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Nilpotent L-subgroups and the set product of L-subsets

Naseem Ajmal[1], Iffat Jahan[2,*], Bijan Davvaz[3]

[1] *Department of Mathematics, Zakir Husain College, University of Delhi, Delhi, India*
[2] *Department of Mathematics, Ramjas College, University of Delhi, Delhi, India*
[3] *Department of Mathematics, Yazd University, Iran*

Abstract. In this paper, we study the set product of a pair of nilpotent normal L-subgroups of a given L-group. A necessary mechanism is developed in order to establish this result. Moreover, the tail of L-subgroups is used effectively while developing this mechanism.

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Key Words and Phrases: L-algebra; L-subgroup; Generated L-subgroup; Normal L-subgroup; Nilpotent L-subgroup.

1. Introduction

Rosenfeld [14] applied the notion of fuzzy subsets in algebra and introduced the concept of fuzzy subgroups in 1971. As a result a new discipline of fuzzy algebraic structures emerged which contains the extensions of various concepts and notions of classical algebra. However, the progress of this discipline could not sustain the impact of metatheorem which was developed by Tom Head [8] during the year 1995. This is due to the fact that the various notions and concepts formulated in the areas of fuzzy semigroups, fuzzy groups and fuzzy rings are generically defined and hence the extension of results from classical algebra to fuzzy algebra became just simple instances of this indigenous result. Therefore for further growth of the subject, a need was felt to develop a framework for these investigations which is beyond the purview of the metatheorem. The notion of lattice valued fuzzy subsets was introduced by Goguen [7] in the year 1967 which was later applied by Wang Jin Liu [10] to define the notions of lattice valued fuzzy subgroup of a group and lattice valued ideals of a ring. It is in this framework that the notions and the concepts extended from classical algebra do not remain projection closed which is a prerequisite for an application of Tom Head metatheorem. The theory of L-subrings is sufficiently developed by Mordeson and Malik [13] along with several other researchers [11, 12]. However, in the area of fuzzy groups such an effort is found lacking. This

*Corresponding author.

Email addresses: nasajmal@yahoo.com (N. Ajmal), ij.umar@yahoo.com (I. Jahan), davvaz@yazd.ac.ir (B. Davvaz)

motivated us to formulate several concepts in the studies of lattice valued fuzzy subgroups (L-subgroups) which appeared in a series of papers [2, 3, 4, 5, 6]. In all the above mentioned papers, we not only replaced the evaluation lattice [0,1] by a completely distributive lattice, we also replaced the parent structure of an ordinary group by an L-group. Therefore an application of metatheorem in our studies become further remote. Consequently, the normality of an L-subgroup of an L-group due to Wu has been used in these studies instead of the concept of normality due to Liu. This allows us to construct the chains of normal L-subgroups in the same fashion as that of ordinary normal subgroups in classical group theory. We have introduced and studied the notions of nilpotent L-subgroup, solvable L-subgroup, normalizer of an L-subgroup and normal closure of an L-group having the parent structure of an L-group. In the continuation of the development of L-group, the authors in this paper, after developing a necessary mechanism, prove that the set product of a pair of nilpotent normal L-subgroups of an L-group is again a nilpotent L-subgroup.

2. Preliminaries

Throughout this paper, the system $\langle L, \leq, \vee, \wedge \rangle$ denotes a completely distributive lattice where $\leq$ denotes the partial ordering of L, the join (sup) and meet (inf) of the elements of L are denoted by $\vee$ and $\wedge$ respectively. Also, we write 1 and 0 for the maximal and the minimal elements of L, respectively. Moreover, our work is carried out by using the definition of L-subset as formulated by Goguen. The definition of a completely distributive lattice is well known in the literature and can be found in any standard text on the subject.

Let $\{J_i : i \in I\}$ be any family of subsets of a complete lattice L and F denotes the set of choice functions for J_i, i.e., functions $f : I \to \prod_{i \in I} J_i$ such that $f(i) \in J_i$ for each $i \in I$. Then, we say that L is a completely distributive lattice, if

$$\bigwedge \left\{ \bigvee_{i \in I} J_i \right\} = \bigvee_{f \in F} \left\{ \bigwedge_{i \in I} f(i) \right\}.$$

The above law is known as the complete distributive law. Moreover, a lattice L is said to be infinitely meet distributive if for every subset $\{b_\beta : \beta \in B\}$ of L, we have

$$a \bigwedge \{ \bigvee_{\beta \in B} b_\beta \} = \bigvee_{\beta \in B} \{ a \bigwedge b_\beta \},$$

provided L is join complete. The above law is known as the infinitely meet distributive law. The definition of infinitely join distributive lattice is dual to the above definition i.e. a lattice L is said to be infinitely join distributive if for every subset $\{b_\beta : \beta \in B\}$ of L, we have

$$a \bigvee \{ \bigwedge_{\beta \in B} b_\beta \} = \bigwedge_{\beta \in B} \{ a \vee b_\beta \},$$

provided L is meet complete. The above law is known as the infinitely join distributive law. Clearly, both these laws follow from the definition of a completely distributive lattice. Here we also mention that the dual of completely distributive law is valid in a completely distributive lattice whereas the infinitely meet and join distributive laws are independent from each other. Next we recall the following from [1-6, 9, 15]:

