overline{g}i = g$, where $i : V \times W \longrightarrow V \otimes_K W$ is the canonical middle linear map. $V \otimes_K W$ is uniquely determined up to isomorphism by this property. In other words $i : V \times W \longrightarrow V \otimes_K W$ is universal in the category $\mathcal{ML}(V, W)$ of all middle linear maps on $V \times W$.*

Proof. Let F be the free abelian group on the set $V \times W$, and let H be the subgroup described in Definition 19. Since F is free, the assignment $(v, w) \mapsto g(v, w) \in Z$ determines a unique group homomorphism $g_1 : F \longrightarrow Z$ by Theorem 2. Use the fact that g is middle linear to show that g_1 maps every generator of H to 0. Hence $H \subset Kerg_1$. g_1 induces a homomorphism $\overline{g} : F/H \longrightarrow Z$ such that $\overline{g}[(v, w) + H] = g_1(v, w) = g(v, w)$. But $F/H = V \otimes_K W$ and $(v, w) + H = v \otimes w$. Therefore, $\overline{g} : V \otimes_K W \longrightarrow Z$ is a homomorphism such that $\overline{g}i(v, w) = \overline{g}(v \otimes w) = g(v, w)$ for all $(v, w) \in V \times W$; that is, $\overline{g}i = g$. If $h : V \otimes_K W \longrightarrow Z$ is any homomorphism with $hi = g$, then for any generator $v \otimes w$ of $V \otimes_K W$,

$$h(v \otimes w) = hi(v, w) = g(v, w) = \overline{g}i(v, w) = \overline{g}(v \otimes w).$$

Since h and $\overline{g}$ are homomorphisms that agree on the generators of $V \otimes_K W$, we must have $h = \overline{g}$, whence $\overline{g}$ is unique. This proves that $i : V \times W \longrightarrow V \otimes_K W$ is a universal object in the category of all middle linear maps on $V \times W$, whence $V \otimes_K W$ is uniquely determined up to isomorphism (equivalence).

Corollary 1. *If V, V', W, and W' are hyperspaces over a field K and $f : V \longrightarrow V'$, $g : W \longrightarrow W'$ are K-hyperspace homomorphisms, then there is a unique group homomorphism $V \otimes_K W \longrightarrow V' \otimes_K W'$ such that $(v, w) \longmapsto f(v) \otimes g(w)$ for all $v \in V, w \in W$.*

Proof. Verify that the assignment $(v, w) \longmapsto f(v) \otimes g(w)$ defines a middle linear map $h : V \times W \longrightarrow C = V' \otimes_K W'$. By Theorem 4 there is a unique homomorphism $\overline{h} : V \otimes_K W \longrightarrow V' \otimes W'$ such that $\overline{h}(v \otimes w) = \overline{h}i(v, w) = h(v, w) = f(v) \otimes g(w)$ for all $v \in V,\ w \in W$.

The unique homomorphism of Corollary 1 is denoted $f \otimes g : V \otimes_K W \longrightarrow V' \otimes_K W'$. If $f' : V' \longrightarrow V''$ and $g' : W' \longrightarrow W''$ are also K-hyperspace homomorphisms, then it is easy to verify that

$$(f' \otimes g')(f \otimes g) = (f'f \otimes g'g) : V \otimes_K W \longrightarrow V'' \otimes_K W''.$$

It follows readily that if f and g are K-hyperspace isomorphisms, then $f \otimes g$ is a group isomorphism with inverse $f^{-1} \otimes g^{-1}$.

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