An L-subset of X is a function from X into L. The set of L -subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$, the set $\{\mu(x) : x \in X\}$ is called the image of μ and is denoted by $Im\mu$ and the tip of μ is defined as $\bigvee_{x\in X} \mu(x)$. Moreover, the tail of μ is defined as $\bigwedge_{x\in X} \mu(x)$. We say that an L-subset μ of X is contained in an L-subset η of X if $\mu(x) \leq \eta(x)$ for $x \in X$ and is denoted by $\mu \subseteq \eta$. For a family $\{\mu_i : i \in I\}$ of L-subsets in X, where I is a nonempty index set, the union $\bigcup_{i\in I} \mu_i$ and the intersection $\bigcap_{i\in I} \mu_i$ of $\{\mu_i : i \in I\}$ are, respectively, defined by:

$$\bigcup_{i\in I} \mu_i(x) = \bigvee_{i\in I} \mu(x) \text{ and } \bigcap_{i\in I} \mu_i(x) = \bigwedge_{i\in I} \mu(x),$$

for each $x \in X$. If $\mu \in L^X$ and $a \in L$, then the notion of level subset μ_a of μ is defined as:

$$\mu_a = \{x \in X : \mu(x) \geq a\}.$$

The set product $\mu \circ \eta$ of $\mu, \eta \in L^S$, where S is a groupoid, is an L-subset of S defined by

$$\mu \circ \eta(x) = \bigvee_{x=yz} \{\mu(y) \wedge \eta(z)\}.$$

Again recall that if x cannot be factored as $x = yz$ in S, then $\mu\circ\eta(x)$ being the least upper bound of the empty set is zero.It can be verified easily that the set product is associative in L^S if S is a semigroup.

Throughout this paper G denotes an ordinary group with the identity element 'e', and I denotes a nonempty indexing set.

Definition 1. *Let $\mu \in L^G$. Then, μ is called an L-subgroup of G if for each $x, y \in G$*

(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(x^{-1}) = \mu(x)$.

The set of L-subgroups of G is denoted by $L(G)$. Clearly, the tip of an L-subgroup is attained at the identity element e of G.

Definition 2. *Let $\mu \in L(G)$. Then, μ is called a normal L- subgroup of G if, $\mu(xy) = \mu(yx)$ for all $x, y \in G$.*

It is well known that the intersection of any arbitrary family of L-subgroups of a group is an L-subgroup of the given group.

Definition 3. *Let $\mu \in L^G$. Then, the L-subgroup of G generated by μ is defined as the smallest L-subgroup of G which contains μ. It is denoted by $\langle\mu\rangle$ i.e.*

$$\langle\mu\rangle = \cap\{\mu_i \in L(G) : \mu \subseteq \mu_i\}.$$

If $\mu, \eta \in L(G)$ and $\eta \subseteq \mu$, then we say that η is an L-subgroup of G. Further, if η is non-constant and $\mu \neq \eta$, then η is said to be a proper L-subgroup of μ. Clearly, η is a proper L-subgroup of μ if and only if η has distinct tip and tail and $\eta \neq \mu$.

Also, η is said to be a trivial L-subgroup of μ if its chain of level subgroups contains only e and G. Thus, an L-subgroup may contain several trivial L-subgroups.

Let η be an L-subgroup of μ. Then, we define the following L-subgroup of μ contained in η, denoted by $\eta_{t_0}^{a_0}$, as follows:

$$\eta_{t_0}^{a_0}(y) = \begin{cases} a_0, & \text{if } y = e, \\ t_0, & \text{if } y \neq e, \end{cases}$$

where $a_0 = \eta(e)$ and $t_0 = \inf \eta$. Here $\eta_{t_0}^{a_0}$, a trivial L-subgroup of μ, is called the trivial L-subgroup of η.

Henceforth μ denotes an L-subgroup of G and we call the parent L-subgroup simply an L-group. The set of L-subgroups of μ is denoted by $L(\mu)$.

Remark 1. *If $\eta \in L^\mu$, then it can be easily verified that $\langle\eta\rangle_\mu = \langle\eta\rangle$, where $\langle\eta\rangle_\mu$ denotes the L-subgroup of μ generated by η.*

We recall the definition of a normal L-subgroup of an L-group.

Definition 4. *Let $\eta \in L(\mu)$. Then, we say that η is a normal L-subgroup of μ if*

$$\eta(yxy^{-1}) \geq \eta(x) \wedge \mu(y) \text{ for all } x, y \in G.$$

The set of normal L-subgroups of μ is denoted by $NL(\mu)$.

Proposition 1. *Let $\eta \in L(\mu)$ and $\theta \in NL(\mu)$. Then,*
(i) $\eta \circ \theta \in L(\mu)$.
(ii)$\eta \circ \theta \in NL(\mu)$ *if* $\eta \in NL(\mu)$.

Proposition 2. *Let $\eta, \theta \in L(\mu)$. Then, $\eta \subseteq \eta \circ \theta$ and $\theta \subseteq \eta \circ \theta$ if and only if $\eta(e) = \theta(e)$.*

Now, recall the following from [3]:

Definition 5. *Let $\eta, \theta \in L^\mu$. Then, the commutator of η and θ is an L-subset (η, θ) of G defined as follows:*

$$(\eta, \theta)(x) = \begin{cases} \vee\{\eta(y) \wedge \theta(z)\}, & \text{if } x = [y, z] \text{ for some } y, z \in G, \\ \inf \eta \wedge \inf \theta, & \text{if } x \neq [y, z] \text{ for any } y, z \in G. \end{cases}$$

The commutator L-subgroup of $\eta, \theta \in L^\mu$ is defined as the L-subgroup of G generated by (η, θ). It is denoted by $[\eta, \theta]$. Clearly, $\inf(\eta, \theta) = \inf \eta \wedge \inf \theta$ and $[\eta, \theta] \in L(\mu)$.

3. Nilpotent L-subgroup

In [6], Ajmal and Jahan extended the construction of a fuzzy subgroup generated by a fuzzy subset to L-setting. They proved for an L-subset of a group, the subgroup generated by its level subset is the level subset of the subgroup generated by that L-subset provided the given L-subset possesses sup-property. Firstly, we recall from [6] a construction for generating an L-subgroup by a given L-subset of an L-group.

Theorem 1. *Let $\eta \in L^\mu$ and $a_0 = \bigvee_{x\in G}\{\eta(x)\}$.Define an L-subset $\hat{\eta}$ of G by:*

$$\hat{\eta}(x) = \bigvee_{a\leq a_0}\{a : x \in \langle \eta_a \rangle\}.$$

Then, $\hat{\eta} \in L(\mu)$ and $\hat{\eta} = \langle \eta \rangle$.

Proposition 3. *Let $\eta, \theta \in L(\mu)$. Then*

$$[\eta, \theta]\,(e) = \eta(e) \wedge \theta(e).$$

Proposition 4. *Let $\eta, \theta \in L(\mu)$ and $\eta \subseteq \theta$. Then, $[\eta, \sigma] \subseteq [\theta, \sigma]$ for each $\sigma \in L^\mu$.*

Proposition 5. *Let $\eta, \theta \in NL(\mu)$. Then, $[\eta, \theta] \in NL(\mu)$.*

Next we recall the notion of nilpotent L-subgroup [3]:

Let $\eta \in L(\mu)$ and define $Z_0(\eta) = \eta$, $Z_1(\eta) = [Z_0(\eta), \eta]$. And in general, for each i, we define $Z_i(\eta) = [Z_{i-1}(\eta), \eta]$.It is easy to verify that $Z_i(\eta) \subseteq Z_{i-1}(\eta)$. Moreover, $Z_i(\eta)$ and η have identical tips and identical tails.

Definition 6. *Let $\eta \in L(\mu)$ with tip a_0 and tail t_0 and $a_0 \neq t_0$. If the descending central chain*

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_i(\eta) \supseteq \cdots$$

terminates finitely to the trivial L-subgroup $\eta_{t_0}^{a_0}$, then η is known as a nilpotent L-subgroup of μ. More precisely, η is said to be nilpotent of class c if c is the least non-negative integer such that $Z_c(\eta) = \eta_{t_0}^{a_0}$. In this case, the series

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq \cdots \supseteq Z_c(\eta) = \eta_{t_0}^{a_0}$$

is called the descending central series of η. If it is a nilpotent L-subgroup of μ, then we simply write η is nilpotent.

Proposition 6. *Let $\eta \in NL(\mu)$. Then, $Z_i(\eta) \in NL(\mu)$.*

Here we give an example of an L-subgroup of an L-group:

Example 1. *Let G be the quaternian group Q_8 given by :*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

where $i^2 = j^2 = k^2 = -1, ij = k, jk = i, kj = i$ *. Let* $C = \{1, -1\}$ *be the center of* G *and* $H = \{\pm 1, \pm i\}$*. Let the evaluation lattice* L *be the chain given by :*

$$L: \ f \leq a \leq b \leq d$$

Define L*-subsets* μ *and* η *of* G *as follows:*

$$\mu(x) = \begin{cases} d & \text{if } x \in C \\ b & \text{if } x \in H \setminus C \\ a & \text{if } x \in Q_8 \setminus H. \end{cases}$$

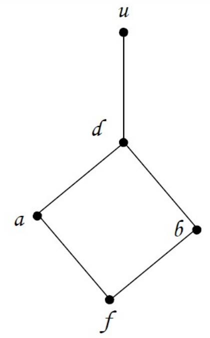
and

$$\eta(x) = \begin{cases} d & \text{if } x = 1 \\ b & \text{if } x \in C \setminus \{1\} \\ a & \text{if } x \in H \setminus C \\ f & \text{if } x \in Q_8 \setminus H \end{cases}$$

Since the level subsets of η *and* μ *are subgroups of* G*,* η *and* μ *are* L*-subgroups of* G*. As* $\eta \subseteq \mu$*,* η *is an* L*-subgroup of* μ*. Now in view of the fact that every subgroup of a nilpotent group is nilpotent, it follows that all the level subsets of* η *are nilpotent subgroups of the corresponding level subsets of* μ*. Therefore the converse of Theorem 4.1[3], implies that* η *is a nilpotent* L*-subgroup of* μ*.*

Next we show that just by changing the evaluation lattice L and keeping the parent group as Q_8, we obtain various types of L-subgroups.

Example 2. *Let* $G = Q_8$ *and the evaluation lattice be given by the diagram :*



Consider the parent L*-subgroup of* G *given by:*

$$\mu(x) = \begin{cases} u & \text{if } x \in C, \\ d & \text{if } x \in G \setminus C. \end{cases}$$

Now define L*-subset* η *of* μ *as given below:*

$$\eta(x) = \begin{cases} u & \text{if } x \in C, \\ d & \text{if } x \in H_1 \setminus C, \\ a & \text{if } x \in H_2 \setminus C, \\ b & \text{if } x \in H_3 \setminus C; \end{cases}$$

where

$$C = \{\pm 1\}, \quad H_1 = \{\pm 1, \pm i\}, \quad H_2 = \{\pm 1, \pm j\}, \quad H_3 = \{\pm 1, \pm k\}.$$

Since the level subsets of η are normal subgroups of G, η is a normal L-subgroup of G and hence of μ. Now that η is a nilpotent L-subgroup of μ in view of Definition 3.5.We demonstrate this as follows:

Note that $G' = \{1, -1\}$. In order to obtain the members of descending central series of η, we set $Z_0(\eta) = \eta$ and consider the commutator (η, η)

$$(\eta, \eta)(x) = \begin{cases} u & \text{if } x = 1, \\ d & \text{if } x \in C \setminus \{1\}, \\ f & \text{if } x \in G \setminus C. \end{cases}$$

As the level subsets of (η, η) are subgroups of G,

$$Z_1(\eta) = [\eta, \eta] = (\eta, \eta).$$

Next, we calculate the commutator :

$$((\eta, \eta), \eta)(x) = \begin{cases} u & \text{if } x = 1, \\ f & \text{if } x \in G \setminus \{1\}. \end{cases}$$

Again by the reasons as given above

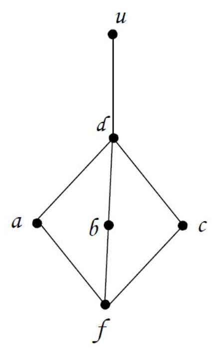
$$Z_2(\eta) = [[\eta, \eta], \eta] = ((\eta, \eta), \eta).$$

Observe that $Z_2(\eta)$ is not only an L-subgroup, it is the trivial L-subgroup of η and so the descending central series terminates at $Z_2(\eta)$, i. e.

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) \supseteq Z_2(\eta) = \eta_f^u.$$

Consequently η is a nilpotent $L-$subgroup of μ having nilpotent length 2.

Example 3. *Let $G = Q_8$ and the evaluation lattice be given by the diagram :*



Consider the parent L-subgroup of G given by:

$$\mu(x) = \begin{cases} u & \text{if } x \in C, \\ d & \text{if } x \in G \setminus C. \end{cases}$$

Now define L-subsets of μ, η, θ and ϕ as given below:

$$\eta(x) = \begin{cases} d & \text{if } x \in C, \\ a & \text{if } x \in H_1 \setminus C, \\ b & \text{if } x \in H_2 \setminus C, \\ c & \text{if } x \in H_3 \setminus C; \end{cases}$$

$$\theta(x) = \begin{cases} d & \text{if } x \in C, \\ b & \text{if } x \in H_1 \setminus C, \\ a & \text{if } x \in H_2 \setminus C, \\ c & \text{if } x \in H_3 \setminus C; \end{cases}$$

and

$$\phi(x) = \begin{cases} d & \text{if } x \in C, \\ a & \text{if } x \in H_1 \setminus C, \\ c & \text{if } x \in H_2 \setminus C, \\ b & \text{if } x \in H_3 \setminus C; \end{cases}$$

where

$$C = \{\pm 1\}, \quad H_1 = \{\pm 1, \pm i\}, \quad H_2 = \{\pm 1, \pm j\}, \quad H_3 = \{\pm 1, \pm k\}.$$

Here

$$\eta_a = H_1, \ \eta_b = H_2 \ \text{and} \ \eta_c = H_3.$$

Since the level subsets of η are normal subgroups of G, η is a normal L-subgroup of G. Similarly, it can be seen that θ and ϕ are also normal $L-$subgroups of G and hence of μ. Now η is a nilpotent L-subgroup of μ, in view of Definition 3.2, can be seen as follows:

Note that $G' = \{1, -1\}$. In order to obtain the members of descending central series of η, we set $Z_0(\eta) = \eta$ and consider the commutator (η, η)

$$(\eta, \eta)(x) = \begin{cases} d & \text{if } x = 1, \\ f & \text{if } x \in G \setminus \{1\}. \end{cases}$$

As the level subsets of (η, η) are subgroups of G,

$$Z_1(\eta) = [\eta, \eta] = (\eta, \eta).$$

Observe that $Z_1(\eta)$ is not only an L-subgroup, it is the trivial L-subgroup of η and so the descending central series terminates at $Z_1(\eta)$, i. e.

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) = \eta_0^d.$$

Consequently η is a nilpotent $L-$subgroup of μ having nilpotent length 1. Now in order to continue our studies further, we mention that the set product of L-subgroups of η, θ and ϕ are given by

$$\eta \circ \theta = \theta \circ \phi = \eta \circ \phi = \psi$$

where ψ is the constant function taking whole of Q_8 to d. Obviously $\eta, \theta \subseteq \psi \subseteq \mu$ but ψ being a constant function is not a nilpotent L-subgroup of μ.

Now, we ascertain the tail of the set product of L-subgroups:

Proposition 7. *Let $\eta, \theta \in L(\mu)$. If $\eta(e) = \theta(e)$, then $inf\eta \circ \theta \geq inf\eta \vee inf\theta$.*

The following theorem sufficiently exhibits the application of the notion of infimums:

Theorem 2. *Let $\eta, \theta \in NL(\mu)$ and $\sigma \in L(\mu)$. If either η and θ or θ and σ have the same tails, then*

$$[\eta \circ \sigma, \theta] \subseteq [\eta, \theta] \circ [\theta, \sigma].$$

Moreover if $\eta(e) = \theta(e)$, then the equality holds.

Proof. Let $x \in G$. If x is not a commutator and η and θ have the same tails, then

$$\begin{aligned}
[\eta, \theta] \circ [\theta, \sigma](x) &\geq [\eta, \theta](x) \wedge [\theta, \sigma](e) \\
&\geq inf\eta \wedge inf\theta \wedge \theta(e) \wedge \sigma(e) \\
&= inf\theta \wedge \sigma(e) \\
&\qquad\qquad (as\ inf\theta = inf\eta \text{ and } inf\theta \wedge \theta(e) = inf\theta) \\
&= inf\theta \wedge \eta(e) \wedge \sigma(e) \\
&\qquad\qquad (as\ inf\theta \wedge \eta(e) = inf\eta \wedge \eta(e) = inf\eta) \\
&= inf\theta \wedge \eta \circ \sigma(e) \\
&\geq inf\theta \wedge inf\eta \circ \sigma \\
&= (\eta \circ \sigma, \theta)(x).
\end{aligned}$$

If θ and σ have the same tails, then also

$$(\eta \circ \sigma, \theta)(x) \leq [\eta, \theta] \circ [\theta, \sigma](x). \tag{1}$$

Suppose that x is a commutator in G. Now, for any $u \in G$, define the following subsets of $G \times G$ by:

$$C(u) = \{(y, z) \in G \times G : u = [y, z]\} \text{ and } P(u) = \{(y, z) \in G \times G : u = yz\}.$$

Now consider

$$
\begin{aligned}
(\sigma \circ \eta, \theta)(x) &= \bigvee_{(y,z)\in C(x)} \{\sigma \circ \eta(y) \wedge \theta(z)\} \\
&= \bigvee_{(y,z)\in C(x)} \{ \bigvee_{(u,v)\in P(y)} \{\sigma(u) \wedge \eta(v)\} \wedge \theta(z)\} \\
&= \bigvee_{(y,z)\in C(x)} \{ \bigvee_{(u,v)\in P(y)} \{\sigma(u) \wedge \eta(v) \wedge \theta(z)\}\} \\
&= \bigvee_{\substack{(y,z)\in C(x)\\(u,v)\in P(y)}} \{\{\sigma(u) \wedge \theta(z)\} \wedge \{\eta(v) \wedge \theta(z)\}\}.
\end{aligned}
$$

As $\sigma \subseteq \mu,$ we have $\sigma(u) \wedge \mu(v) = \sigma(u)$. This implies that

$$
\begin{aligned}
(\sigma \circ \eta, \theta)(x) &\leq \bigvee_{\substack{([v,z]^u,[u,z\in P(x)\\(uv,z)\in C(x)}} \{\{\sigma(u) \wedge \mu(u) \wedge \theta(z)\} \wedge \{\eta(v) \wedge \theta(z)\}\} \\
&= \bigvee_{\substack{([v,z]^u,[u,z])\in P(x)\\(uv,z)\in C(x)}} \{\{\sigma(u) \wedge \theta(z)\} \wedge \{\eta(v) \wedge \theta(z) \wedge \mu(u)\}\}.
\end{aligned}
$$

Now, we have

$$[\sigma, \theta]([u, z]) \geq \sigma(u) \wedge \theta(z),$$

and since $\eta, \theta \in NL(\mu)$, by Proposition 5, $[\eta, \theta] \in NL(\mu)$. Therefore,

$$
\begin{aligned}
[\eta, \theta]([v, z]^u) &\geq [\eta, \theta]([v, z]) \wedge \mu(u) \\
&\geq \eta(v) \wedge \theta(z) \wedge \mu(u).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
(\sigma \circ \eta, \theta)(x) &\leq \bigvee_{\substack{([v,z]^u,[u,z])\in P(x)\\(uv,z)\in C(x)}} \{[\sigma, \theta]([u, z]) \wedge [\eta, \theta]([v, z]^u)\} \\
&\leq [\eta, \theta] \circ [\sigma, \theta](x). \qquad (2)
\end{aligned}
$$

Thus, by (1) and (2)

$$(\sigma \circ \eta, \theta) \leq [\eta, \theta] \circ [\theta, \sigma].$$

Also, as $[\eta, \theta] \in NL(\mu)$ and $[\sigma, \theta] \in L(\mu)$, by Proposition 2.6 $[\eta, \theta] \circ [\sigma, \theta]$ is an $L-$subgroup of μ. Again, by Proposition 2.6, $\eta \circ \sigma \in L(\mu)$ so that $\eta \circ \sigma = \sigma \circ \eta$. Hence

$$[\eta \circ \sigma, \theta] = [\sigma \circ \eta, \theta] \leq [\eta, \theta] \circ [\theta, \sigma].$$

Lastly, let $\eta(e) = \sigma(e)$. We show that

$$[\eta, \theta] \circ [\theta, \sigma] \subseteq [\sigma \circ \eta, \theta].$$

By Proposition 2, $\eta \subseteq \eta \circ \theta$ and $\sigma \subseteq \eta \circ \theta$. By Lemma 5

$$[\eta,\theta]\subseteq[\eta\circ\sigma,\theta] \text{ and } [\sigma,\theta]\subseteq[\eta\circ\sigma,\theta].$$

Therefore,

$$[\eta,\theta]\circ[\theta,\sigma]\subseteq[\eta\circ\sigma,\theta].$$

The proof of the following result can be obtained as in classical group theory which exhibits a routine application of the Principle of Mathematical Induction.

Lemma 1. *Let $\eta,\eta_1,...,\eta_{n+1}\in NL(\mu)$ having identical tails. If $\eta_i=\eta$ for k+1 distinct values of i where $0\leq k\leq n$, then $[\eta_1,\eta_2,...,\eta_{n+1}]\subseteq Z_k(\eta)$.*

Next result provides a necessary and sufficient condition for the set product of two trivial L-subgroups of μ to be a trivial L-subgroup.

Lemma 2. *Let η and θ be trivial L-subgroups of μ. Then, the set product $\eta\circ\theta$ is also a trivial L-subgroup of μ defined by*

$$\eta\circ\theta(x)=\begin{cases}\eta(e)\wedge\theta(e) & \textit{if } \ x=e,\\ inf\eta\vee inf\theta & \textit{if } \ x\neq e,\end{cases}$$

if and only if $inf\eta\vee inf\theta<\eta(e)\wedge\theta(e)$.

Proof. Since η and θ are trivial L-subgroups, it follows that

$$Im\eta=\{inf\eta,\eta(e)\} \text{ and } Im\theta=\{inf\theta,\theta(e)\}.$$

Thus, if $x=e$ then

$$\eta\circ\theta(x)=\eta(e)\wedge\theta(e).$$

Suppose that $x\neq e$. If $\eta(e)=a_0$,$\theta(e)=a_0^*$ and $inf\eta=t_0$, $inf\theta=t_0^*$, then

$$\begin{aligned}\eta\circ\theta(x)&=\bigvee_{x=yz}\{\eta(y)\wedge\theta(z)\}\\ &=\{\eta(x)\wedge\theta(e)\}\vee\{\eta(e)\wedge\theta(x)\}\vee\{\bigvee_{\substack{b\in G\\ b\neq e,b\neq x}}\eta(xb^{-1})\wedge\theta(b)\}\\ &=\{t_0\wedge a_0^*\}\vee\{a_0\wedge t_0^*\}\vee\{t_0\wedge t_0^*\}\\ &=\{t_0\wedge a_0^*\}\vee\{a_0\wedge t_0^*\}, \quad (\text{as } \ a_0\wedge t_0^*\geq t_0\wedge t_0^*)\\ &=\{t_0\vee\{a_0\wedge t_0^*\}\}\vee\{a_0^*\vee\{a_0\wedge t_0^*\}\}\\ &=\{t_0\vee\{a_0\wedge t_0^*\}\}\wedge a_0^* \qquad (\text{as } \ a_0^*\geq t_0^*\geq a_0\wedge t_0^*)\\ &=\{a_0\wedge\{t_0\vee t_0^*\}\}\wedge a_0^* \qquad (\text{as } L \text{ is modular})\\ &=\{a_0\wedge a_0^*\}\wedge\{t_0\vee t_0^*\}.\end{aligned}$$

Thus, if $inf\eta\vee inf\theta<\eta(e)\wedge\theta(e)$, then $\eta\circ\theta$ is the trivial L-subgroup given by

$$\eta\circ\theta(x)=\begin{cases}\eta(e)\wedge\theta(e) & \text{if } \ x=e,\\ inf\eta\vee inf\theta & \text{if } \ x\neq e.\end{cases}$$

On the other hand if $\eta\circ\theta$ is a trivial L-subgroup as given above, then for any $x\neq e$

$$\eta \circ \theta(x) = inf\eta \vee inf\theta = \{\eta(e) \wedge \theta(e)\} \wedge \{inf\eta \vee inf\theta\}.$$

Thus, $inf\eta \vee inf\theta \leq \eta(e) \wedge \theta(e)$. Since $\eta \circ \theta$ is a trivial L-subgroup

$$inf\eta \vee inf\theta \neq \eta(e) \wedge \theta(e).$$

Therefore, $inf\eta \vee inf\theta < \eta(e) \wedge \theta(e)$.

Theorem 3. *Let $\eta, \theta \in NL(\mu)$ with common tail t_0 such that $t_0 < \eta(e) \wedge \theta(e)$ and $inf\eta \circ \theta = t_0$. If η and θ are nilpotent of classes c and d respectively, then $\eta \circ \theta$ is a nilpotent L-subgroup of μ of nilpotent class at most $c + d$.*

Proof. Since $\eta, \theta \in NL(\mu)$, by Proposition 1, we conclude that $\eta \circ \theta \in NL(\mu)$. In view of Proposition 6, $Z_i(\eta \circ \theta) \in NL(\mu)$. Now, let η and θ be nilpotent of classes c and d respectively. In order to show that the set product $\eta \circ \theta$ is nilpotent, we show that the descending central series of $\eta \circ \theta$ terminates finitely. Set $\lambda = \eta \circ \theta$ so that

$$inf\lambda = t_0. \tag{1}$$

In view of Lemma 2, the set product of two trivial L-subgroups is a trivial L-subgroup provided the join of their tails is different from the meet of their tips. Thus, as $t_0 < \eta(e) \wedge \theta(e)$ and by (1), we have

$$\eta_{t_0}^{a_o} \circ \theta_{t_0}^{a_0^*} = \lambda_{t_0}^{a_o \wedge a_0^*},$$

where a_o and a_0^* denote the tips of η and θ respectively. To achieve our aim, we demonstrate that

$$Z_n(\eta \circ \theta) = \lambda_{t_0}^{a_o \wedge a_0^*},$$

for some integer $n \geq 0$. As η and θ are nilpotent of classes c and d respectively, we get

$$Z_c(\eta) = \eta_{t_0}^{a_o} \text{ and } Z_d(\theta) = \theta_{t_0}^{a_0^*}. \tag{2}$$

To prove the result, it is sufficient to show that for some positive integer n, $Z_n(\eta \circ \theta)$ is contained in the set product of trivial L-subgroups $\eta_{t_0}^{a_o}$ and $\theta_{t_0}^{a_0^*}$. Firstly, we claim that for any positive integer n, $Z_n(\eta \circ \theta)$ is contained in the set product of L-subgroups of the form $[\lambda_1, \lambda_2, ..., \lambda_{n+1}]$, where $\lambda_i = \eta$ or θ. As $\eta \circ \theta \in NL(\mu)$, in view of (1) and Theorem 2, it follows that

$$Z_1(\eta \circ \theta) = [\eta \circ \theta, \eta \circ \theta] \subseteq [\eta, \eta] \circ [\eta, \theta] \circ [\theta, \theta].$$

Suppose that for some positive integer k, $Z_k(\eta \circ \theta)$ is contained in the set product of L-subgroups of the form $[\lambda_1, \lambda_2, ..., \lambda_{k+1}]$, where $\lambda_i = \eta$ or θ. Also,

$$infZ_k(\eta \circ \theta) = inf\eta \circ \theta.$$

(3)

Hence in view of (1) and Theorem 2, we have

$$Z_{k+1}(\eta \circ \theta) = [Z_k(\eta \circ \theta), \eta \circ \theta] \subseteq [Z_k(\eta \circ \theta), \eta] \circ [Z_k(\eta \circ \theta), \theta].$$

By the hypothesis $Z_k(\eta \circ \theta)$ is contained in the set product of L-subgroups of the form $[\lambda_1, \lambda_2, ..., \lambda_{k+1}]$, where $\lambda_i = \eta$ or θ. Thus, it follows that $Z_{k+1}(\eta \circ \theta)$ is contained in the set product of L-subgroups of the form $[\lambda_1, \lambda_2, ..., \lambda_{k+2}]$, where $\lambda_i = \eta$ or θ. Thus, by the Principle of Mathematical Induction our claim is established for every positive integer n. Now, let $n = c + d$. Then, in any commutator L-subgroup of the form $[\lambda_1, \lambda_2, ..., \lambda_{n+1}]$ if the number of occurrences of η is greater than c, then by Lemma 1 and by (2)

$$[\lambda_1, \lambda_2, ..., \lambda_{n+1}] \subseteq Z_c(\eta) = \eta_{t_0}^{a_o}.$$

On the other hand, if the number of occurrences of η is less than or equal to c, then the number of occurrences of θ is greater than or equal to $d + 1$. Hence, again by Lemma 1 and by (2)

$$[\lambda_1, \lambda_2, ..., \lambda_{n+1}] \subseteq Z_d(\theta) = \theta_{t_0}^{a_o^*}.$$

Thus, each L-subgroup of the form $[\lambda_1, \lambda_2, ..., \lambda_{n+1}]$, where $\lambda_i = \eta$ or θ is contained in $\eta_{t_0}^{a_o}$ or $\theta_{t_0}^{a_o^*}$. Therefore, $Z_k(\eta \circ \theta)$ is contained in the set product of finitely many trivial L-subgroups $\eta_{t_0}^{a_o}$ and $\theta_{t_0}^{a_o^*}$.This product turns out to be $\eta_{t_0}^{a_o} \circ \theta_{t_0}^{a_o^*} = \lambda_{t_0}^{a_0 \wedge a_0^*}$.On the other hand, $Z_n(\eta \circ \theta)(e) = \eta \circ \theta(e)$. Also, in view of (1) and (3)

$$inf Z_n(\eta \circ \theta) = t_0.$$

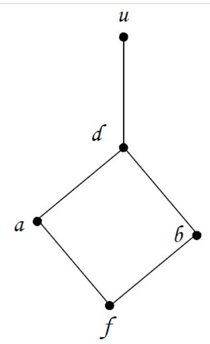
This implies

$$\lambda_{t_0}^{a_0 \wedge a_0^*} \subseteq Z_n(\eta \circ \theta) \subseteq \lambda_{t_0}^{a_0 \wedge a_0^*}.$$

Hence $Z_n(\eta \circ \theta) = \lambda_{t_0}^{a_0 \wedge a_0^*}$.

Below we illustrate the above theorem with the help of an example:

Example 4. *Let $G = Q_8$ as in Example 1.Let L be the evaluation lattice given by the diagram :*



Let C,H_1, H_2 and H_3 be the following subgroups of G:

$$C = \{\pm 1\}, \quad H_1 = \{\pm 1, \pm i\}, \quad H_2 = \{\pm 1, \pm j\}, \quad H_3 = \{\pm 1, \pm k\}.$$

Consider the parent L-subgroup of G, defined as follows:

$$\mu(x) = \begin{cases} u & \text{if } x = 1, \\ d & \text{if } x \in G \setminus \{1\}. \end{cases}$$

Now define L-subsets η and θ of G, as given below:

$$\eta(x) = \begin{cases} d & \text{if } x = 1, \\ a & \text{if } x \in H_1 \setminus \{1\}, \\ f & \text{if } x \in G \setminus H_1; \end{cases}$$

and

$$\theta(x) = \begin{cases} u & \text{if } x = 1, \\ b & \text{if } x \in H_2 \setminus \{1\}, \\ f & \text{if } x \in G \setminus H_2. \end{cases}$$

Since the level subsets of η are normal subgroups of G, η is a normal L-subgroup of G. Similarly, it can be seen that θ is also a normal $L-$subgroup of G and hence of μ. Now η is a nilpotent L-subgroup of μ , in view of the Definition 3.5 follows as given below:

Note that $G' = C$. In order to obtain the members of descending central series, we start with $Z_0(\eta) = \eta$. Next consider the commutator (η, η):

$$(\eta, \eta)(x) = \begin{cases} d & \text{if } x = 1, \\ f & \text{if } x \in G \setminus \{1\}. \end{cases}$$

As the level subsets of (η, η) are subgroups,

$$Z_1(\eta) = [\eta, \eta] = (\eta, \eta).$$

Note that $Z_1(\eta)$ is the trivial L-subgroup η_f^d of η and so the descending central series terminates at $Z_1(\eta)$ i.e.

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) = \eta_f^d.$$

Consequently η is a nilpotent $L-$subgroup of μ having nilpotent length 1. Similarly it can be shown that θ is also a nilpotent $L-$subgroup of μ having nilpotent length 1. Next, we exhibit the set product of L-subgroups η and θ. It can be verified that

$$\eta \circ \theta = \theta \circ \eta = \phi,$$

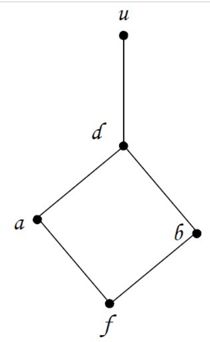
where ϕ is the $L-$subgroup of μ given by

$$\phi(x) = \begin{cases} d & \text{if } x \in C, \\ a & \text{if } x \in H_1 \setminus C, \\ b & \text{if } x \in H_2 \setminus C, \\ f & \text{if } x \in G \setminus \{H_1 \cup H_2\}. \end{cases}$$

Note that here $inf\eta = inf\theta = inf\eta \circ \theta$. Hence in view of the above theorem $\eta \circ \theta$ is a nilpotent L-subgroup of μ.

The following example exhibits that the condition $inf\eta = inf\theta = inf\eta \circ \theta$ is only sufficient:

Example 5. *Let $G = Q_8$ as in Example 1.Let L be the evaluation lattice given by the diagram :*



Let C,H_1, H_2 and H_3 be the following subgroups of G:

$$C = \{\pm 1\}, \quad H_1 = \{\pm 1, \pm i\}, \quad H_2 = \{\pm 1, \pm j\}, \quad H_3 = \{\pm 1, \pm k\}.$$

Consider the parent L-subgroup of G, defined as follows:

$$\mu(x) = \begin{cases} u & \text{if } x = 1, \\ d & \text{if } x \in G \setminus \{1\}. \end{cases}$$

Now define L-subsets η, and θ of G, as given below:

$$\eta(x) = \begin{cases} d & \text{if } x \in C, \\ a & \text{if } x \in H_1 \setminus C, \\ b & \text{if } x \in H_2 \setminus C, \\ f & \text{if } x \in G \setminus \{H_1 \cup H_2\}; \end{cases}$$

and

$$\theta(x) = \begin{cases} d & \text{if } x \in C, \\ a & \text{if } x \in H_1 \setminus C, \\ b & \text{if } x \in H_3 \setminus C, \\ f & \text{if } x \in G \setminus \{H_1 \cup H_3\}. \end{cases}$$

Since the level subsets of η are normal subgroups of G, η is a normal L-subgroup of G. Similarly, it can be seen that θ is also a normal $L-$subgroup of G and hence of μ. Now that η is a nilpotent L-subgroup of μ , in view of the Definition 3.5, can be seen as follows:

Note that $G' = C$. In order to obtain the members of descending central series, we start with $Z_0(\eta) = \eta$. Next consider the commutator (η, η):

$$(\eta, \eta)(x) = \begin{cases} d & \text{if } x = 1, \\ f & \text{if } x \in G \setminus \{1\}. \end{cases}$$

As the level subsets of (η, η) are subgroups,

$$Z_1(\eta) = [\eta, \eta] = (\eta, \eta).$$

Note that $Z_1(\eta)$ is the trivial L-subgroup η_f^d of η and so the descending central series terminates at $Z_1(\eta)$ i.e.

$$\eta = Z_0(\eta) \supseteq Z_1(\eta) = \eta_f^d.$$

Consequently η is a nilpotent $L-$subgroup of μ having nilpotent length 1. Similarly it can be shown that θ is also a nilpotent $L-$subgroup of μ having nilpotent length 1. Next, we exhibit the set product of L-subgroups η and θ. It can be verified that

$$\eta \circ \theta = \theta \circ \eta = \phi,$$

where ϕ is the $L-$subgroup of μ given by

$$\phi(x) = \begin{cases} d & \text{if } x \in H_1, \\ b & \text{if } x \in G \setminus H_1. \end{cases}$$

Again in view of the converse of Theorem 4.1[3], it follows that ϕ is a nilpotent L-subgroup but

$$inf\phi \neq inf\eta \text{ and } inf\theta.$$

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